Arkansas Tech University
MATH 3243: Differential Equations I
Dr. Marcel B Finan

1.1 Definitions and Terminology

In many models, we will have equations involving the derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables and are interested in discovering these functions. Such equations are referred to as **differential equations** (abbreviated DE). They arise in many applications such as population growth, decay of radioactive substance, the motion of a falling object, electrical network, Newton's law of cooling and many more models.

A First Source of Differential Equations: Projectile Motion
Suppose that an object initially at height y_0 is thrown up with initial velocity v_0. Let $y(t)$ denote the distance of the object from the ground , $v(t)$ the object's velocity, and $a(t)$ the object's acceleration at time t. If air resistance is neglected, then by **Newton's second law**, which states that the net force is equal to the product of mass and acceleration, we have $ma(t) = -mg$. The negative sign on the right-hand of the equation is due to the fact that acceleration due to gravity is pointing downward. Using the fact that $a(t) = y''(t)$ and eliminating the mass, we obtain the equation

$$y'' = -g.$$

To find the velocity $v(t)$ we integrate for a first time and obtain

$$v(t) = y'(t) = -gt + C_1.$$

Since the initial velocity is v_0, we obtain $C_1 = v(0) = v_0$ so that

$$v(t) = -gt + v_0.$$

Integrating for the second time we find the position function

$$y(t) = -\frac{1}{2}gt^2 + v_0t + C_2.$$

Since y_0 is the initial height, we find $C_2 = y_0$ and so

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0.$$

Example 1.1.1
An object is dropped from the top of a cliff that is 144 feet above ground level.
(a) When will the object reach ground level?
(b) What is the velocity with which the object strikes the ground?

Solution.
(a) The motion of the object translates to the differential equation $y'' = -32$ with solution $y(t) = -16t^2 + 144$. The object reaches ground level when $y(t) = 0$ or $16t^2 = 144$. Solving for t we find $t = \sqrt{\frac{144}{16}} = 3$ sec. The object will reach the ground 3 seconds after it is dropped from the cliff.
(b) The object strikes the ground with velocity $v(3) = -32(3) = -96$ ft/sec ■

Basic Terms of Differential Equations
We next discuss some basic notions of differential equations. There are two types of differential equations: ordinary and partial differential equations.
A differential equation is an **ordinary differential equation** (abbreviated ODE) if the unknown functions (the dependent variables) are functions of a single variable (the independent variable). When the unknown functions are functions of two or more independent variables then the differential equation is called a **partial differential equation** (abbreviated PDE). For example, the **wave equation** is a partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

In this course, when we use the term differential equation, we'll mean an ordinary differential equation.

Remark 1.1.1
The notation $y^{(n)}$ for the derivative is known as the **prime notation**. An alternative notation which displays both the dependent and the independent variables is **Leibniz notation** $\frac{d^n y}{dx^n}$. When the independent variable is time then a third notation is used, known as the **dot notation**. Thus, $\ddot{s}$ stands for $\frac{d^2 s}{dt^2}$.
For partial differentials, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$, and $\frac{\partial^2 u}{\partial x \partial t}$ can be represented by u_x, u_{xx}, and u_{xt}.

Example 1.1.2
Identify which variables are dependent and which are independent for the following differential equations.
(a) $\frac{d^4y}{dx^4} - x^2 + y = 0.$
(b) $u_{tt} + xu_{tx} = 0.$
(c) $x\frac{dx}{dt} = 4.$
(d) $\frac{\partial y}{\partial u} - 4\frac{\partial y}{\partial v} = u + 3y.$

Solution.
(a) Independent variable is x and the dependent variable is y.
(b) Independent variables are x and t and the dependent variable is u.
(c) Independent variable is t and the dependent variable is x.
(d) Independent variables are u and v and the dependent variable is y ■

Example 1.1.3
Classify the following as either ODE or PDE.
(a) $u_t = c^2 u_{xx}.$
(b) $y'' - 4y' + 5y = 0.$
(c) $z_t + cz_x = 5.$

Solution.
(a) A PDE with dependent variable u and independent variables t and x.
(b) An ODE with dependent variable y and independent variable x.
(c) A PDE with dependent variable z and independent variables t and x ■

The highest order derivative that appears in a differential equation is known as the **order** of the equation. Thus, an n^{th} **order ordinary differential equation** is an equation of the form

$$g(x, y, y', y'', \cdots, y^n) = 0 \tag{1.1.1}$$

or alternatively

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}). \tag{1.1.2}$$

We call Equation (1.1.2) the **normal form** of Equation (1.1.1).
A **first-order ordinary differential equation**, for example, takes the form $g(x, y(x), y'(x)) = 0$, and may alternatively be written as

$$y'(x) = f(x, y(x)) \tag{1.1.3}$$

for all x in the interval of existence of y. Equation (1.1.3) can be written in the form

$$M(x,y)dx + N(x,y)dy = 0.$$

We refer to this form as the **differential form** of a first order differential equation.

Example 1.1.4
Find the differential form of the equation $4x\frac{dy}{dx} + x = y$.

Solution.
The differential form is $(x-y)dx + 4xdy = 0$ ■

Example 1.1.5
Determine the order of each equation.
(a) $y' + 2xy = e^{-x^2}$.
(b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y(x) = 0$.
(c) $y'' + 3xy' + 2y = \sin(5x)$.

Solution.
(a) This is a first order differential equation because the highest derivative is the first derivative.
(b) and (c) are second order differential equations since the highest derivative in each equation is the second order derivative ■

Any differential equation that can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x) \tag{1.1.4}$$

where $a_1(x), \cdots, a_0(x)$ and $g(x)$ are functions with common domain $a < x < b$, is called an n^{th} **order linear differential equation**. The term linear is used because $\mathcal{L}(y) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y$ is linear in y. That is, $\mathcal{L}(\alpha y_1 + \beta y_2) = \alpha\mathcal{L}(y_1) + \beta\mathcal{L}(y_2)$. An ODE that is not linear is called **non-linear.**
It follows from the above definition, that an n^{th} order differential equation is linear when the unknown function and/or its derivatives appear with power 1;, are not part of a composite function, and the coefficients of y and its derivatives depend at most on the independent variable x.
If $g(x) = 0$ in (1.1.4) then the equation is said to be **homogeneous**. Otherwise, the equation is **non-homogeneous.** Keep in mind that the concept of homogeneity applies only for linear differential equations.

Example 1.1.6
Classify each of the following differential equations as linear or non-linear.
(a) $\frac{dy}{dx} + \frac{y}{10} = xy$.
(b) $x^2 - 3y^2 + 2xy\frac{dy}{dx} = 0$.

Solution.
(a) Notice that the given equation can be written as $\frac{dy}{dx} + (\frac{1}{10} - x)y = 0$ which is a first order linear DE where $a_1(x) = 1,\ a_0(x) = \frac{1}{10} - x$ and $g(x) = 0$.
(b) This is a first order non-linear because of the term y^2 and a_1 depends on both x and y ■

A **solution** to a differential equation is a function that satisfies the equation: When you substitute this function and its derivatives into the differential equation, you get a true mathematical statement or an identity. The function and its derivatives must be defined and continuous on a common domain, called the **interval of existence** or the **domain of the solution**.

Example 1.1.7
Show that $y(x) = \frac{1}{x}$ is a solution to the equation $xy' + y = 0$. Find the interval of existence of this solution.

Solution.
We have $xy' + y = x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = 0$. Thus, $y = \frac{1}{x}$ is a solution to the differential equation. Note that as a function, $y = \frac{1}{x}$ is defined for all $x \neq 0$. However, as a solution to a differential equation, we restrict this function to $(0, \infty)$ since the derivative of $y = \frac{1}{x}$ is not continuous in an interval containing 0. Thus, an interval of existence of the solution is $(0, \infty)$. Also, an interval of existence is $(-\infty, 0)$ ■

Example 1.1.8 *(A piecewise-defined solution)*
Consider the differential equation $xy' - 4y = 0$ on the interval $(-\infty, \infty)$. Verify that the piecewise-defined function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

is a solution.

Solution.
For $x < 0$, we have $xy' - 4y = x(-x^4)' - 4(-x^4) = -4x^4 + 4x^4 = 0$. For $x \geq 0$, we have $xy' - 4y = x(x^4)' - 4x^4 = 4x^4 - 4x^4 = 0$. Thus, the given function is a solution with interval of existence $(-\infty, \infty)$ ■

Example 1.1.9 *(Integral-Defined Solution)*

Show that the function $y(x) = \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}} dt$ is a solution to the differential eqaution $2xy' - y = 2x\cos x$.

Solution.
To see, we first find

$$y'(x) = \frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}dt + \sqrt{x}\frac{\cos x}{\sqrt{x}}$$

using the product rule. Thus, by substitution, we find

$$2xy' - y = \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}dt - \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}dt = 0 \blacksquare$$

The solution $y(x) = 0$ for all x in the interval of existence is called the **trivial solution**. Likewise, a solution $y(x) = c$ for all x in the interval of existence is called a **constant solution**.

Solving a differential equation means finding all possible solutions of the equation.

Example 1.1.10
Solve the differential equation:

$$y'' = -2x.$$

Solution.
Integrating twice, all the solutions have the form

$$y(x) = -\frac{x^3}{3} + C_1 x + C_2$$

with interval of solution $(-\infty, \infty)$ ■

Note that the function of the previous example defines all the solutions to the differential equation. Such a function will be referred to as the **general solution**. The constants C_1 and C_2 are called the **parameters**. This general solution is an example of a **two-parameter family of solutions**. Specific values of C_1 and C_2 determine what is called a **particular solution**. To find a particular solution additional conditions on the values of the function or its derivatives must be given. Such conditions are called **initial conditions**. A differential equation together with a set of initial conditions is called an **initial value problem** (abbreviated IVP).

Example 1.1.11
Consider the differential equation $y''(x) - 1 = 0$.
(a) Find the general solution of this equation.
(b) Find the particular solution that satisfies the initial conditions $y(1) = 1$ and $y'(1) = 4$.

Solution.
(a) Integrating twice we find the general solution

$$y(x) = \frac{x^2}{2} + C_1 x + C_2.$$

(b) Since $y'(x) = x + C_1$ and $y'(1) = 4$, we find $4 = 1 + C_1$ so that $C_1 = 3$. Hence, $y(x) = \frac{x^2}{2} + 3x + C_2$. Now, since $y(1) = 1$, we have $1 = \frac{1}{2} + 3 + C_2$. Solving for C_2 we find $C_2 = -\frac{5}{2}$. Hence, the solution to the IVP

$$\begin{cases} y''(x) - 1 = 0 \\ y'(1) = 4, y(1) = 1 \end{cases}$$

is

$$y(x) = \frac{x^2}{2} + 3x - \frac{5}{2} \blacksquare$$

Boundary Value Problems

Initial conditions are prescribed at a single value of the independent variable. However, in some cases we require that conditions are prescribed at multiple values of the independent variable. In this cse, these conditions are called **boundary value conditions**. Boundary value conditions together with the differential equation form what is called **boundary value problems**.

Example 1.1.12
Solve the boundary condition problem: $y'' = -1,\ y(-2) = 0, y(2) = 0$.

Solution.
Solving the differential equation, we find $y(x) = -\frac{1}{2}x^2 + C_1x + C_2$. From the condition $y(-2) = 0$, we obtain $-2C_1 + C_2 = 2$. From the condition $y(2) = 0$, we obtain $2C_1 + C_2 = 0$. Solving this system of equations, we find $C_1 = 0$ and $C_2 = 2$.Hence, the solution to the (BVP) is $y(x) = -\frac{1}{2}x^2 + 2$ ■

The graph of a particular solution is called a **solution curve**. The function $y(x) = Ce^{-3x} + 2x + 1$ is the general solution to the differential equation $y' + 3y = 6x + 5$. It is a **one-parameter** family of solutions. A family of solution curves is shown in Figure 1.1.1. Notice for $C \neq 0$ the solution curves have an oblique asymptote with equation $y(x) = 2x + 1$.

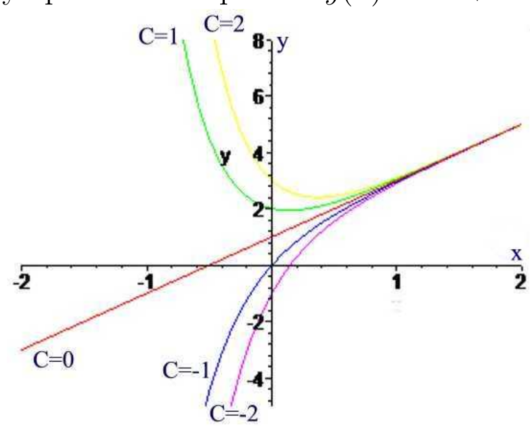

Figure 1.1.1

Sometimes a differential equation possesses a solution that cannot be obtained by assigning values to the parameters in a family of solutions. Such a solution is called a **singular solution**.

Example 1.1.13
The non-zero solutions to the differential equation $y' = xy^{\frac{1}{2}}$ are given by the one-parameter family $y(x) = (\frac{x^2}{4} + C)^2$. Find the singular solution.

Solution.
The function $y(x) \equiv 0$ is a solution to the differential equation. This is a singular solution since it cannot be obtained from the family for any choice

of the parameter C. The general solution consists of all the solutions of the form $y(x) = (\frac{x^2}{4} + C)^2$ together with the zero solution ■

If the dependent variable of solution to a differential equation can be expressed in terms of the independent variable only, for example, $y = f(x)$, then we refer to such a solution as an **explicit solution**. If a solution is written in the form $f(x, y) = 0$ then we say that the solution is defined **implicitly**.

Example 1.1.14
Consider the differential equation $y' = -\frac{x}{y}$.
(a) Show that $x^2 + y^2 = 25$ is an implicit solution to the given equation. What is the interval of existence of such a solution.
(b) Find two explicit solutions to the given differential equation.

Solution.
(a) Using implicit differentiation, we find $2x + 2yy' = 0$. Solving for y', we find $y' = -\frac{x}{y}$. The solution curve is a circle centered at the origin and with radius 5. Hence, the interval of solution is the interval $(-5, 5)$.
(b) Solving the equation $x^2 + y^2 = 25$ for y we find $y = \pm\sqrt{25 - x^2}$. The functions $y = f_1(x) = \sqrt{25 - x^2}$ and $y = f_2(x) = -\sqrt{25 - x^2}$ are explicit solutions to the given differential equation. The solution curve of $f_1(x)$ is the upper circle centered at the origin and with radius 5. The solution curve of $f_2(x)$ is the lower circle centered at the origin and with radius 5 ■

A **system of ordinary differential equations** consists of two or more ODEs involving the derivatives of two or more unknown functions in a single variable. A **solution** to a system of ordinary differential equations with unknown functions $x_1, x_2, \cdots, x_n$ of a single variable t consists of n differentiable functions $x_1 = f_1(t), x_2 = f_2(t), \cdots, x_n = f_n(t)$ defined on a common interval, that satisfy each equation of the system on that interval.

Example 1.1.15
Verify that the functions $x = \cos(2t) + \sin(2t) + \frac{1}{5}e^t$ and $y = -\cos(2t) - \sin(2t) - \frac{1}{5}e^t$ are solutions to the system

$$\begin{aligned}\frac{d^2x}{dt^2} &= 4y + e^t \\ \frac{d^2y}{dt^2} &= 4x - e^t\end{aligned}$$

on the interval $(-\infty, \infty)$.

Solution.
By differentiation, we have

$$\begin{aligned}
\frac{dx}{dt} =& -2\sin(2t) + 2\cos(2t) + \frac{1}{5}e^t \\
\frac{d^2x}{dt^2} =& -4\cos(2t) - 4\sin(2t) + \frac{1}{5}e^t = 4y + e^t \\
\frac{dy}{dt} =& 2\sin(2t) - 2\cos(2t) - \frac{1}{5}e^t \\
\frac{d^2y}{dt^2} =& 4\cos(2t) + 4\sin(2t) - \frac{1}{5}e^t = 4x - e^t \blacksquare
\end{aligned}$$