Undergraduate Notes in Mathematics

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An Introductory Single Variable Real Analysis: A Learning Approach through Problem Solving

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Preface

The present manuscript is designed for an introductory course in real analysis suitable to junior or beginning senior level students who already had the calculus sequel as well as a course in discrete mathematics or an equivalent course in mathematical proof. The content is considered of a moderate level of difficulty.
The manuscript evolved from a class I taught at Arkansas Tech University in undergraduate real analysis. The class consisted of both Mathematics majors and Mathematics education majors. The approach adopted in this book is a modified Moore method also known as Inquiry-Based Learning (IBL). The basic results in single-variable analysis were submitted to the students in the form of definitions and short problems that the students were asked to solve and present their findings to their peers during class time with a little mentoring by the instructor. By leaving routine parts of a proof to the students constitutes a recommendable tactic since it will include the students in the enterprise of establishing the proof of a theorem, and thus strengthen their conviction in the end.
The objectives of these notes are: Firstly, enhancing the student's mathematical thinking and problem-solving ability. Secondly, it improves the students' skills in writing and presenting mathematical content, two essential components for future mathematics educators.

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Chapter 0

Requirements

For this book, we require three things from the users: (1) Understanding the vocabulary of a mathematical system such as axiom, definition, lemma, etc. (2) understanding the basics of mathematical logic, and (3) writing elegant mathematics.

0.1 Vocabulary of Mathematical Systems

What is a mathematical system? We will define a **mathematical system** to be a triplet $(\mathcal{S}, \mathcal{O}, \mathcal{A})$, where $\mathcal{S}$ is a non-empty sets of elements, $\mathcal{O}$ is a collection of operations on the elements of $\mathcal{S}$, and $\mathcal{A}$ is a set of axioms involving the elements of $\mathcal{S}$ and the operations $\mathcal{O}$. For example, $\mathcal{S} = \mathbb{R}$ is the set of all real numbers. The basic operations on the real numbers are addition and multiplication. Examples of axioms in $\mathbb{R}$ are the commutative property of both addition and multiplication of real numbers.

What is a definition? By a **mathematical definition** we mean a mathematical statement that involve the elements of $\mathcal{S}$ and $\mathcal{O}$. For example, we define the average of two real numbers a and b to be the real number $\frac{1}{2}(a+b)$.

What is an axiom? An **axiom** is a statement that is accepted without questioning. For example, the axioms of the real lines are:
(i) Commutativity: $a + b = b + a$ and $ab = ba$.
(ii) Associativity: $(a+b)+c = a+(b+c) = a+b+c$ and $a(bc) = (ab)c = abc$.
(iii) Distributivity of multiplication over addition: $a(b + c) = ab + ac$.

(iv) Zero element: $a + 0 = 0 + a = a$.
(v) Identity element: $a \cdot 1 = 1 \cdot a = a$.
(vii) Opposite element: $a + (-a) = (-a) + a = 0$.
(viii) Inverse element: $a \cdot \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \cdot a = 1$, where $a \neq 0$.

What is a proof? A **proof** is an argument establishing either the truth or falsity of a mathematical statement.

What is a theorem? A **theorem** is a mathematical statement that requires a proof.

What is a lemma? A **lemma** is a minor statement that requires a proof but whose sole purpose is to help proving a major result such as a theorem.

What is a Corollary A **corollary** is a theorem that is the result of previously proved major theorem.

0.2 Some Vocabulary of Logic

A **proposition** is any meaningful statement that is either true or false, but not both. We will use lowercase letters, such as $p, q, r, \cdots$, to represent propositions. New propositions called **compound propositions** or **propositional functions** can be obtained from old ones by using **symbolic connectives** which we discuss next.

Let p and q be propositions. The **conjunction** of p and q, denoted by $p \wedge q$ (read "p wedge q"), is the proposition: p *and* q. This proposition isdefined to be true only when both p and q are true and it is false otherwise. The **disjunction** of p and q, denoted by $p \vee q$ (read "p vee q"), is the proposition: p *or* q. The "or" is used in an inclusive way. This proposition is false only when both p and q are false, otherwise it is true. The **negation** of p, denoted $\sim p$, is the proposition that is false when p is true and true when p is false.

Two propositions are **logically equivalent** if they have exactly the same truth values under all circumstances. We write $p \equiv q$.

Let p and q be propositions. The **conditional proposition** $p \to q$ is the proposition that is false only when p is true and q is false; otherwise it is true. p is called the **hypothesis** and q is called the **conclusion.** It is easy to show that $p \to q \equiv (\sim p) \vee q$.
The proposition $p \to q$ is always true if the hypothesis p is false, regardless of the truth value of q. We say that $p \to q$ is **true by default** or **vacuously true**.
In terms of words the proposition $p \to q$ also reads:
(a) if p then q.
(b) p implies q.
(c) p is a sufficient condition for q.
(d) q is a necessary condition for p.
(e) p only if q.

The **converse** of $p \to q$ is the proposition $q \to p$. The **opposite** or **inverse** of $p \to q$ is the proposition $\sim p \to \sim q$. The **contrapositive** of $p \to q$ is the proposition $\sim q \to \sim p$. It can be shown that $[p \to q] \equiv [\sim q \to \sim p]$. This is referred to as the **proof by contrapositive**.

The **biconditional proposition** of p and q, denoted by $p \leftrightarrow q$, is the propositional function that is true when both p and q have the same truth values and false if p and q have opposite truth values. Also reads, "p if and only if q" or "p is a necessary and sufficient condition for q." It can be shown that $p \leftrightarrow q \equiv [p \rightarrow q] \wedge [q \rightarrow p]$.

To say that statements $p_1, p_2, \cdots, p_n$ are all equivalent means that either they are all true or all false. To prove that they are equivalent, one assumes p_1 to be true and proves that p_2 is true, then assumes p_2 to be true and proves that p_3 is true, continuing in this fashion, assume that p_{n-1} is true and prove that p_n is true and finally, assume that p_n is true and prove that p_1 is true. This is known as the **proof by circular argument**.

An indirect method of proving $p \rightarrow q$ is the **proof by contradiction**: We want to show that q is true. Instead, we assume it is not, i.e., $\sim q$ is true, and derive that a proposition of the form $r \wedge \sim r$ is true. But $r \wedge \sim r$ is a contradiction which is always false. Hence, the assumption $\sim q$ must be false, so the original proposition q must be true. The method of proof by contradiction is not limited to just proving conditional propositions of the form $p \rightarrow q$, it can be used to prove any kind of statement whatsoever.

Finally, we recall the reader of the method of **proof by induction**: We want to prove that a statement $P(n)$ is true for any non-negative integer $n \geq n_0$. The steps of mathematical induction are as follows:
(i) (Basis of induction) Show that $P(n_0)$ is true.
(ii) (Induction hypothesis) Assume $P(k)$ is true for $n_0 \leq k \leq n$.
(iii) (Induction step) Show that $P(n+1)$ is true.

0.3 Writing Proofs Elegantly

An elegant proof is a proof that uses words and sentences and not just a sequence of symbols. A weak proof is one that consists mainly of symbols. To be more precise. Suppose that we want to prove that the sum of two even integers m and n is always even. An elegant proof proceeds as follows: Since m and n are even integers, there exist integers k_1 and k_2 such that $m = 2k_1$ and $n = 2k_2$. Adding the two numbers together, we find $m+n = 2k_1+2k_2 = 2(k_1 + k_2) = 2k$, where $k = k_1 + k_2 \in \mathbb{Z}$. From the definition of an even integer, we conclude that $m + n$ is even.
An example of a weak or non-explanatory proof of the above result can be as follows:

$$m = 2k_1, n = 2k_2 \Rightarrow m + n = 2(k_1 + k_2) = 2k \Rightarrow m + n \text{ is even.}$$

Chapter 1

Properties of Real Numbers

In this chapter we review the important properties of real numbers that are needed in this course.

1.1 Basic Properties of Absolute Value

In this section, we introduce the absolute value function and we discuss some of its properties. The use of absolute value will be apparent in many of the discussions of this course.

Definition 1.1.1
Let $a \in \mathbb{R}$. We define the **absolute value** of a, denoted by $|a|$, to be the largest of the two numbers a and $-a$. That is,

$$|a| = \max\{-a, a\}.$$

Exercise 1.1.1
Show that $|a| \geq a$ and $|a| \geq -a$.

Exercise 1.1.2
Show that

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

That is, the absolute value function is a piecewise defined function. Graph this function in the rectangular coordinate system.

Exercise 1.1.3
Show that $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$.

Exercise 1.1.4
Show that if $|a| = |b|$ then $a = \pm b$.

Exercise 1.1.5
Solve the equation $|3x - 2| = |5x + 4|$.

Exercise 1.1.6
Show that $|-a| = |a|$.

Exercise 1.1.7
Show that $|ab| = |a| \cdot |b|$.

Exercise 1.1.8
Show that $\left|\frac{1}{a}\right| = \frac{1}{|a|}$, where $a \neq 0$.

Exercise 1.1.9
Show that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $b \neq 0$.

Exercise 1.1.10
Show that for any two real numbers a and b we have $ab \leq |a| \cdot |b|$.

Exercise 1.1.11
Recall that a number $b \geq 0$ is the **square root** of a number a, written $\sqrt{a} = b$, if and only if $a = b^2$. Show that

$$\sqrt{a^2} = |a|.$$

Exercise 1.1.12
Suppose that A and B are points on a coordinate line that have coordinates a and b, respectively. Show that $|a - b|$ is the distance between the points A and B. Thus, if $b = 0$, $|a|$ measures the distance from the number a to the origin.

Exercise 1.1.13
Graph the portion of the real line given by the inequality
(a) $|x - a| < \delta$
(b) $0 < |x - a| < \delta$
where $\delta > 0$. Represent each graph in interval notation.

Exercise 1.1.14
Show that $|x - a| < k$ if and only if $a - k < x < a + k$, where $k > 0$.

Exercise 1.1.15
Show that $|x - a| > k$ if and only if $x < a - k$ or $x > a + k$, where $k \geq 0$.

The statements in the above two exercises remain true if $<$ and $>$ are replaced by $\leq$ and $\geq$.

Exercise 1.1.16
Solve each of the following inequalities: (a) $|2x - 3| < 5$ and (b) $|x + 4| > 2$. Write your answer in interval notation.

Exercise 1.1.17 *(Triangle inequality)*
Use Exercise 1.1.1, Exercise 1.1.7, and the expansion of $|a + b|^2$ to establish the inequality

$$|a + b| \leq |a| + |b|,$$

where a and b are arbitrary real numbers.

Exercise 1.1.18
Show that for any real numbers a and b we have $|a| - |b| \leq |a - b|$. Hint: Note that $a = (a - b) + b$.

Practice Problems

Exercise 1.1.19
Let $a \in \mathbb{R}$. Show that $\max\{a, 0\} = \frac{1}{2}(a + |a|)$ and $\min\{a, 0\} = \frac{1}{2}(a - |a|)$.

Exercise 1.1.20
Show that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

Exercise 1.1.21
Suppose $0 < x < \frac{1}{2}$. Simplify $\frac{x+3}{|2x^2+5x-3|}$.

Exercise 1.1.22
Write the function $f(x) = |x + 2| + |x - 4|$ as a piecewise defined function (i.e. without using absolute value symbols). Sketch its graph.

Exercise 1.1.23
Prove that $||a| - |b|| \leq |a - b|$ for any real numbers a and b.

Exercise 1.1.24
Solve the equation $4|x - 3|^2 - 3|x - 3| = 1$.

Exercise 1.1.25
What is the range of the function $f(x) = \frac{|x|}{x}$ for all $x \neq 0$?

Exercise 1.1.26
Solve $3 \leq |x - 2| \leq 7$. Write your answer in interval notation.

Exercise 1.1.27
Simplify $\frac{\sqrt{x^2}}{|x|}$.

Exercise 1.1.28
Solve the inequality $\left|\frac{x+1}{x-2}\right| < 3$. Write your answer in interval notation.

Exercise 1.1.29
Suppose x and y are real numbers such that $|x - y| < |x|$. Show that $xy > 0$.

1.2 Important Properties of the Real Numbers

In this section we will discuss some of the important properties of real numbers. We assume that the reader is familiar with the basic operations of real numbers (i.e. addition, subtraction, multiplication, division, and inequalities) and their properties (i.e. commutative, associative, reflexive, symmetry, etc.)

Definition 1.2.1
A set $A \subset \mathbb{R}$ is said to be **bounded from below** if and only if there is a real number m such that $m \leq x$ for all $x \in A$. We call m a **lower bound** of A. A set A is said to be **bounded from above** if and only if there is a real number M such that $x \leq M$ for all $x \in A$. In this case, we call M an **upper bound**. A is said to be **bounded** if and only if it is bounded from below and from above.

Exercise 1.2.1
Prove that A is bounded if and only if there is a positive constant C such that $|x| \leq C$ for all $x \in A$.

Exercise 1.2.2
Let $A = [0, 1]$.
(a) Find an upper bound of A. How many upper bounds are there?
(b) Find a lower bound of A. How many lower bounds are there?

By the previous exercise, we see that a set might have an infinite number of both upper bounds and lower bounds. This leads to the following definition.

Definition 1.2.2
Suppose A is a bounded subset of $\mathbb{R}$. A number α that satisfies the two conditions
(i) α is an upper bound of A,
(ii) for every upper bound γ of A we have $\alpha \leq \gamma$
is called the **supremum** or the **least upper bound** of A and is denoted by $\alpha = \sup A$. Thus, the supremum is the smallest upper bound of A.
A number β that satisfies the two conditions
(i) β is a lower bound of A,

(ii) for every lower bound γ of A we have $\gamma \leq \beta$
is called the **infimum** or the **greatest lower bound** of A and is denoted by $\beta = \inf A$. Thus, the infimum is the largest lower bound of A.
The supremum may or may not be an element of A. If it is in A then the supremum is called the **maximum value** of A. Likewise, if the infimum is in A then we call it the **minimum value** of A.

Exercise 1.2.3
Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Exercise 1.2.4
Consider the set $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.
(a) Show that 1 is an upper bound of A.
(b) Suppose $L < 1$ is another upper bound of A. Let n be a positive integer such that $n > \frac{1}{1-L}$. Such a number n exist by the Archimedian property which we will discuss below. Show that this leads to a contradiction. Thus, $L \geq 1$. This shows that 1 is the least upper bound of A and hence $\sup A = 1$.

Among the most important fact about the real number system is the so-called **Completeness Axiom of $\mathbb{R}$**:
Any subset of $\mathbb{R}$ that is bounded from above has a least upper bound and any subset of $\mathbb{R}$ that is bounded from below has a greatest lower bound.

The first consequence of this axiom is the so-called **Archimedean Property**. This is the property responsible for the fact that given any real number we can find an integer which exceeds it.

Exercise 1.2.5
Let $a, b \in \mathbb{R}$ with $a > 0$.
(a) Suppose that $na \leq b$ for all $n \in \mathbb{N}$. Show that the set $A = \{na : n \in \mathbb{N}\}$ has a supremum. Call it c.
(b) Show that $na \leq c - a$ for all $n \in \mathbb{N}$. That is, $c - a$ is an upper bound of A. Hint: $n + 1 \in \mathbb{N}$ for all $n \in \mathbb{N}$.
(c) Conclude from (b) that there must be a positive integer n such that $na > b$.

A consequence of the Archimedean property is the fact that between any two real numbers there is a rational number. We say that the set of rationals is **dense** in $\mathbb{R}$. We prove this result next.

Exercise 1.2.6
Let a and b be two real numbers such that $a < b$.
(a) Let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a. We call $\lfloor \cdot \rfloor$ the **floor function**.Show that $\lfloor a \rfloor - 1 < a < \lfloor a \rfloor + 1$.
(b) Let n be a positive integer such that $n > \frac{1}{b-a}$. Show that $na + 1 < nb$.
(c) Let $m = \lfloor na \rfloor + 1$. Show that $na < m < nb$. Thus, $a < \frac{m}{n} < b$. We see that between any two distinct real numbers there is a rational number.

Practice Problems

Exercise 1.2.7
Consider the set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Exercise 1.2.8
Consider the set $A = \{x \in \mathbb{R} : 1 < x < 2\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Exercise 1.2.9
Consider the set $A = \{x \in \mathbb{R} : x^2 > 4\}$.
(a) Show $x \in A$ and $x < 2$ lead to a contradiction. Hence, we must have $x \geq 2$ for all $x \in A$. That is, 2 is a lower bound of A.
(b) Let L be a lower bound of A such that $L > 2$. Let $y = \frac{L+2}{2}$. Show that $2 < y < L$.
(c) Use (a) to show that $y \in A$ and $L \leq y$. Show that this leads to a contradiction. Hence, we must have $L \leq 2$ which means that 2 is the infimum of A.

Exercise 1.2.10
Show that for any real number x there is a positive integer n such that $n > x$.

Exercise 1.2.11
Let a and b be any two real numbers such that $a < b$.
(a) Let w be a fixed positive irrational number. Show that there is a rational number r such that $a < wr < b$.
(e) Show that wr is irrational. Hence, between any two distinct real numbers there is an irrational number.

Exercise 1.2.12
Suppose that $\alpha = \sup A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\alpha - \epsilon < x$.

Exercise 1.2.13
Suppose that $\beta = \inf A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\beta + \epsilon > x$.

Exercise 1.2.14
For each of the following sets S, find $\sup\{S\}$ and $\inf\{S\}$, if they exist.
(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.
(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$.
(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$.

Chapter 2

Sequences

2.1 Sequences and their Convergence

In this section, we introduce sequences and study their convergence.

Definition 2.1.1
A **sequence** is a function with domain $\mathbb{N} = \{1, 2, 3, \cdots\}$ and range a subset of $\mathbb{R}$. That is,

$$\begin{aligned} a :& \mathbb{N} \longmapsto \mathbb{R} \\ & n \longmapsto a(n) = a_n \end{aligned}$$

We write $a = \{a_n\}_{n=1}^{\infty}$ and we call a_n the n^{th} **term** of the sequence.

Exercise 2.1.1
Find a simple expression for the general term of each sequence.
(a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$
(b) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$
(c) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$
(d) $-1, 1, -1, 1, -1, 1, \cdots$

Definition 2.1.2
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to a number L if and only if for every positive number ϵ there exists a positive integer $N = N(\epsilon)$ (depending on ϵ) such that

$$n \geq N \Longrightarrow |a_n - L| < \epsilon.$$

We write

$$\lim_{n\to\infty} a_n = L.$$

We say that the sequence is **convergent**. If a sequence is not convergent then it is said to be **divergent**.

Note that $|a_n - L|$ is the distance between the points a_n and L on the real line. The definition says that no matter how small a positive number ϵ we take, the distance between a_n and L will eventually be smaller than ϵ, i.e., the numbers a_n will eventually lie between $L-\epsilon$ and $L+\epsilon$. Thus the terms of the sequence will eventually lie in the shaded region shown in the Figure 2.1.1.

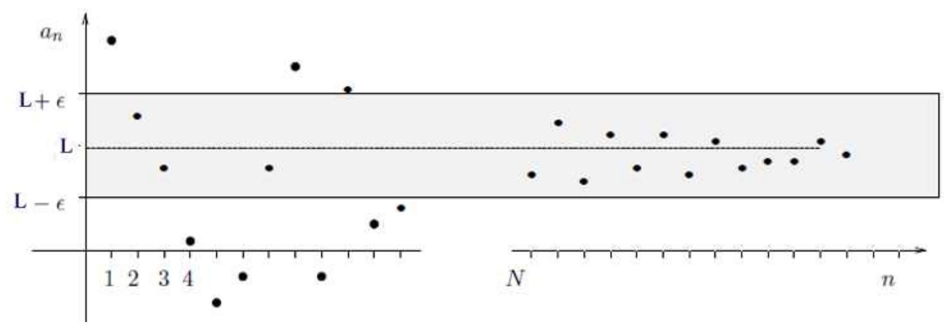


Figure 2.1.1

Exercise 2.1.2
Show that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Exercise 2.1.3
Show that the sequence $\left\{1+\frac{C}{n}\right\}_{n=1}^{\infty}$ converges to 1, where $C \neq 0$ is a constant.

Exercise 2.1.4
Is there a number L with the property that $|(-1)^n - L| < 1$ for all $n \geq N_1$, where N_1 is some positive integer? Hint: Consider the inequality with an even integer n greater than N_1 and an odd integer n greater than N_1.

Exercise 2.1.5
Use the previous exercise to show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

The following exercise shows that the limit of a convergent sequence is unique.

Exercise 2.1.6
Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ with $a < b$. Show that by choosing $\epsilon = \frac{b-a}{2} > 0$ we end up with the impossible inequality $b-a < b-a$. A similar result holds if $b < a$. Thus, we must have $a = b$. Hint: Exercise 1.1.6 and Exercise 1.1.17.

We next introduce the concept of a bounded sequence. This concept provides us with a divergence test for sequences. We will see that if a sequence is not bounded then it is divergent.

Definition 2.1.3
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **bounded** if there is a positive constant M such that $|a_n| \le M$ for all $n \in \mathbb{N}$.

Exercise 2.1.7
Show that each of the following sequences is bounded. Identify M in each case.
(a) $a_n = (-1)^n$.
(b) $a_n = \frac{1}{\sqrt{n}\ln(n+1)}$.

Exercise 2.1.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $|a_n| \le K$ for all $n \ge N$. Show that this sequence is bounded. Identify your M.

Exercise 2.1.9
Show that a convergent sequence is bounded. Hint: use the definition of convergence with $\epsilon = 1$.

The converse of the above result is not always true. That is, a bounded sequence need not be convergent.

Exercise 2.1.10
Give an example of a bounded sequence that is divergent.

The following result is known as the **squeeze rule**.

Exercise 2.1.11
Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ be three sequences with the following conditions:

(1) $b_n \leq a_n \leq c_n$ for all $n \geq K$, where K is some positive real number.
(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$.
Show that $\lim_{n\to\infty} a_n = L$. Hint: Use the definition of convergence along Exercise 1.1.14

Exercise 2.1.12
An expansion of $(a+b)^n$, where n is a positive integer is given by the **Binomial formula**

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^k b^{n-k}$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$.
(a) Use the Binomial formula to establish the inequality

$$(1+x)^{\frac{1}{n}} \leq 1 + \frac{x}{n},\ x \geq 0.$$

(b) Show that if $a \geq 1$ then $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$. Hint: Use Exercise 2.1.3.

Practice Problems

Exercise 2.1.13
Prove that the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$ is divergent.

Exercise 2.1.14
Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by $a_n = n$ for all $n \in \mathbb{N}$. Explain why the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to any limit.

Exercise 2.1.15
(a) Show that for all $n \in \mathbb{N}$ we have

$$\frac{n!}{n^n} \le \frac{1}{n}.$$

(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{n!}{n^n}$ is convergent and find its limit.

Exercise 2.1.16
Using only the definition of convergence show that

$$\lim_{n\to\infty} \frac{\sqrt[3]{n} - 5001}{\sqrt[3]{n} - 1001} = 1.$$

Exercise 2.1.17
Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \in \mathbb{N}$. Show that $a_n \le 2$ for all $n \in \mathbb{N}$.

Exercise 2.1.18
Calculate $\lim_{n\to\infty} \frac{(n^2+1)\cos n}{n^3}$.

Exercise 2.1.19
Calculate $\lim_{n\to\infty} \frac{2(-1)^{n+3}}{\sqrt{n}}$.

Exercise 2.1.20
Suppose that $\lim_{n\to\infty} a_n = L$ with $L > 0$. Show that there is a positive integer N such that $2a_N > L$.

Exercise 2.1.21
Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Clearly, $a < a + \frac{1}{n}$.
(a) Show that there is $a_1 \in \mathbb{Q}$ such that $a < a_1 < a + \frac{1}{n}$. Hint: Exercise 1.2.6(c).
(b) Show that there is $a_2 \in \mathbb{Q}$ such that $a < a_2 < a_1$.
(c) Continuing the above process we can find a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a < a_n < a + \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that this sequence converges to a.
We have proved that if a is a real number then there is a sequence of rational numbers converging a. We say that the set $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

2.2 Arithmetic Operations on Sequences

In this section we discuss the operations of addition, subtration, scalar multiplication, multiplication, reciprocal, and the ratio of two sequences. Our first result concerns the convergence of the sum or difference of two sequences.

Exercise 2.2.1
Suppose that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Show that

$$\lim_{n\to\infty} a_n \pm b_n = A \pm B.$$

The next result concerns the product of two sequences.

Exercise 2.2.2
Suppose that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.
(a) Show that $|b_n| \le M$ for all $n \in \mathbb{N}$, where M is a positive constant.
(b) Show that $a_n b_n - AB = (a_n - A)b_n + A(b_n - B)$.
(c) Let $\epsilon > 0$ be arbitrary and $K = M + |A|$. Show that there exists a positive integer N_1 such that $|a_n - A| < \frac{\epsilon}{2K}$ for all $n \ge N_1$.
(d) Let $\epsilon > 0$ and K be as in (c). Show that there exists a positive integer N_2 such that $|b_n - B| < \frac{\epsilon}{2K}$ for all $n \ge N_2$.
(e) Show that $\lim_{n\to\infty} a_n b_n = AB$.

Exercise 2.2.3
Give an example of two divergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $\{a_n b_n\}$ and $\{a_n + b_n\}$ are convergent.

Exercise 2.2.4
Let k be an arbitrary constant and $\lim_{n\to\infty} a_n = A$. Show that $\lim_{n\to\infty} ka_n = kA$.

Exercise 2.2.5
Suppose that $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}_{n=1}^{\infty}$ is bounded. Show that $\lim_{n\to\infty} a_n b_n = 0$.

Exercise 2.2.6
(a) Use the previous exercise to show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.
(b) Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ using the squeeze rule.

Exercise 2.2.7
Suppose that $\lim_{n\to\infty} a_n = A$ with $A \neq 0$. Show that there is a positive integer N such that $|a_n| > \frac{|A|}{2}$ for all $n \geq N$. Hint: Use Exercise 1.1.18.

Exercise 2.2.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with the following conditions:
(1) $a_n \neq 0$ for all $n \geq 1$.
(2) $\lim_{n\to\infty} a_n = A$, with $A \neq 0$.
(a) Show that there is a positive integer N_1 such that for all $n \geq N_1$ we have

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| < \frac{2}{|A|^2}|a_n - A|.$$

(b) Let $\epsilon > 0$ be arbitrary. Show that there is a positive integer N_2 such that for all $n \geq N_2$ we have

$$|a_n - A| < \frac{|A|^2}{2}\epsilon.$$

(c) Using (a) and (b), show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}.$$

Exercise 2.2.9
Let $0 < a < 1$. Show that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$. Hint: Use Exercise 2.1.12 (b).

Exercise 2.2.10
Show that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ with $b_n \neq 0$ for all $n \geq 1$ and $B \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}.$$

Hint: Note that $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$.

Exercise 2.2.11
Given that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ with $a_n \leq b_n$ for all $n \geq 1$.
(a) Suppose that $B < A$. Let $\epsilon = \frac{A-B}{2} > 0$. Show that there exist positive integers N_1 and N_2 such that $A - \epsilon < a_n < A + \epsilon$ for $n \geq N_1$ and $B - \epsilon < b_n < B + \epsilon$ for $n \geq N_2$.
(b) Let $N = N_1 + N_2$. Show that for $n \geq N$ we obtain the contradiction $b_n < a_n$. Thus, we must have $A \leq B$.

Practice Problems

Exercise 2.2.12
Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\lim_{n\to\infty} b_n = 0$ where $b_n \neq 0$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$.

Exercise 2.2.13
The Fibonacci numbers are defined recursively as follows:

$$a_1 = a_2 = 1 \text{ and } a_{n+2} = a_{n+1} + a_n \text{ for all } n \in \mathbb{N}.$$

Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Find the value of L.

Exercise 2.2.14
Show that the sequence defined by

$$a_n = \frac{n}{n+1} + (-1)^n \frac{n^2+3}{n^2+7}$$

has two limits by finding $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$.

Exercise 2.2.15
Use the properties of this section to find

$$\lim_{n\to\infty} \frac{\sqrt{2n^2+5n}}{n+4}.$$

Exercise 2.2.16
Find the limit of the sequence defined by

$$a_n = n^{\frac{1}{2\ln n}}.$$

Exercise 2.2.17
Consider the sequence defined by

$$a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

(a) Show that $a_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$.
(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent. Hint: Exercise 2.2.11

Exercise 2.2.18
Find the limit of the sequence defined by

$$a_n = \ln(2n + \sqrt{n}) - \ln n.$$

Exercise 2.2.19
Consider the sequence defined by $a_n = \sqrt[n]{3^n + 1}$.
(a) Show that $3 < a_n < 3\sqrt[n]{2}$ for all $n \in \mathbb{N}$.
(b) Find the limit of a_n as $n \to \infty$.

Exercise 2.2.20
Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of nonnegative terms with limit L. Suppose that the terms of sequence satisfy the recursive relation $a_n a_{n+1} = a_n + 2$ for all $N \in \mathbb{N}$. Find L.

Exercise 2.2.21
Find the limit of the sequence defined by

$$a_n = \cos\frac{1}{n} + \frac{\sin n}{n}.$$

Exercise 2.2.22
Suppose that $a_{n+1} = \frac{a_n^2+1}{a_n}$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent.

2.3 Monotone and Bounded Sequences

One of the problems with deciding if a sequence is convergent is that you need to have a limit before you can test the definition. However, it is often the case that it is more important to know if a sequence converges than what it converges to. In this and the next section, we look at two ways to prove a sequence converges without knowing its limit. That is convergence is solely based on the terms of the sequence.

Definition 2.3.1
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **increasing** if and only if $a_n \leq a_{n+1}$ for all $n \geq 1$.
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **decreasing** if and only if $a_n \geq a_{n+1}$ for all $n \geq 1$.
A sequence that is either increasing or decreasing is said to be **monotone.**

Exercise 2.3.1
Show that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing.

Exercise 2.3.2
Show that the sequence $\{\frac{1}{1+e^{-n}}\}_{n=1}^{\infty}$ is increasing.

Definition 2.3.2
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **bounded from below** if and only if there is a constant m such that $m \leq a_n$ for all $n \geq 1$. We call m a **lower bound**.
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **bounded from above** if and only if there is a constant M such that $a_n \leq M$ for all $n \geq 1$. We call M an **upper bound**.

Exercise 2.3.3
Show that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded from below. What is a lower bound? Are there more than one lower bound?

Exercise 2.3.4
Show that the sequence $\{\frac{1}{1+e^{-n}}\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound? Are there more than one upper bound?

Suppose that $N \leq a_n \leq M$ for all $n \geq 1$. That is, the sequence is bounded from above and below. As we have seen from the previous two exercises, the sequence may have many lower and upper bounds.

Definition 2.3.3
The largest lower bound is called the **greatest lower bound** (or the **infimum**) denoted by $\inf\{a_n : n \geq 1\}$. Note that the infimum is a lower bound. Moreover, for any lower bound m of $\{a_n\}_{n=1}^{\infty}$ we have $m \leq \inf\{a_n : n \geq 1\}$. The smallest upper bound is called the **least upper bound** (or the **supremum**) denoted by $\sup\{a_n : n \geq 1\}$. Note that the supremum is an upper bound. Moreover, for any upper bound M of $\{a_n\}_{n=1}^{\infty}$ we have $\sup\{a_n : n \geq 1\} \leq M$.

The next result shows that an increasing sequence that is bounded from above is always convergent.

Exercise 2.3.5
Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence that is bounded from above.
(a) Show that there is a finite number M such that $M = \sup\{a_n : n \geq 1\}$.
(b) Let $\epsilon > 0$ be arbitrary. Show that $M - \epsilon$ cannot be an upper bound of the sequence.
(c) Show that there is a positive integer N such that $M - \epsilon < a_N$.
(d) Show that $M - \epsilon < a_n$ for all $n \geq N$.
(e) Show that $M - \epsilon < a_n < M + \epsilon$ for all $n \geq N$.
(f) Show that $\lim_{n\to\infty} a_n = M$. That is, the given sequence is convergent.

Exercise 2.3.6
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{1}{2}a_n + 1$ for $n \geq 1$.
(a) Show by induction on $n \geq 1$, that $a_{n+1} = a_n + \frac{1}{2^{n+1}}$.
(b) Show that this sequence is increasing.
(c) Show that $\{a_n\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound?
(d) Show that $\{a_n\}_{n=1}^{\infty}$ is convergent. What is its limit? Hint: In finding the limit, use the arithmetic operations of sequences.

Exercise 2.3.7
Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $m \leq a_n$ for all $n \geq 1$. Show that $\{a_n\}_{n=1}^{\infty}$ is convergent. Hint: Let $b_n = -a_n$ and use Exercise 2.3.5 and Exercise 2.2.4.

Exercise 2.3.8
Show that a monotone sequence is convergent if and only if it is bounded.

Practice Problems

Exercise 2.3.9
Let a_n be defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \in \mathbb{N}$.
(a) Show that $a_n \leq 2$ for all $n \in \mathbb{N}$. That is, $\{a_n\}_{n=1}^{\infty}$ is bounded from above.
(b) Show that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. That is, $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent. Find its limit.

Exercise 2.3.10
Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$.
(a) Show that $a_n < 2$ for all $n \in \mathbb{N}$. Hint: Recall that $\sum_{k=1}^{n} \frac{1}{(n+1)n} = 1 - \frac{1}{n+1}$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent.

Exercise 2.3.11
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively as follows

$$a_1 = 2 \text{ and } 7a_{n+1} = 2a_n^2 + 3 \text{ for all } n \in \mathbb{N}.$$

(a) Show that $\frac{1}{2} < a_n < 3$ for all $n \in \mathbb{N}$.
(b) Show that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
(c) Deduce that $\{a_n\}_{n=1}^{\infty}$ is convergent and find its limit.

Exercise 2.3.12
Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence. Define $b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$. Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.

Exercise 2.3.13
Give an example of a monotone sequence that is divergent.

Exercise 2.3.14
Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 3 + \frac{a_n}{2}$ for all $n \in \mathbb{N}$.
(a) Show that $a_n \leq 6$ for all $n \in \mathbb{N}$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that the sequence is convergent. Find its limit.

Exercise 2.3.15
Give an example of two monotone sequences whose sum is not monotone.

2.4 Subsequences and the Bolzano-Weierstrass Theorem

In this section we consider a sequence contained in another sequence. More formally we have

Definition 2.4.1
Consider a sequence $\{a_n\}_{n=1}^{\infty}$. A sequence consisting of terms of the given sequence of the form $\{a_{n_k}\}_{k=1}^{\infty}$ where $n_1 < n_2 < n_3 < \cdots$ is called a **subsequence**.

Exercise 2.4.1
Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$. Use induction on k to show that $n_k \geq k$ for all $k \in \mathbb{N}$.

Exercise 2.4.2
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that converges to a number L. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be any subsequence of $\{a_n\}_{n=1}^{\infty}$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N' such that if $n \geq N'$ then $|a_n - L| < \epsilon$.
(b) Let N be the first positive integer such that $n_N \geq N'$. Show that if $k \geq N$ then $|a_{n_k} - L| < \epsilon$. That is, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L. Hence, every subsequence of a convergent sequence is convergent to the same limit of the original sequence.

The next result shows that every sequence has a monotonic subsequence.

Exercise 2.4.3
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $S = \{n \in N : a_n > a_m \text{for all } m > n\}$.
(a) Suppose that S is infinite. Then there is a sequence $n_1 < n_2 < n_3 < \cdots$ such $n_k \in S$. Show that $a_{n_{k+1}} < a_{n_k}$. Thus, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is decreasing.
(b) Suppose that S is finite. Let n_1 be the first positive integer such that $n_1 \notin S$. Show that the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is increasing.

As a corollary to the previous exercise we obtain the following famous result which says that every bounded sequence of real numbers has a convergent subsequence.

Exercise 2.4.4 *(Bolzano-Weierstrass)*
Prove that every bounded sequence has a convergent subsequence. Hint: Exercise 2.3.8

Exercise 2.4.5
Show that the sequence $\{e^{\sin n}\}_{n=1}^{\infty}$ has a convergent subsequence.

Practice Problems

Exercise 2.4.6
Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \cos\frac{n\pi}{2}$ is divergent.

Exercise 2.4.7
Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \frac{(n^2 + 20n + 35)\sin n^3}{n^2 + n + 1}$$

has a convergent subsequence. Hint: Show that $\{a_n\}_{n=1}^{\infty}$ is bounded.

Exercise 2.4.8
Show that the sequence defined by $a_n = 2\cos n - \sin n$ has a convergent subsequence.

Exercise 2.4.9
True or false: There is a sequence that converges to 6 but contains a subsequence converging to 0. Justify your answer.

Exercise 2.4.10
Give an example of an unbounded sequence with a bounded subsequence.

Exercise 2.4.11
Show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent by using subsequences.

2.5 Cauchy Sequences

The notion of a Cauchy sequence provides us with a characterization of convergence in terms of just the terms in the sequence without explicit reference to the limit.

Definition 2.5.1
A sequence $\{a_n\}_{n=1}^{\infty}$ is called a **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that

$$\text{if } n, m \geq N \text{ then } |a_n - a_m| < \epsilon.$$

Thus, for a sequence to be Cauchy, we don't require that the terms of the sequence to be eventually all close to a certain limit, just that the terms of the sequence to be eventually all close to one another.

Exercise 2.5.1
Consider the sequence whose n^{th} term is given by $a_n = \frac{1}{n}$. Let $\epsilon > 0$ be arbitrary and choose $N > \frac{2}{\epsilon}$. Show that for $m, n \geq N$ we have $|a_m - a_n| < \epsilon$. That is, the above sequence is a Cauchy sequence. Hint: Exercise 1.1.17.

The next result shows that Cauchy sequences are bounded sequences.

Exercise 2.5.2
Show that any Cauchy sequence is bounded. Hint: Let $\epsilon = 1$ and use Exercise 1.1.18.

Exercise 2.5.3
Show that if $\lim_{n\to\infty} a_n = A$ then $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus, every convergent sequence is a Cauchy sequence.

Now, consider a Cauchy sequence $\{a_n\}_{n=1}^{\infty}$. Create new sequences as follows: For each $n \geq 1$, a new sequence is obtained by deleting the previous $n-1$ terms from the original sequence. For example, if $n = 1$, the new sequence is just the original sequence, for $n = 2$ the new sequence is $\{a_2, a_3, \cdots\}$, for $n = 3$ the new sequence is $\{a_3, a_4 \cdots\}$ and so on.

Exercise 2.5.4
(a) Using Exercise 2.5.2, show that for each $n \geq 1$, the sequence $\{a_n, a_{n+1}, \cdots\}$ is bounded.
(b) Show that for each $n \geq 1$ the infimum of $\{a_n, a_{n+1}, \cdots\}$ exists. Call it b_n.

Exercise 2.5.5
(a) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded from above.
(b) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing. Hint: Show that b_n is a lower bound of the sequence $\{a_{n+1}, a_{n+2}, \cdots\}$.

Exercise 2.5.6
Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent. Call the limit B.

Exercise 2.5.7
(a) Let $\epsilon > 0$ be arbitrary. Using the definition of Cauchy sequences and Exercise 1.1.14, show that there is a positive integer N such that $a_N - \frac{\epsilon}{2} < a_n < a_N + \frac{\epsilon}{2}$ for all $n \geq N$.
(b) Using (a), show that $a_N - \frac{\epsilon}{2}$ is a lower bound of the sequence $\{a_N, a_{N+1}, \cdots\}$ Thus, $a_N - \frac{\epsilon}{2} \leq b_n$ for all $n \geq N$.
(c) Again, using (a) show that $b_n < a_N + \frac{\epsilon}{2}$ for all $n \geq N$. Thus, combining (b) and (c), we obtain $a_N - \frac{\epsilon}{2} \leq b_n < a_N + \frac{\epsilon}{2}$.
(d) Using Exercise 2.2.11, show that $a_N - \frac{\epsilon}{2} \leq B \leq a_N + \frac{\epsilon}{2}$.
(e) Using (a), (d), and Exercise 1.1.17, show that $\lim_{n\to\infty} a_n = B$. Thus, every Cauchy sequence is convergent.

Practice Problems

Exercise 2.5.8
(a) Show that if $\{a_n\}_{n=1}^{\infty}$ is Cauchy then $\{a_n^2\}_{n=1}^{\infty}$ is also Cauchy.
(b) Give an example of Cauchy sequence $\{a_n^2\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty}$ is not Cauchy.

Exercise 2.5.9
Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence such that a_n is an integer for all $n \in \mathbb{N}$. Show that there is a positive integer N such that $a_n = C$ for all $n \geq N$, where C is a constant.

Exercise 2.5.10
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that satisfies

$$|a_{n+2} - a_{n+1}| < c^2|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}$$

where $0 < c < 1$.
(a) Show that $|a_{n+1} - a_n| < c^n|a_2 - a_1|$ for all $n \geq 2$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 2.5.11
What does it mean for a sequence $\{a_n\}_{n=1}^{\infty}$ to not be Cauchy?

Exercise 2.5.12
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Define $c_n = |a_n - b_n|$. Show that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 2.5.13
Suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose $a_n \geq 0$ for infinitely many n and $a_n \leq 0$ for infinitely many n. Prove that $\lim_{n\to\infty} a_n = 0$.

Exercise 2.5.14
Explain why the sequence defined by $a_n = (-1)^n$ is not a Cauchy sequence.

Chapter 3

Limits

3.1 The Limit of a Function

A fundamental concept in single variable calculus is the concept of the limit of a function. In this section, we introduce the definition of limit and discuss some of its properties.

Definition 3.1.1
Let f be a function with domain $D \subset \mathbb{R}$. Let a be a point in D. We say that f has a **limit** L at a if and only if for every $\epsilon > 0$ there exists a positive number δ depending on ϵ such that for any $x \in D$ with the property $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$. In symbol we write

$$\lim_{x \to a} f(x) = L.$$

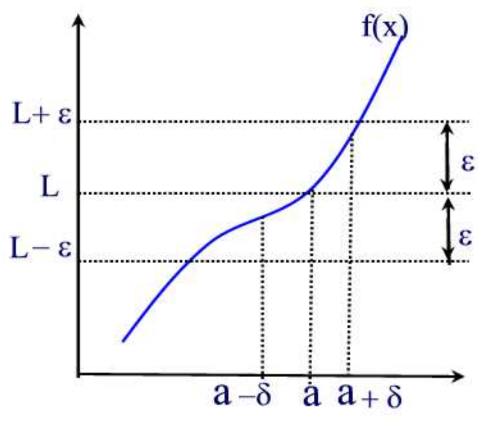


Figure 3.1.1

Geometrically, the definition says that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if the distance between a point x and a is less than δ (i.e. the point x is inside the tiny interval around a) then the distance between $f(x)$ and L is less than ϵ (i.e, the point $f(x)$ is inside the tiny interval around L) as shown in Figure 3.1.1.

Exercise 3.1.1
Show that $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

The limit of a function may not exist as shown in the next exercise.

Exercise 3.1.2
Let $f(x) = \frac{|x|}{x}$. Suppose that $\lim_{x\to 0} f(x) = L$.
(a) Show that there is a positive number δ such that if $0 < |x| < \delta$ then $\left|\frac{|x|}{x} - L\right| < \frac{1}{4}$.
(b) Let $x_1 = \frac{\delta}{4}$ and $x_2 = -\frac{\delta}{4}$. Compute the value of $|f(x_1) - f(x_2)|$.
(c) Use (a) to show that $|f(x_1) - f(x_2)| < \frac{1}{2}$.
(d) Conclude that L does not exist. That is, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Exercise 3.1.3
Let $f(x) = \sin\left(\frac{1}{x}\right)$. Suppose that $\lim_{x\to 0} f(x) = L$.
(a) Show that there is a positive number δ such that if $0 < |x| < \delta$ then $\left|\sin\left(\frac{1}{x}\right) - L\right| < \frac{1}{4}$.
(b) Let n be a positive integer such that $x_1 = \frac{2}{(2n+1)\pi} < \delta$ and $x_2 = \frac{1}{(2n+1)\pi} < \delta$. Compute the value of $|f(x_1) - f(x_2)|$.
(c) Use (a) to show that $|f(x_1) - f(x_2)| < \frac{1}{2}$.
(d) Conclude that L does not exist. That is, $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

The next exercise shows that a function can have only one limit, if such a limit exists.

Exercise 3.1.4
Suppose that $\lim_{x\to a} f(x)$ exists. Also, suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$. So either $L_1 = L_2$ or $L_1 \neq L_2$.
(a) Suppose that $L_1 \neq L_2$. Show that there exist positive constants δ_1 and δ_2 such that if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \frac{|L_1-L_2|}{2}$ and if $0 < |x-a| < \delta_2$ then $|f(x) - L_2| < \frac{|L_1-L_2|}{2}$.

(b) Let $\delta = \min\{\delta_1, \delta_2\}$ so that $\delta \leq \delta_1$ and $\delta \leq \delta_2$. Show that if $0 < |x-a| < \delta$ then $|L_1 - L_2| < |L_1 - L_2|$ which is impossible.
(c) Conclude that $L_1 = L_2$. That is, whenever a function has a limit, that limit is unique.

Practice Problems

Exercise 3.1.5
Using the $\epsilon\delta$ definition of limit to show that

$$\lim_{x\to -1}(2x^2 + x + 1) = 2.$$

Exercise 3.1.6
Prove directly from the definition that $\lim_{x\to 1}\frac{x}{x+3} = \frac{1}{4}$.

Exercise 3.1.7
In this exercise we discuss the concept of sided limits.
(a) We say that L is the **left side limit** of f as x approaches a from the left if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$$

and we write $\lim_{x\to a^-} f(x) = L$. Show that $\lim_{x\to 0^-}\frac{|x|}{x} = -1$.
(b) We say that L is the **right side limit** of f as x approaches a from the right if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$$

and we write $\lim_{x\to a^+} f(x) = L$. Show that $\lim_{x\to 0^+}\frac{|x|}{x} = 1$.

Exercise 3.1.8
Prove that $L = \lim_{x\to a} f(x)$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.

Exercise 3.1.9
Using ϵ and δ, what does it mean that $\lim_{x\to a} f(x) \neq L$?

3.2 Properties of Limits

When computing limits, one uses some established properties rather than the $\epsilon\delta$ definition of limit. In this section, we discuss these basic properties.

Exercise 3.2.1
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Show that

$$\lim_{x\to a}[f(x) \pm g(x)] = L_1 \pm L_2.$$

Exercise 3.2.2
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Show the following:
(a) There is a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x)| < 1 + |L_1|.$$

Hint: Notice that $f(x) = (f(x) - L_1) + L_1$.
(b) Given $\epsilon > 0$, there is a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2(1+|L_2|)}.$$

Exercise 3.2.3
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$.
(a) Show that $f(x)g(x) - L_1L_2 = f(x)(g(x) - L_2) + L_2(f(x) - L_1)$.
(b) Show that $|f(x)g(x) - L_1L_2| \le |f(x)||g(x) - L_2| + |L_2||f(x) - L_1|$.
(c) Show that $\lim_{x\to a} f(x)g(x) = L_1L_2$. Hint: Use the previous exercise.

Exercise 3.2.4
(a) Suppose that $|f(x)| \le M$ for all x in its domain and $\lim_{x\to a} g(x) = 0$. Show that

$$\lim_{x\to a} f(x)g(x) = 0.$$

Hint: Recall Exercise 2.2.5
(b) Show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

The following exercise says that when a function approaches a nonzero number as the variable x approaches a, then there is an open interval around a where the function is always different from zero in that interval.

Exercise 3.2.5
Suppose that $\lim_{x\to a} f(x) = L$ with $L \neq 0$. Show that there exists a $\delta > 0$ such that
$$0 < |x-a| < \delta \Longrightarrow |f(x)| > \frac{|L|}{2} > 0.$$
Hint: Recall the solution to Exercise 2.2.7

Exercise 3.2.6
Let $g(x)$ be a function with the following conditions:
(1) $g(x) \neq 0$ for all x in the domain of g.
(2) $\lim_{x\to a} g(x) = L_2$, with $L_2 \neq 0$.
(a) Show that there is a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then
$$\left|\frac{1}{g(x)} - \frac{1}{L_2}\right| < \frac{2}{|L_2|^2}|g(x) - L_2|.$$
(b) Let $\epsilon > 0$ be arbitrary. Show that there is $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then
$$|g(x) - L_2| < \frac{|L_2|^2}{2}\epsilon.$$
(c) Using (a) and (b), show that
$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{L_2}.$$
Hint: Recall Exercise 2.2.8

Exercise 3.2.7
Show that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ where $g(x) \neq 0$ in its domain and $L_2 \neq 0$ then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$
Hint: Recall Exercise 2.2.10.

Exercise 3.2.8
Let $f(x)$ and $g(x)$ be two functions with a common domain D and a a point in D. Suppose that $f(x) \leq g(x)$ for all x in D. Show that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then $L_1 \leq L_2$. Hint: Recall Exercise 2.2.11

Practice Problems

Exercise 3.2.9
Let D be the domain of a function $f(x)$. Suppose that $f(x) \geq 0$ for all x in D and $\lim_{x \to a} f(x) = L$ with $L > 0$.
(a) Show that

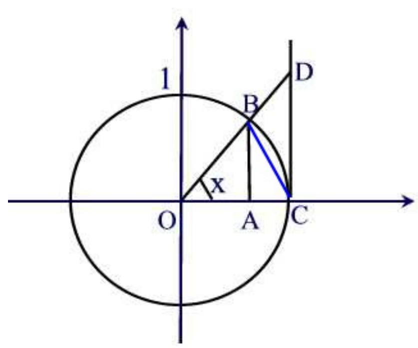
$$\sqrt{f(x)} - \sqrt{L} = \frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}}.$$

(b) Let $\epsilon > 0$. Show that there exists $\delta > 0$ such that $|f(x) - L| < \epsilon\sqrt{L}$ whenever $0 < |x - a| < \delta$.
(c) Show that

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}.$$

Exercise 3.2.10 *(Squeeze Rule)*
Let $f(x), g(x)$ and $h(x)$ be three functions with common domain D and a be a point in D. Suppose that
(1) $g(x) \leq f(x) \leq h(x)$ for all x in D.
(2) $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$.
Show that $\lim_{x \to a} f(x) = L$. Hint: Recall Exercise 2.1.11

Exercise 3.2.11
Consider the following figure.



where $0 < x < \frac{\pi}{2}$.
(a) Using geometry, establish the inequality

$$0 < \sin x < x.$$

Hint: The area of a circular sector with radius r and central angle θ is given by the formula $\frac{1}{2}r^2\theta$.
(b) Show that $\lim_{x\to 0^+}\sin x = 0$.
(c) Show that $\lim_{x\to 0^-}\sin x = 0$. Thus, we conclude that $\lim_{x\to 0}\sin x = 0$.
Hint: Recall that the sine function is an odd function.
(d) Show that $\lim_{x\to 0}\cos x = 1$. Hint: $\cos^2 x + \sin^2 x = 1$.
(e) Using geometry, establish the double inequality

$$\frac{\sin x\cos x}{2} < \frac{x}{2} < \frac{\tan x}{2}.$$

(f) Using (a) show that

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}.$$

(g) Show that

$$\lim_{x\to 0^+}\frac{\sin x}{x} = 1.$$

(h) Show that for $-\frac{\pi}{2} < x < 0$ we have also

$$\lim_{x\to 0^-}\frac{\sin x}{x} = 1.$$

Exercise 3.2.12
Find each of the following limits:
(1) $\lim_{x\to 1}\frac{\sqrt{x^2+3}-2\sqrt{x}}{x^2-1}$.
(2) $\lim_{x\to 2^-}\frac{x-2}{|x^2-5x+6|}$.

Exercise 3.2.13
Find $\lim_{x\to\infty}\frac{x^2+x}{x^2-x}$ by using the change of variable $u = \frac{1}{x}$.

Exercise 3.2.14
Find $\lim_{x\to 0}\sqrt[3]{x}\sin\frac{1}{x}$.

Exercise 3.2.15
Find $\lim_{x\to 0}x^2\tan x$.

Exercise 3.2.16
Let n be a positive integer. Prove that $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$.

3.3 Connection Between the Limit of a Function and the Limit of a Sequence

The limit of a function given so far is known as the $\epsilon\delta$ definition. In this section, we will establish an equivalent definition that involves the limit of a sequence.

Exercise 3.3.1
Suppose that $\lim_{x\to a} f(x) = L$, where a is in the domain of f. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence whose terms belong to the domain of f and are different from a and suppose that $\lim_{n\to\infty} a_n = a$.
(a) Let $\epsilon > 0$ be arbitrary. Show that there exist a positive integer N and a positive number δ such that for $n \geq N$ we have $|a_n - a| < \delta$ and for $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$.
(b) Use (a) to conclude that for a given $\epsilon > 0$ there is a positive integer N such that if $n \geq N$ then $|f(a_n) - L| < \epsilon$. That is, $\lim_{n\to\infty} f(a_n) = L$.

Using Definition 3.1.1, what do we mean by $\lim_{x\to a} f(x) \neq L$? This means that there is an interval centered at L such that for any interval centered at a we can find a point x in that interval and in the domain of f such that $f(x)$ is not in the interval centered at L. This is the same thing as saying that we can find an $\epsilon > 0$ such for all $\delta > 0$ there is x_δ (in the domain of f) with the property that $0 < |x_\delta - a| < \delta$ but $|f(x_\delta) - L| \geq \epsilon$.

Exercise 3.3.2
Let $f : D \to \mathbb{R}$ be a function with the property that for any sequence $\{a_n\}_{n=1}^{\infty}$ (with $a_n \neq a$ for all $n \geq 1$) if $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} f(a_n) = L$. We want to show that

$$\lim_{x\to a} f(x) = L$$

(a) Suppose first that $\lim_{x\to a} f(x) \neq L$. Show that there is an $\epsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of terms in the domain of f such that $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \epsilon$.
(b) Use the squeeze rule to show that $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Use the fact that $-|a| \leq a \leq |a|$ for any number a and the squeeze rule to show that $\lim_{n\to\infty}(a_n - a) = 0$.
(d) Use Exercise 2.2.1 to show that $\lim_{n\to\infty} a_n = a$.
(e) Using (a), (d), the given hypothesis and Exercise 2.2.11, show that $\epsilon \leq 0$.

Thus, this contradiction shows that $\lim_{x\to a} f(x) \neq L$ cannot happen. We conclude that

$$\lim_{x\to a} f(x) = L.$$

The above two exercises establish that the sequence version and the $\epsilon\delta$ version are equivalent.
In many cases, one is interested in knowing that the limit of a function exist without the need of knowing the value of the limit. In what follows, we will establish a result that uses Cauchy sequences to provide a test for establishing that the limit of a function exists.

Exercise 3.3.3
Let f be a function with domain D and a be a point in D. Suppose that f satisfies the following Property:

(P) If $\{a_n\}_{n=1}^{\infty}$, with a_n in D, $a_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = a$ then $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

(a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of D such that $a_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = a$. Show that the sequence $\{f(a_n)\}_{n=1}^{\infty}$ is convergent. Call the limit L. Hint: See Exercise 2.5.7
(b) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of elements of D such that $b_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} b_n = a$. Show that the sequence $\{f(b_n)\}_{n=1}^{\infty}$ converges to some number L'.

Exercise 3.3.4
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the two sequences of the previous exercise. Define the sequence

$$\{c_n\} = \{b_1, a_1, b_2, a_2, b_3, a_3, \cdots\}.$$

That is, $c_n = a_k$ if $n = 2k$ and $c_n = b_k$ if $n = 2k + 1$ where $k \geq 0$.
(a) Show that for all $n \geq 1$ we have $c_n \in D$ and $c_n \neq a$.
(b) Let $\epsilon > 0$. Show that there exist positive integers N_1 and N_2 such that if $n \geq N_1$ then $|a_n - a| < \epsilon$ and if $n \geq N_2$ then $|b_n - a| < \epsilon$.
(c) Let $N = 2N_1 + 2N_2 + 1$. Show that if $n \geq N$ then $|c_n - a| < \epsilon$. Hence, $\lim_{n\to\infty} c_n = a$. Hint: Consider the cases $n = 2k$ or $n = 2k + 1$.
(d) Show that $\lim_{n\to\infty} f(c_n) = L''$ for some number L''.

The next exercise establishes the fact that the two sequences $\{f(a_n)\}_{n=1}^{\infty}$ and $\{f(b_n)\}_{n=1}^{\infty}$ converge to the same limit.

Exercise 3.3.5
Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be as in the previous exercise.
(a) Compare $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$.
(b) Let $\epsilon > 0$ be arbitrary. Show that there is a positive integer N such that if $n \geq N$ then $|f(c_n) - L''| < \epsilon$.
(c) Let N_1 be a positive integer such that $N_1 \geq \frac{N}{2}$. Show that if $n \geq N_1$ then $|f(a_n) - L''| < \epsilon$. Hence, $\lim_{n\to\infty} f(a_n) = L''$.
(d) Show that $\lim_{n\to\infty} f(b_n) = L''$. Thus, by Exercise 2.1.6, we must have $L = L' = L''$.

Exercise 3.3.6
Prove that if a function f satisfies property (P) then $\lim_{x\to a} f(x)$ exists. Hint: Use Exercise 3.3.2.

Practice Problems

Exercise 3.3.7
Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the two sequences defined by $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Clearly, $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$, $a_n \to 0$ and $b_n \to 0$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Exercise 3.3.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n \neq 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 2$.
(a) Find $\lim_{n\to\infty} \frac{a_n^2-4}{a_n+2} = 4$.
(b) Find $\lim_{x\to 2} \frac{x^2-4}{x+2}$.

Exercise 3.3.9
Consider the floor function $f : [0,1] \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Find $\lim_{x\to 1} \lfloor x \rfloor$ using sequences.

Exercise 3.3.10
Consider the floor function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denote the largest integer less than or equal to x.
(a) Let $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} f(a_n)$ and $\lim_{n\to\infty} f(b_n)$.
(b) Does $\lim_{x\to 1} \lfloor x \rfloor$ exist?

Chapter 4

Continuity

4.1 Continuity of a Function

In this section we introduce the notion of continuity of a function and study the various equivalent definitions of this notion.

Definition 4.1.1
Let f be a real-valued function with domain D and a a point in D. We say that f is **continuous** at a if and only if for any given ϵ we can find $\delta = \delta(\epsilon) > 0$ such that

$$\text{for all } x \text{ in } D \text{ if } |x - a| < \delta \text{ then } |f(x) - f(a)| < \epsilon.$$

If f is continuous at every point in D, then we say that f is continuous in D.

Exercise 4.1.1
Show that the function $f(x) = x^2$ is continuous at $x = 0$.

The next result provides a definition of continuity in terms of limits.

Exercise 4.1.2
Show that f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

Definition 4.1.2
A function f that is not continuous at a is said to be **discontinuous** there. In terms of Definition 4.1.1, f is discontinuous at $x = a$ if and only if there is an $\epsilon > 0$ such that for all $\delta > 0$ there is an $x = x_\delta$ in D such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$.

Exercise 4.1.3
Consider the function

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2. \end{cases}$$

Show that f is discontinuous at $x = 2$.

Exercise 4.1.4
Suppose that f is discontinuous at $x = a$.
(a) Show that there is a sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D such that $0 \leq |a_n - a| < \frac{1}{n}$ and $|f(a_n) - f(a)| \geq \epsilon$.
(b) Show that $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Show that $\lim_{n\to\infty} a_n = a$.

The next two results provide a definition of continuity in terms of sequences.

Exercise 4.1.5
Suppose that f is continuous at $x = a$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in D converging to a.
(a) Let $\epsilon > 0$ be given. Show that there is a $\delta > 0$ such that for any x in D such that $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$.
(b) With the ϵ and δ as in (a), show that there is a positive integer N such that if $n \geq N$ then $|a_n - a| < \delta$.
(c) Conclude that $\lim_{n\to\infty} f(a_n) = f(a)$.

Exercise 4.1.6
Suppose that for any sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D that converges to a, the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(a)$. Then either f is continuous at a or f is discontinuous at a.
(a) Suppose that f is discontinuous at a. Show that there is an $\epsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D such that $\lim_{n\to\infty} a_n = a$ and $|f(a_n) - f(a)| \geq \epsilon$ for all $n \geq 1$.
(b) Show that $\lim_{n\to\infty} f(a_n) = f(a)$.
(c) Show that by (a) and (b) we conclude that $\epsilon \leq 0$, a contradiction. Thus, f must be continuous at $x = a$.

Practice Problems

Exercise 4.1.7
Consider the function
$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$
(a) Let $a_n = -\frac{1}{n}$. Find $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} f(a_n)$.
(b) Is f continuous at $x = 0$?

Exercise 4.1.8
Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f(a_n)$ exists, but $\lim_{n\to\infty} a_n$ does not exist.

Exercise 4.1.9
Determine the values of a and b that makes the function f continuous everywhere.
$$f(x) = \begin{cases} 2\frac{\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0. \end{cases}$$

Exercise 4.1.10
Using the ϵ-δ definition of continuity show that $f(x) = x^3$ is continuous at $x = 1$. Hint: $x^3 - 1 = (x-1)(x^2 + x + 1)$.

Exercise 4.1.11
Consider the function $f(x) = \cos\left(\frac{1}{x}\right)$.
(a) Let $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{(n+\frac{1}{2})\pi}$. Find $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n, \lim_{n\to\infty} f(a_n)$, and $\lim_{n\to\infty} f(b_n)$.
(b) Is f continuous at $x = 0$?

Exercise 4.1.12
Consider the function
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Show that this function is continuous at $x = 0$ by using the ϵ-δ definition.

Exercise 4.1.13
Prove that if f is continuous at $x = a$ so does $|f|$. Hint: Exercise 1.1.23.

Exercise 4.1.14
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$. Suppose $h : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}$. If $f(c) = g(c)$, prove that h is continuous at c.

Exercise 4.1.15
Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Show that f is continuous on $[0, \infty)$.

4.2 Properties of Continuous Functions

In this section, we discuss the various properties that continuous functions enjoy.

Exercise 4.2.1
Let $f(x)$ and $g(x)$ be two functions with common domain D . Suppose that f and g are continuous at a point a in D. Show the following properties:
(i) $f \pm g$ is continuous at a.
(ii) $f \cdot g$ is continuous at a.
(iii) $\frac{f}{g}$ is continuous at a provided that $g(a) \neq 0$.

Exercise 4.2.2
Let f be continuous at a point a in its domain with $f(a) \neq 0$. Show that there exists a $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow |f(x)| > \frac{|f(a)|}{2}.$$

That is, there is an open interval centered at a where the function is always different from zero there. Hint: Look at Exercise 2.2.7

Exercise 4.2.3
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ with the range of f contained in D'. Thus, $g \circ f : D \to \mathbb{R}$ is a function with domain D. Suppose that f is continuous at a and g is continuous at $f(a)$.
(a) Let $\epsilon > 0$ be given. Show that there is a $\delta' > 0$ such that for all y in D' satisfying $|y - f(a)| < \delta'$ we have $|g(y) - g(f(a))| < \epsilon$.
(b) Show that there is a $\delta'' > 0$ such that if $|x-a| < \delta''$ then $|f(x)-f(a)| < \delta'$.
(c) Show that there is a $\delta > 0$ such that if $|x-a| < \delta$ then $|g(f(x))-g(f(a))| < \epsilon$. In other words, the composite function $g(f(x))$ is continuous at a. Hence, the composition of two continuous functions is a continuous function.

Exercise 4.2.4
In Exercise 3.2.11, we established that $\lim_{x\to 0} \sin x = 0 = \sin 0$. That is, the sine function is continuous at 0.
(a) Using the trigonometric identity

$$\sin{(a + b)} = \sin a \cos b + \cos a \sin b$$

show that the sine function is continuous at every number a. Hint: Use the substitution $u = x - a$ and note that $u \to 0$ as $x \to a$.

(b) Show that the cosine function is continuous for every number a. Hint: Note that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and use Exercise 4.2.3.

Practice Problems

Exercise 4.2.5
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Hint: Exercise 2.1.21

Exercise 4.2.6
Consider the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is continuous at $x = 0$.
(b) Let $a \neq 0$. Prove that f is discontinuous at $x = a$.

Exercise 4.2.7
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

Exercise 4.2.8
Use continuity to evaluate $\lim_{x \to \pi} \sin(x + \sin x)$.

Exercise 4.2.9
Give an example of two functions f and g that are not continuous on the interval $(0, 1)$ but their sum $f + g$ is continuous on $(0, 1)$.

Exercise 4.2.10
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$.
(a) Show that $f(0) = 0$ and $f(n) = an$ for all $n \in \mathbb{N}$ where $a = f(1)$.
(b) Show $f\left(\frac{m}{n}\right) = a \cdot \frac{m}{n}$ where m and n are integers with $n \neq 0$. That is, $f(x) = ax$ for all $x \in \mathbb{Q}$.
(c) Show that $f(x) = ax$ for all $x \in \mathbb{R}$. Hint: Exercise 4.2.5 applied to the function $g(x) = f(x) - ax$.

Exercise 4.2.11
Prove that if f is continuous on $[a, b]$, then either $f(x) = 0$ for some $x \in [a, b]$, or there is a number $\epsilon > 0$ such that $|f(x)| \geq \epsilon$ for all $x \in [a, b]$.

4.3 Uniform Continuity

Recall that a function $f : D \to \mathbb{R}$ is continuous at point a in D if and only if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{if } |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon.$$

The δ in this definition depends on ϵ and the point a. That is, for the same ϵ but with a different point b the δ might be different. Is there a function f such that for all x_1 and x_2 in D with distance less than a fixed δ, we have $|f(x_1) - f(x_2)| < \epsilon$? The answer is yes. We say that such a function is uniformly continuous. More formally, we have

Definition 4.3.1
A function $f : D \to \mathbb{R}$ is **uniformly continuous** if and only if for any $\epsilon > 0$ there is a $\delta > 0$ (depending only on ϵ) such that for all x_1 and x_2 in D

$$\text{if } |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \epsilon.$$

A graphical illustration is given in Figure 4.3.1.

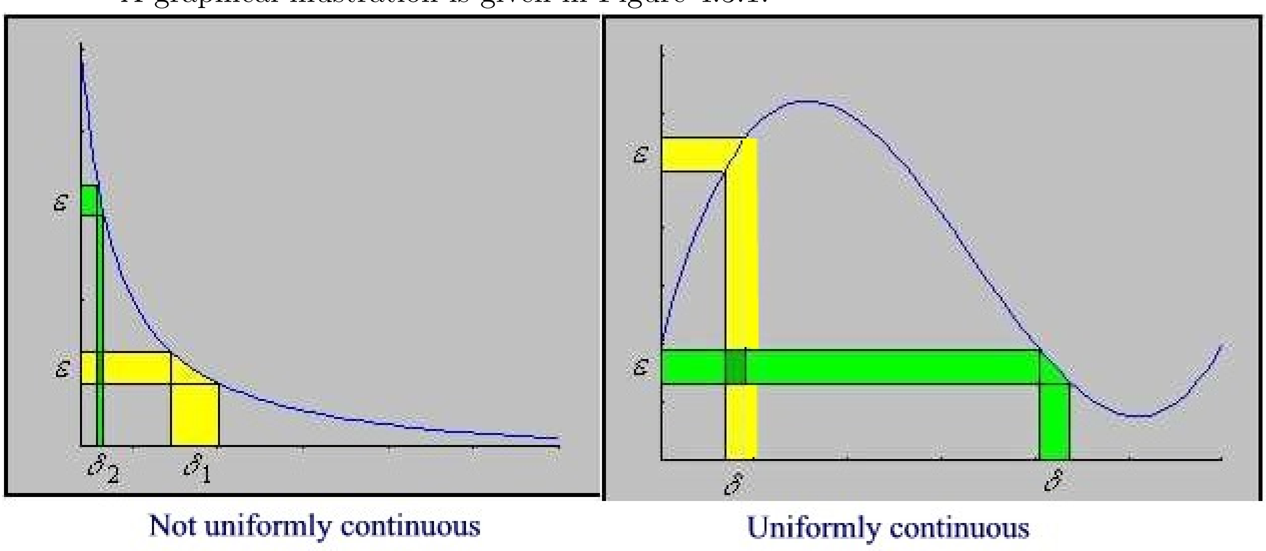


Figure 4.3.1

Continuity of a function at a point is a local property of the function. In contrast, uniform continuity is a global property of the function.

Exercise 4.3.1
Show that the function $f(x) = x$ is uniformly continuous.

Exercise 4.3.2
Consider the function $f(x) = \frac{1}{x}$ on the set $x > 0$. Let $\delta > 0$ be any number and define $\alpha = \min\{2, \delta\}$. Then $\alpha \leq 2$ and $\alpha \leq \delta$. Let $x_1 = \frac{\alpha}{3} > 0$ and $x_2 = \frac{\alpha}{6} > 0$.
(a) Show that $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \geq 1$.
(b) Conclude from (a) that f is not uniformly continuous on the interval $0 < x < \infty$.

Exercise 4.3.3
(a) Show that if f is uniformly continuous on D then f is continuous at every point in D.
(b) Using properties of continuous functions, show that the function $f(x) = \frac{1}{x}$ is continuous on the interval $0 < x < \infty$.
(c) Is the converse of (a) always true? That is, every continuous function is uniformly continuous?

Exercise 4.3.4
Show that if $f, g : D \to \mathbb{R}$ are uniformly continuous then $f + g : D \to \mathbb{R}$ is also uniformly continuous.

Exercise 4.3.5
Let $f(x) = x^2$. Suppose that there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ for all real numbers x_1 and x_2. In addition, suppose we want $|x_1^2 - x_2^2| = 1$. That is, $|x_1 - x_2||x_1 + x_2| = 1$. One way to achieve that is by setting $x_1 - x_2 = \frac{\delta}{2}$ and $x_1 + x_2 = \frac{2}{\delta}$.
(a) Find x_1 and x_2 in terms of δ.
(b) Show that f is not uniformly continuous. Hint: Let $\epsilon = \frac{1}{2}$ in Definition 12.

Exercise 4.3.6
Give an example of two functions $f, g : D \to \mathbb{R}$ that are uniformly continuous but the product function $f \cdot g$ is not.

Exercise 4.3.7
Let $f, g : D \to \mathbb{R}$ be uniformly continuous and bounded, say $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all x in D. Let $\epsilon > 0$ be arbitrary.
(a) Show that there is a $\delta_1 > 0$ such that

$$\text{if } |x-u| < \delta_1 \Longrightarrow |f(x)-f(u)| < \tfrac{\epsilon}{2M_2} \text{ for all } x, u \text{ in } D.$$

(b) Show that there is a $\delta_2 > 0$ such that

$$\text{if } |x-u| < \delta_2 \Longrightarrow |g(x)-g(u)| < \tfrac{\epsilon}{2M_1} \text{ for all } x, u \text{ in } D.$$

(c) Show that $f \cdot g : D \to \mathbb{R}$ is also uniformly continuous. Note that boundedness is crucial in this result. Hint: Note that $f(x)g(x) - f(u)g(u) = (f(x) - f(u))g(x) + f(u)(g(x) - g(u))$.

Exercise 4.3.8
Suppose that $f : D \to \mathbb{R}$ is uniformly continuous. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence of terms in D.
(a) Let $\epsilon > 0$ be arbitrary. Show that there is a $\delta > 0$ such that

$$\text{If } |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \epsilon \text{ for all } x_1, x_2 \text{ in } D.$$

(b) Show that there is a positive integer N such that

$$\text{If } n, m \geq N \Longrightarrow |a_n - a_m| < \delta.$$

(c) Show that $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (and therefore by Exercise 2.5.7 is convergent).

Exercise 4.3.9
Consider the function $f(x) = \tan x$ on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(a) Show that the sequence $\{\frac{\pi}{2} - \frac{1}{n}\}_{n=1}^{\infty}$ is convergent.
(b) Show that the sequence in (a) is also Cauchy.
(c) Show that the sequence $\{f\left(\frac{\pi}{2} - \frac{1}{n}\right)\}_{n=1}^{\infty}$ is not Cauchy.
(d) Show that the function $f(x) = \tan x$ is not uniformly continuous on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Exercise 4.3.10
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two uniformly continuous functions with the range of f contained in D'. Looking closely at Exercise 4.2.3, show that the composite function $g(f(x))$ is also uniformly continuous.

Practice Problems

Exercise 4.3.11
Consider the function $f(x) = \sin x$ defined on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(a) Use Exercise 3.2.11(a) to show that $|\sin x| \leq |x|$ on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(b) Using the trigonometric identity $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$ and (a) to show that

$$|\sin a - \sin b| \leq |a - b|.$$

(c) Show that f is uniformly continuous on the $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Exercise 4.3.12
Using Exercise 4.3.10 and Exercise 4.3.11, show that the function $g(x) = \cos x$ is uniformly continuous in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Exercise 4.3.13
Give an example of two uniformly continuous functions f and g such that $\frac{f(x)}{g(x)}$ is not uniformly continuous.

Exercise 4.3.14
Let $g : D \to \mathbb{R}$ be a uniformly continuous function with $|g(x)| \geq M > 0$ for all $x \in D$. Hence, the function $\frac{1}{g(x)}$ is bounded and $g(x) \neq 0$ for all x in D. Show that $\frac{1}{g(x)}$ is uniformly continuous.

Exercise 4.3.15
Let $f, g : D \to \mathbb{R}$ be two uniformly continuous functions such that $f(x)$ is bounded and $|g(x)| \geq M > 0$ for all $x \in D$. Show that the function $\frac{f(x)}{g(x)}$ is uniformly continuous on D.

Exercise 4.3.16
A function $f : D \to \mathbb{R}$ is said to be **Lipschitz** if there is a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in D$. Show that a Lipschitz function is uniformly continuous.

4.4 Under What Conditions a Continuous Function is Uniformly continuous?

We have seem (Exercise 4.3.3(a)) that a function $f : D \to \mathbb{R}$ uniformly continuous on D is continuous on D. However, the converse is not always true as seen from Exercise 4.3.3(c). In this section, we will show that any continuous function on the interval $[a, b]$ is uniformly continuous there.
Suppose not, then there is an $\epsilon > 0$ such that for all $\delta > 0$ there are u and v in $[a, b]$ such that

$$\text{if } |u - v| < \delta \Longrightarrow |f(u) - f(v)| \geq \epsilon.$$

In particular, for each positive integer n we can let $\delta = \frac{1}{n}$ and thus obtain two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ of numbers in $[a, b]$ such that

$$|u_n - v_n| < \frac{1}{n} \Longrightarrow |f(u_n) - f(v_n)| \geq \epsilon. \tag{4.4.1}$$

Exercise 4.4.1
(a) Let $c_0 = \frac{a+b}{2}$. Then either $[a, c_0]$ or $[c_0, b]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_1, b_1]$. Show that $b_1 - a_1 = \frac{b-a}{2}$.
(b) Let $c_1 = \frac{a_1+b_1}{2}$. Then either $[a_1, c_1]$ or $[c_1, b_1]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_2, b_2]$. Show that $b_2 - a_2 = \frac{b-a}{2^2}$. Compare a_1 and a_2. Compare b_1 and b_2.
(c) Let $c_2 = \frac{a_2+b_2}{2}$. Then either $[a_2, c_2]$ or $[c_2, b_2]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_3, b_3]$. Show that $b_3 - a_3 = \frac{b-a}{2^3}$. Compare a_1, a_2 and a_3. Compare b_1, b_2 and b_3.

Continuing the process of the previous exercise we can find intervals $[a_n, b_n] \subset [a, b]$ such that $b_n - a_n = \frac{b-a}{2^n}$ with the sequence $\{a_n\}_{n=1}^{\infty}$ being increasing and the sequence $\{b_n\}_{n=1}^{\infty}$ being decreasing. Moreover, the interval $[a_n, b_n]$ contains an infinite terms of the sequence $\{v_n\}_{n=1}^{\infty}$.

Exercise 4.4.2
(a) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound?
(b) Show that there is a constant M such that $M = \sup\{a_1, a_2, \cdots\}$.
(c) Show that $a \leq M \leq b$.

Exercise 4.4.3
(a) Show that there is $\delta > 0$ such that for any $a \le x \le b$ if $|x - M| < \delta$ then $|f(x) - f(M)| < \frac{\epsilon}{2}$.
(b) Show that for all u and v in $[a, b]$ if $|u - M| < \delta$ and $|v - M| < \delta$ then $|f(u) - f(v)| < \epsilon$.

Exercise 4.4.4
(a) Let $w_n = \frac{b-a}{2^n}$. Show that $\lim_{n\to\infty} w_n = 0$. Hint: Squeeze rule.
(b) Show that there is a positive integer N such that $\frac{b-a}{2^N} < \frac{\delta}{2}$ and $|x - M| < \frac{\delta}{2}$ for all $a_N \le x \le b_N$.

Exercise 4.4.5
(a) Using Exercise 4.4.4, show that there is a large n such that $\frac{1}{n} < \frac{\delta}{2}$ and $a_N \le v_n \le b_N$.
(b) For the n found in (a), show that $|u_n - v_n| < \frac{1}{n} < \frac{\delta}{2}$ and $|v_n - M| < \frac{\delta}{2}$.
(c) For the n found in (a), Show that $|u_n - M| < \delta$.
(d) Using (b), (c), and Exericse 4.4.3(b), show that $|f(u_n) - f(v_n)| < \epsilon$.
Conclusion: The result in (d), contradicts (4.4.1). Hence, f must be uniformly continuous.

Practice Problems

Exercise 4.4.6
Show that the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

Exercise 4.4.7
(a) A function $f : D \to \mathbb{R}$ is said to be **Lipschitz** if there is a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in D$. Show that the function $f(x) = \sqrt{x}$ is not Lipschitz on $[0,1]$. Hint: Assume the contrary and get a contradiction.
(b) Give an example of a uniformly continuous function that is not Lipschitz. Thus, the converse to Exercise 4.3.16 is false.

Exercise 4.4.8
Show, using the definition of uniform continuity ($\epsilon - \delta$ definition) the function $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0,2]$.

Exercise 4.4.9
Conisder the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0. \end{cases}$$

Show that f is uniformly continuous on $[0,1]$.

Exercise 4.4.10
Show that the function $f : (-2,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is Lipschitz on $(-2,1]$.

4.5 More Continuity Results: The Intermediate Value Theorem

In this section we proceed with establishing more properties of continuous and uniformly continuous functions. We first define what we mean by a bounded set.

Definition 4.5.1
A set $D \subseteq \mathbb{R}$ is said to be **bounded** if and only if there is a positive real number M such that

$$\text{for all } x \text{ in } D \text{ we have } |x| \leq M.$$

That is, for all x is D we have $-M \leq x \leq M$. This says that D is contained in the closed interval $[-M, M]$.

The first result shows that a continuous function does not necessarily map bounded sets to bounded sets.

Exercise 4.5.1
Give an example of a continuous $f : D \to \mathbb{R}$ with D a bounded set (i.e. $|x| \leq M$ for some $M > 0$ and for all x in D) but $f(D)$ is not bounded.

The following result shows that uniformly continuous functions preserve boundedness. That is, the range of a bounded set under a uniformly continuous function is bounded.

Exercise 4.5.2
Let D be a bounded subset of $\mathbb{R}$ with $|x| \leq M$ for all $x \in D$. Suppose that $f : D \to \mathbb{R}$ is uniformly continuous.
(a) Show that there is a $\delta > 0$ such that if u and v belong to D such that $|u - v| < \delta$ then $|f(u) - f(v)| < 1$.
(b) Let n be a positive integer such that $n > \frac{2M}{\delta}$. Divide the interval $[-M, M]$ into n equal subintervals:$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$. Show that $x_k - x_{k-1} < \delta$ for all $k = 1, 2, \cdots, n$
(c) Let $[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]$ be those intervals in (b) that intersect D. That is, $D \subseteq [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$. For $1 \leq i \leq k$ let $u_i \in [a_i, b_i] \cap D$. Show that if v is in D then there is an $1 \leq i \leq k$ such that $|v - u_i| < \delta$ and $|f(v)| < 1 + |f(u_i)|$.
(d) Show that $|f(v)| \leq M$ for all v in D. That is, $f(D)$ is bounded.

Exercise 4.5.3
Show that if $f : [a, b] \to \mathbb{R}$ is continuous then $f([a, b])$ is bounded. Hint: Exercises 4.4.5 and 4.5.2.

If D is a bounded set, then by the Completeness Axiom of real numbers there exist finite numbers I and S such that

$$I = \inf\{x \in D\} \text{ and } S = \sup\{x \in D\}$$

Exercise 4.5.4
Show that if $f : [a, b] \to \mathbb{R}$ is continuous then $\inf\{f(x) : a \le x \le b\}$ and $\sup\{f(x) : a \le x \le b\}$ exist.

Exercise 4.5.5
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $I = \inf\{f(x) : a \le x \le b\}$. Note that I exists by Exercise 4.5.4. Suppose that $I < f(x)$ for all $x \in [a, b]$. That is, the infimum can not be attained in $[a, b]$. Define the function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \frac{1}{f(x) - I}.$$

(a) Show that g is continuous on $[a, b]$.
(b) Show that there is a positive number M such that $|g(x)| \le M$ for all $x \in [a, b]$.
(c) Show that $I + \frac{1}{M}$ is a lower bound of $f([a, b])$ and this leads to a contradiction.
Conclusion: There must be a number $x_1 \in [a, b]$ such that $f(x_1) = \inf\{f(x) : a \le x \le b\}$.

Exercise 4.5.6
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $S = \sup\{f(x) : a \le x \le b\}$. Note that S exists by Exercise 4.5.4. Show that there exists $x_2 \in [a, b]$ such that $f(x_2) = S$. Hint: Mimic Exercise 4.5.5.

From the previous two exercises, we have seen that extreme values of a function continuous on $[a, b]$ are attained in $[a, b]$. What can we say about possible values between these? The following result, known as the **intermediate value theorem**, addresses this question.

Exercise 4.5.7
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $f(a) \leq c \leq f(b)$.
(a) Let $D = \{x \in [a, b] : f(x) \leq c\}$. Show that D is non-empty and that D is bounded from above. By the Completeness Axiom of real numbers there is a number d such that $d = \sup\{x \in D\}$.
(b) Show that $d \in [a, b]$.
(c) Suppose that $f(d) > c$. Show that there is a $\delta > 0$ such that if $|x - d| < \delta$ then $|f(x) - f(d)| < f(d) - c$.
(d) Show that for $x \in [a, b]$ and $|x - d| < \delta$ we must have $f(x) > c$. Hint: Exercise 1.1.14.
(e) Using (d), show that $d - \delta$ is an upper bound of D. Thus, $f(d) > c$ leads to a contradiction.
(f) Suppose that $f(d) < c$. Show that there is a $\delta > 0$ such that if $d - \delta < x < d + \delta$ and $x \in [a, b]$ we must have $f(x) < c$.
(g) Show that $f(d + \frac{\delta}{2}) < c$. Why this leads to a contradiction?
Conclusion: We must have $f(d) = c$.

Exercise 4.5.8
Let $f : [a, b] \to \mathbb{R}$ be continuous. By Exercise 4.5.5, there exist $x_1 \in [a, b]$ and $x_2 \in [a, b]$ such that $m = f(x_1) = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$.
(a) Show that $f([a, b]) \subseteq [m, M]$.
(b) Use Exercise 4.5.7 (restricted to the interval $[x_1, x_2]$) to show that $[m, M] \subseteq f([a, b])$.
Conclusion: $f([a, b]) = [m, M]$.

Practice Problems

Exercise 4.5.9
Prove that there exists a number $c \in \left(0, \frac{\pi}{2}\right)$ such that $2c - 1 = \sin\left(c^2 + \frac{\pi}{4}\right)$.

Exercise 4.5.10
Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there is $c \in [a, b]$ such that $f(c) = c$. We call c a **fixed point** of f. Hint: Intermediate Value Theorem applied to a specific function F (to be found) defined on $[a, b]$.

Exercise 4.5.11
Using the Intermediate Value Theorem, show that
(a) the equation $3\tan x = 2 + \sin x$ has a solution in the interval $[0, \frac{\pi}{4}]$;
(b) the polynomial $p(x) = -x^4 + 2x^3 + 2$ has at least two real roots.

Exercise 4.5.12
Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Show that there is a $c \in [a, b]$ such that $f(c) = g(c)$.

Exercise 4.5.13
Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) \leq a$ and $f(b) \geq b$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. We call c a **fixed point** of f.

Exercise 4.5.14
Let $f : [a, b] \to \mathbb{Q}$ be continuous. Prove that f must be a constant function. Hint: Exercise 1.2.6(c).

Exercise 4.5.15
Prove that a polynomial of odd degree considered as a function from the reals to the reals has at least one real root.

Exercise 4.5.16
Suppose $f(x)$ is continuous on the interval $[0, 2]$ and $f(0) = f(2)$: Prove there must be a number c between 0 and 1 so that $f(c + 1) = f(c)$. Hint: Consider the function $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

Chapter 5

Derivatives

5.1 The Derivative of a Function

In this section we introduce the concept of the derivative of a function and discuss some of its properties.

Definition 5.1.1
Let $f : D \to \mathbb{R}$ and $x \in D$. We say that f is **differentiable** at a if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Symbolically, we write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We call $f'(a)$ the **derivative** of f at a. A function that is not differentiable at a is said to be **non-diffferentiable**. If $f'(a)$ exists for every $a \in D$, we say that f is **differentiable on** D. The process of finding the derivative is referred to as **differentiation.**

Exercise 5.1.1
Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is not differentiable at $a = 0$.

Exercise 5.1.2
Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable at $a = 0$. What is $f'(0)$?

Exercise 5.1.3
Show that $f(x) = |x|$ is not differentiable at 0.

Exercise 5.1.4
Find the derivative of $f(x) = \sin x$. Hint: Recall the trigonometric identity $\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ and use Exercise 3.2.11.

The following exercise shows that every differentiable function is continuous.

Exercise 5.1.5
Let $f : D \to \mathbb{R}$ be differentiable at a.
(a) Show that

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{h\to 0}[f(h+a) - f(a)].$$

(b) Show that f is continuous at a. That is,

$$\lim_{x\to a}[f(x) - f(a)] = 0.$$

Exercise 5.1.6
Give an example of a function $f : D \to \mathbb{R}$ that is continuous at a but not differentiable there. That is, the converse to the result of Exercise 5.1.5 is false in general.

Exercise 5.1.7
Suppose that $f, g : D \to \mathbb{R}$ are differentiable at a. Show that the functions $f \pm g$ are also differentiable at a.

Exercise 5.1.8 *(Product Rule)*
Suppose that $f, g : D \to \mathbb{R}$ are differentiable at a.
(a) Show that $(fg)(a+h) - (fg)(a) = [f(a+h) - f(a)]g(a+h) + f(a)[g(a+h) - g(a)]$.
(b) Show that the function $f \cdot g$ is also differentiable at a.

Exercise 5.1.9 *(Quotient Rule)*
Suppose that $f, g : D \to \mathbb{R}$ are differentiable at a with $g(a) \neq 0$.
(a) Show that

$$\frac{\left(\frac{f}{g}\right)(a+h) - \left(\frac{f}{g}\right)(a)}{h} = \frac{f(a+h)-f(a)}{h}\cdot\frac{1}{g(a+h)} - \frac{g(a+h)-g(a)}{h}\cdot\frac{f(a)}{g(a)}\cdot\frac{1}{g(a+h)}.$$

(b) Show that

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Exercise 5.1.10 *(Chain Rule)*
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two functions with $f(D) \subseteq D'$. Suppose that f is differentiable at a and g is differentiable at $f(a)$.
(a) Define $w : D' \to \mathbb{R}$ by

$$w(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & \text{if } y \neq f(a) \\ g'(f(a)) & \text{if } y = f(a). \end{cases}$$

Show that w is continuous at $f(a)$. That is,

$$\lim_{h\to 0} w(h + f(a)) = w(f(a)).$$

(b) Show that $(w \circ f)(x)$ is continuous at a.
(c) Show that

$$\frac{(g\circ f)(a+h) - (g\circ f)(a)}{h} = (w\circ f)(a+h)\cdot\frac{f(a+h)-f(a)}{h}.$$

(d) Show that

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a).$$

Exercise 5.1.11
Let $f(x) = x^n$ where n is a non-negative integer.
(a) By letting $h = ax - x$, show that

$$f'(x) = \lim_{a\to 1}\frac{f(ax)-f(x)}{ax-x}.$$

(b) What is the quotient of the division of $a^n - 1$ by $a - 1$? Hint: Use synthetic division.
(c) Use (a) and (b) to show that $f'(x) = nx^{n-1}$.

Practice Problems

Exercise 5.1.12
(a) Show that the derivative of a constant function is zero and that the derivative of $f(x) = x$ is $f'(x) = 1$.
(b) Show that the function $h(x) = x \sin\left(\frac{1}{x}\right)$ is differentiable for all $x \neq 0$.

Exercise 5.1.13
Let $f(x) = \sqrt{2x-1}$. Find $f'(2)$ by using only the definition of the derivative.

Exercise 5.1.14
Let

$$f(x) = \begin{cases} 2x+5 & \text{if } x \leq 1 \\ 9x^2 - 2 & \text{if } x > 1. \end{cases}$$

Show that $f(x)$ is continuous but not differentiable at $x = 1$.

Exercise 5.1.15
Find constants a and b such that the piecewise defined function

$$f(x) = \begin{cases} ax^2 - 4 & \text{if } x \leq 1 \\ bx + a & \text{if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

Exercise 5.1.16
Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

Exercise 5.1.17
(a) Let $f(x) = x^n$ with n a negative integer. Prove that $f'(x) = nx^{n-1}$.
(b) Let $f(x) = x^{\frac{p}{q}}$ where p and q are integers with $q \neq 0$. Prove that $f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$. Hint: Let $y = x^{\frac{p}{q}}$ so that $y^q = x^p$ and use Exercise 5.1.10.

Exercise 5.1.18
We define the number e to be the unique number satisfying

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

It is an irrational number whose value is approximately 2.718281828459045. Define the function $f(x) = e^x$. Find $f'(x)$ using the definition of the derivative.

Exercise 5.1.19
The **natural logarithmic function** is the function $f(x) = \ln x$ defined as follows: $y = \ln x$ if and only if $x = e^y$. Find the derivative of f. Hint: Differentiate $x = e^y$ with respect to x.

Exercise 5.1.20
Consider the function $f(x) = x^n$ where n is a real number.
(a) Suppose that $x > 0$ and x in the domain of f. Using the fact that $x^n = e^{n \ln x}$, show that $f'(x) = nx^{n-1}$.
(b) Suppose that $x < 0$ and x in the domain of f. Show that $f'(x) = nx^{n-1}$. Hint: $x^n = (-1)^n(-x)^n$.

5.2 Extreme values of a Function and Related Theorems

Points of interest on the graph of a function are those points that are the highest on the curve, or the lowest, in a specific interval. Such points are called **local extrema.**

Definition 5.2.1
Let $f : D \to \mathbb{R}$. We say that f has a **local maximum** or a **relative maximum** at $a \in D$ if there is an $\epsilon > 0$ such that $f(x) \leq f(a)$ for all $x \in (a-\epsilon, a+\epsilon) \cap D$.
We say that f has a **local minimum** or a **relative minimum** at $a \in D$ if there is an $\epsilon > 0$ such that $f(a) \leq f(x)$ for all $x \in (a-\epsilon, a+\epsilon) \cap D$.

Exercise 5.2.1
(a) Find the local extrema (if they exist) of the function $f(x) = |x|$.
(b) Find the local extrema (if they exist) of the function $f(x) = x^3$.
(c) Find the local extrema (if they exist) of the function $f(x) = x$ on the interval $[0, 1]$.

The following exercise shows that if a differentiable function has a local extrema (that is not a boundary point) then the derivative at that point must be zero.

Exercise 5.2.2
Let $f : [a, b] \to \mathbb{R}$. Suppose that $c \in (a, b)$ is a local maximum (or local minimum) of f such that $f'(c)$ exists. Let $\epsilon > 0$ such that $f(x) \leq f(c)$ for all $x \in (c-\epsilon, c+\epsilon) \subseteq [a, b]$.
(a) Let $h > 0$ be small enough so that $c + h \in (c-\epsilon, c+\epsilon)$. Using Exercise 3.2.8, show that $f'(c) \leq 0$.
(b) Let $h < 0$ be large enough so that $c + h \in (c-\epsilon, c+\epsilon)$. Using Exercise 3.2.8, show that $0 \leq f'(c)$ and therefore $f'(c) = 0$.

Exercise 5.2.3
The condition $a < c < b$ is critical in the previous exercise. Give an example of a function $f : [a, b] \to \mathbb{R}$ such that either a or b is a local extremum but with non-zero derivative there.

Definition 5.2.2
A point $(c, f(c))$ such that either $f'(c)$ does not exist or $f'(c) = 0$ is called a **critical point**. Exercise 5.2.2 tells us that potential local extrema are critical points.

By Exercise 4.5.8, if $f : [a, b] \to \mathbb{R}$ is continuous then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$. That is, x_1 is a local minimum and x_2 is a local maximum. The following exercise tells us where to look for x_1 and x_2.

Exercise 5.2.4
Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$. Show that x_1 and x_2 are either the endpoints of $[a, b]$ or critical points of f in $a < x < b$.

Exercise 5.2.5 *(Rolle's Theorem)*
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. By Exercise 4.5.8 there exist $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$. Suppose that $f(a) = f(b)$.
(a) Show that if $f(x) = C$ for all $a \leq x \leq b$ then there is at least a number $a < c < b$ such that $f'(c) = 0$.
(b) Suppose that f is a non-constant function. Let $d \in [a, b]$ such that $f(d) \neq f(a)$. Show that if $f(d) < f(a)$ then $a < x_1 < b$. What can you say about the value of $f'(x_1)$?
(c) Show that if $f(a) < f(d)$ then $a < x_2 < b$. What can you say about the value of $f'(x_2)$?

Geometrically, Rolle's theorem claims that if $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$ and $f(a) = f(b)$, somewhere between a and b the graph of f has a horizontal tangent line. See Figure 5.2.1.

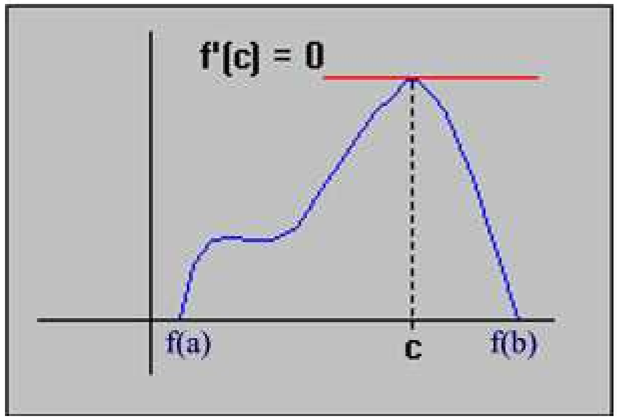


Figure 5.2.1

Exercise 5.2.6
Find the number c of Rolle's theorem for the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x} - x$.

Practice Problems

Exercise 5.2.7
Assume $a_0, a_1, \cdots, a_n$ are real numbers such that

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0.$$

Show that the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has at least one root in $(0, 1)$.

Exercise 5.2.8
(a) Show that the function $f(x) = x^3 - 4x^2 - 3x + 1$ has a root in $[0, 2]$.
(b) Use Rolle's theorem to show that there is exactly one root in $[0, 2]$.

Exercise 5.2.9
Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable, and let $a, b \in \mathbb{R}$ be such that $a < b$. Show that there is a $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Hint: Apply Rolle's theorem to the function $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$.

Exercise 5.2.10
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Apply Rolle's theorem to the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

5.3 The Mean Value Theorem and its Applications

The Mean Value Theorem is behind many of the important results in calculus that we will discuss in this section. The Mean Value Theorem is a generalization of Rolle's Theorem.

Exercise 5.3.1 *(Mean Value Theorem)*
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Use Exercise 5.2.5 with the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Geometrically, the mean value theorem claims that if $f : [a, b] \to \mathbb{R}$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$, somewhere between a and b the graph of f has a tangent line parallel to the line connecting $(a, f(a))$ and $(b, f(b))$. See Figure 5.3.1.

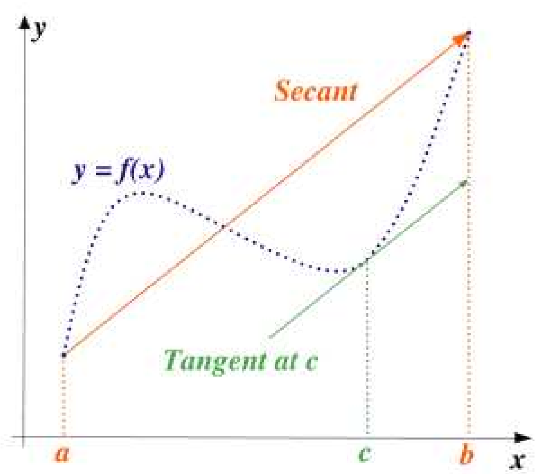


Figure 5.3.1

Exercise 5.3.2 *(Cauchy Mean Value Theorem)*
Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous for $a \le x \le b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c).$$

Hint: Use Exercise 5.2.5 with the function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Exercise 5.3.3
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. We say that f is one-to-one if and only if for any $a \le x_1 \le b$ and $a \le x_2 \le b$ such that $f(x_1) = f(x_2)$ we must have $x_1 = x_2$. Suppose that $f'(x) \ne 0$ for all $a < x < b$.
(a) Let $a \le x_1 \le b$ and $a \le x_2 \le b$ such that $f(x_1) = f(x_2)$. Show that if $x_1 < x_2$ then there is $a < x_1 < c < x_2 < b$ such that $f'(c) = 0$ which contradicts the assumption that $f'(x) \ne 0$ for all $a < x < b$. Hint: Use the Mean Value Theorem on the interval $[x_1, x_2]$.
(b) Answer the same question for $x_2 < x_1$.
Conclusion: We must have $x_1 = x_2$. This shows that f is 1-1.

Exercise 5.3.4
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. We say that f is **increasing** in $[a, b]$ if and only if for every x_1 and x_2 in $[a, b]$, if $x_1 \le x_2$ then $f(x_1) \le f(x_2)$. Show that if $f'(x) \ge 0$ for all $a < x < b$ then $f(x)$ is increasing in $[a, b]$. Hint: Use the MVT restricted to the interval $[x_1, x_2]$.

Definition 5.3.1
We say that $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ if and only if f is differentiable in $a < x < b$ and the following limits exist

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ and } f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable such that $f'(x) \ne 0$ for all $a < x < b$. We know from Exercise 5.3.3 that f is one-to-one on $[a, b]$. We want to show that f is monotone as well on $[a, b]$.
To say that f is not monotone on $[a, b]$ means that one of the following cases

applies:
(i) There are $x, y, z \in [a, b]$ such that $x < y < z$ and $f(x) < f(y), f(z) < f(y)$. That is the graph of f is increasing on $[x, y]$ and decreasing on $[y, z]$.
(ii) There are $x, y, z \in [a, b]$ such that $x < y < z$ and $f(x) > f(y), f(y) < f(z)$. That is the graph of f is decreasing on $[x, y]$ and increasing on $[y, z]$.

Exercise 5.3.5
Consider Case (i). We have either $f(x) < f(z) < f(y)$ or $f(z) < f(x) < f(y)$.
(a) Suppose that $f(z) < f(x) < f(y)$. Use the Intermediate Value theorem restricted to $[y, z]$ to show that such a double inequality can not occur.
(b) Suppose that $f(x) < f(z) < f(y)$. Use the Intermediate Value theorem restricted to $[x, y]$ to show that such a double inequality can not occur.
We conclude that Case (i) does not hold.

Exercise 5.3.6
Consider Case (ii). We have either $f(y) < f(x) < f(z)$ or $f(y) < f(z) < f(x)$.
(a) Suppose that $f(y) < f(x) < f(z)$. Use the Intermediate Value theorem restricted to $[y, z]$ to show that such a double inequality can not occur.
(b) Suppose that $f(y) < f(z) < f(x)$. Use the Intermediate Value theorem restricted to $[x, y]$ to show that such a double inequality can not occur.
We conclude that Case (ii) does not hold.

We conclude from the previous two exercises that f must be monotone in $[a, b]$.

Exercise 5.3.7
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable such that $f'(x) \neq 0$ for all $a < x < b$. We know from the above discussion that f is monotone.
(a) Show that if f is increasing in $[a, b]$ then $f'(x) \geq 0$ for all $a \leq x \leq b$. Hint: Let $x \in [a, b)$ and choose $h > 0$ small enough so that $x + h \in [a, b)$. If $x = b$, choose $h < 0$ so that $b + h < b$. Now use the definition of the derivative.
(b) Show that if f is decreasing in $[a, b]$ then $f'(x) \leq 0$ for all $a \leq x \leq b$.

Practice Problems

Exercise 5.3.8
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. We say that f is a constant function on $[a, b]$ if and only if there is a constant C such that $f(x) = C$ for all $a \le x \le b$. Suppose that $f'(x) = 0$ for all $a < x < b$.
Let x_1 and x_2 be any two numbers in the interval $[a, b]$ with $x_1 < x_2$. Suppose that $f(x_1) \neq f(x_2)$. Show that by applying the Mean Value Theorem on the interval $[x_1, x_2]$ we obtain the contradiction $f(x_1) = f(x_2)$. Thus, we must have $f(x_1) = f(x_2) = C$ for any x_1 and x_2 in $[a, b]$. That is, $f(x) = C$ for all $a \le x \le b$.

Exercise 5.3.9
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. Suppose that $f'(x) = g'(x)$ for all $a < x < b$. Show that $f(x) = g(x) + C$ for all $a \le x \le b$, where C is a constant. Hint: Exercise 5.3.8

Exercise 5.3.10
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. We say that f is **decreasing** in $[a, b]$ if and only if for every x_1 and x_2 in $[a, b]$, if $x_1 \le x_2$ then $f(x_1) \ge f(x_2)$. Show that if $f'(x) \le 0$ for all $a < x < b$ then $f(x)$ is decreasing in $[a, b]$. Hint: Use the MVT restricted to the interval $[x_1, x_2]$.

Exercise 5.3.11
Consider the function $f(x) = (1 + x)^p$ where $0 < p < 1$. Let $h > 0$.
(a) Apply the MVT to the interval $[0, h]$ to show that $f(h) = p(1+t)^{p-1}h+1$ for some $0 < t < h$.
(b) Use (a) to show that $(1 + h)^p < 1 + ph$.
In annuity theory, $(1 + h)^p$ may represent compound interest and $1 + ph$ represent simple interest. A result in annuity theory says that for time p less than a year compound interest formula can be estimated by the simple interest formula.

Exercise 5.3.12
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$. Let λ be a real number such that either $f'(a) < \lambda < f'(b)$ or $f'(b) < \lambda < f'(a)$.
(a) Define $g(x) = f(x) - \lambda x$. Show that if $f'(a) < \lambda < f'(b)$ then $g'(x)$

changes sign between a and b.
(b) Establish the same result for $f'(b) < \lambda < f'(a)$.
(c) Show that the condition $g'(c) \neq 0$ for all $c \in [a,b]$ leads to a contradiction. Hint: Exercise 5.3.7. Conclude that there must be a $a < c < b$ such that $f'(c) = \lambda$.

Exercise 5.3.13
Let $f, g : [a,b] \to \mathbb{R}$ be two differentiable functions on $[a,b]$ such that $f(a) = g(a)$. Show that if $f'(x) = g'(x)$ for all $x \in (a,b)$ then $f(x) = g(x)$ for all $x \in [a,b]$. Hint: Exercise 5.3.8.

Exercise 5.3.14
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Show that f can have at most one fixed point. That is, There is at most one $c \in \mathbb{R}$ such that $f(c) = c$. Hint: Mean Value Theorem.

Exercise 5.3.15
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and that $f'(a) < 0$ and $f'(b) > 0$ for some $a < b$. Prove that there is a $c \in (a,b)$ such that $f'(c) = 0$.

Exercise 5.3.16
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| \leq K < 1$ for all $x \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$. Define the numbers $a_n = f(a_{n-1})$.
(a) Show that $|a_{n+1} - a_n| \leq K^n |a_1 - a_0|$ for all $n \in \mathbb{N}$.
(b) Show that for all $m, n \in \mathbb{N}$ such that $m > n$ we have

$$|a_m - a_n| \leq \frac{K^n}{1-K}.$$

Exercise 5.3.17
Show that if $0 < a < b$ then $1 - \frac{a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1$. Hint: Apply the MVT for the function $f(x) = \ln x$.

5.4 L'Hôpital's Rule and the Inverse Function Theorem

The following result known as **L'Hôpital's Rule** uses derivatives to evaluate limits of the ratio of two functions with limit of the form $\frac{0}{0}$.

Exercise 5.4.1
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $a < x < b$ with $g'(x) \neq 0$ for all $a < x < b$. Suppose that $f(c) = g(c) = 0$ for some $a \leq c \leq b$. Also, suppose that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = A.$$

(a) Let $\{c_n\}_{n=1}^{\infty} \subset [a, b]$ be an arbitrary sequence with the properties $c_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = c$. Show that there is a d_n between c_n and c such that

$$[f(c_n) - f(c)]g'(d_n) = [g(c_n) - g(c)]f'(d_n).$$

(b) Show that $d_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} d_n = c$.
(c) Show that $g(d_n) \neq g(c)$ for all $n \geq 1$. Hint: Exercise 5.3.3.
(d) Show that

$$\frac{f'(d_n)}{g'(d_n)} = \frac{f(c_n)}{g(c_n)}.$$

(e) Show that $\lim_{n\to\infty} \frac{f'(d_n)}{g'(d_n)} = A$. Hint: See Exercise 3.3.1.
(f) Show that $\lim_{n\to\infty} \frac{f(c_n)}{g(c_n)} = A$.
(g) Show that $\lim_{x\to c} \frac{f(x)}{g(x)} = A$.

Exercise 5.4.2
Find

$$\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2} + \sqrt{x-2}}{\sqrt{x^2 - 4}}.$$

Definition 5.4.1
Let $f : D \to \mathbb{R}$. We say that f is **invertible** if and only if there is a function $g : D' \to \mathbb{R}$ such that the following two statements are true

$$f(g(x)) = x \text{ for all } x \text{ in } D'$$

and

$$g(f(x)) = x \text{ for all } x \text{ in } D.$$

We Write $g = f^{-1}$ and we call f^{-1} the **inverse function** of f.

Exercise 5.4.3
Let $f : [a, b] \to \mathbb{R}$ be a **one-to-one** function. That is, if $f(x) = f(y)$ then $x = y$, where $x, y \in [a, b]$.
(a) Define $g : f([a, b]) \to [a, b]$ by $g(y) = x$ if and only if $f(x) = y$. Show that g is indeed a function. That is, if $y_1, y_2 \in f([a, b])$ are such that $y_1 = y_2$ then $g(y_1) = g(y_2)$.
(b) Show that $f(g(y)) = y$ for all $y \in f([a, b])$ and $g(f(x)) = x$ for all $x \in [a, b]$. Thus, conclude that f is invertible.

Exercise 5.4.4
Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in $[a, b]$ with $f'(x) \neq 0$ for all $a < x < b$. Let the range of f be denoted by $[m, M]$.
(a) Show that f is one-to-one, monotone, and invertible with inverse $f^{-1} : [m, M] \to [a, b]$.
(b) Assume that f is strictly increasing. That is, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. In this case, $[m, M] = [f(a), f(b)]$. Let $f(a) < y_0 < f(b)$. Show that there is a $a < x_0 < b$ such that $f(x_0) = y_0$.
(c) Let $\epsilon > 0$ be given. Let $\epsilon_1 = \min\{\epsilon, x_0 - a, b - x_0\}$. Show that if x satisfies $|x - x_0| < \epsilon_1$ then $a < x < b$ and $|x - x_0| < \epsilon$.
(d) Let $y_1 = f(x_0 - \epsilon_1)$ and $y_2 = f(x_0 + \epsilon_1)$. Show that $f[(x_0 - \epsilon_1, x_0 + \epsilon_1)] = (y_1, y_2)$.
(e) Choose a $\delta > 0$ so that $(y_0 - \delta, y_0 + \delta) \subset (y_1, y_2)$. Show that if $|y - y_0| < \delta$ then $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. This shows that f^{-1} is continuous in $(f(a), f(b))$.
(f) Show that f^{-1} is right continuous at $f(a)$ and left continuous at $f(b)$.
We conclude from this problem that f^{-1} is continuous on the closed interval $[f(a), f(b)]$.

Remark 5.4.1
A similar result holds if f is strictly decreasing.

Exercise 5.4.5 *(Inverse Function Theorem)*
Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in $[a, b]$ with $f'(x) \neq 0$ for all $a < x < b$. Let $c \in f([a, b])$. Then there is a $d \in [a, b]$ such

that $f(d) = c$.
(a) Let $\{c_n\}_{n=1}^{\infty} \subseteq f([a, b])$ such that $c_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = c$. Show that there is a sequence $\{d_n\}_{n=1}^{\infty} \subseteq [a, b]$ such that

$$\lim_{n\to\infty} d_n = d.$$

Hint: Exercise 5.4.4(b).
(b) Show that $d_n \neq d$ for all $n \geq 1$.
(c) Show that

$$\lim_{n\to\infty} \frac{f(d_n) - f(d)}{d_n - d} = f'(d).$$

Hint: Exercise 3.3.1.
(d) Show that $\frac{f(d_n)-f(d)}{d_n-d} \neq 0$ for all $n \geq 1$. Hint: Exercise 5.3.3.
(e) Show that

$$\lim_{n\to\infty} \frac{f^{-1}(c_n) - f^{-1}(c)}{c_n - c} = \frac{1}{f'(d)}.$$

Thus, conclude that

$$(f^{-1})'(f(d)) = \frac{1}{f'(d)}$$

for all $d \in [a, b]$. That is f^{-1} is differentiable in $f([a, b])$. Hint: Exercise 3.3.2.

Practice Problems

Exercise 5.4.6
Find $\lim_{x\to\infty}\left(\frac{\ln x}{x}\cdot\sin\left(\frac{x\pi+2}{2x}\right)\right)$.

Exercise 5.4.7
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $a < x < b$ with $g'(x) \neq 0$ for all $a < x < b$. Suppose that $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$ for some $a \leq c \leq b$. Also, suppose that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} = A.$$

Prove that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = A.$$

Exercise 5.4.8
Use L'Hôpital's rule to evaluate $\lim_{x\to 0^+} x^x$. Note that 0^0 is an undeterminate form.

Exercise 5.4.9
Let f and g be invertible differentiable functions such that

$$f(1) = 2; g(2) = 1; f'(1) = g'(2) = 3.$$

Find the derivative $(f^{-1}\circ g^{-1})'(1)$.

Exercise 5.4.10
Let $f(x) = x\tan^2 x$ for $x \in (0, \frac{\pi}{2})$. Calculate $(f^{-1})'(\pi)$. Note that $f(\frac{\pi}{3}) = \pi$.

Chapter 6

Riemann Integrals

6.1 The Theory of Riemann Integral

The Riemann integral, as it is called today, is the one usually discussed in introductory calculus. Throughout this section, it is assumed that we are working with a bounded function f on a closed interval $[a, b]$, meaning that there exist real numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition 6.1.1
A **partition** P of $[a, b]$ is a finite, ordered set

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

If Q is another partition of $[a, b]$ such that $P \subset Q$ then we call Q a **refinement** of P.
For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The **Riemann lower sum** of f with respect to P is given by

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

Likewise, we define the **Riemann upper sum** of f with respect to P by

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

Exercise 6.1.1
(a) Show that $m \le m_i(f) \le M_i(f) \le M$.
(b) Show that $m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$.

Exercise 6.1.2
Let Q be a refinement of P. Suppose that $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ and $Q = \{a = x_0 < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < x_n = b\}$.
(a) Show that $U(f,Q) \le U(f,P)$.
(b) Show that $L(f,P) \le L(f,Q)$.

Definition 6.1.2
We define

$$S_U = \{U(f,P) : P \text{ a partition of } [a,b]\}$$

and

$$S_L = \{L(f,P) : P \text{ a partition of } [a,b]\}.$$

Then S_U is bounded from below by $m(b-a)$. By the completeness axiom of $\mathbb{R}$, $\inf S_U$ is a finite number. We define the **upper Riemann integral** to be

$$\overline{\int_a^b} f(x)dx = \inf S_U.$$

Likewise, S_L is bounded from above by $M(b-a)$. By the completeness axiom of $\mathbb{R}$, $\sup S_L$ is a finite number. We define the **lower Riemann integral** to be

$$\underline{\int_a^b} f(x)dx = \sup S_L.$$

Exercise 6.1.3
Suppose that f is bounded on $[a,b]$. Show that $\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$. Hint: Exercise 6.1.2.

Definition 6.1.3
We say that a bounded function $f : [a,b] \to \mathbb{R}$ is **Riemann integrable** on $[a,b]$ if and only if

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

We write

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$$

and we call $\int_a^b f(x)dx$ the **Riemann integral** of f on $[a, b]$.

Exercise 6.1.4
Consider the function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } a \leq x < b \\ 3 & \text{if } x = b. \end{cases}$$

(a) Find two numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$.
(b) Show that for any partition P of $[a, b]$ we have $L(f, P) = 2(b - a)$. Conclude that

$$\underline{\int_a^b} f(x)dx = 2(b - a).$$

(c) Show that $\overline{\int_a^b} f(x)dx \geq 2(b - a)$.
(d) Suppose $\overline{\int_a^b} f(x)dx > 2(b - a)$. Let $\epsilon = \overline{\int_a^b} f(x)dx - 2(b - a) > 0$. Let Q be the partition

$$Q = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

such that $b - x_{n-1} < \epsilon$. Show that $U(f, Q) < \overline{\int_a^b} f(x)dx$. Why this is impossible?
(e) Is $f(x)$ Riemann integrable? If so, what is the value of the integral $\int_a^b f(x)dx$?

The above example shows that a discontinuous function can be Riemann integrable.
Next, we present an example of a function that is not Riemann integrable.

Exercise 6.1.5
Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational.
(a) Compute the upper Riemann integral and the lower Riemann integral. Hint: Exercise 1.2.6(c).
(b) Is f Riemann integrable on $[0, 1]$?

Exercise 6.1.6
Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable. We want to show that f satisfies the following property, known as **Riemann criterion**:

(P)$\forall \epsilon > 0$, there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

(a) Let $\epsilon > 0$ be given. Show that there is a partition P of $[a,b]$ such that

$$\underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} < L(f,P).$$

Hint: Assume the contrary and get a contradiction.
(b) Show that there is a partition Q of $[a,b]$ such that

$$U(f,Q) < \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}.$$

(c) Let $R = P \cup Q$. Use Exercise 6.1.2 to show that

$$\int_a^b f(x)dx - \frac{\epsilon}{2} < L(f,R) \leq U(f,R) < \int_a^b f(x)dx + \frac{\epsilon}{2}.$$

(d) Show that

$$\left| L(f,R) - \int_a^b f(x)dx \right| < \tfrac{\epsilon}{2} \text{ and } \left| U(f,R) - \int_a^b f(x)dx \right| < \tfrac{\epsilon}{2}.$$

(e) Use the triangle inequality to show that $U(f,R) - L(f,R) < \epsilon$.

Exercise 6.1.7
Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f satisfies property (P) above.
(a) Show that for each positive integer n, there is a partition P_n such that

$$U(f,P_n) - L(f,P_n) < \frac{1}{n}.$$

(b) Using (a), show that

$$L(f,P_n) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx < L(f,P_n) + \frac{1}{n}.$$

(c) Show that

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \frac{1}{n}.$$

(d) Show that

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Hint: Squeeze rule. We conclude that any bounded function that satisfies property (P) is Riemann integrable.

Exercise 6.1.8
Let $f : [0, 1] \to \mathbb{R}$ be the function $f(x) = x^2$. For any $\epsilon > 0$, choose a partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ such that

$$x_i - x_{i-1} < \tfrac{\epsilon}{2} \text{ for all } 1 \leq i \leq n.$$

Show that $U(f, P) - L(f, P) < \epsilon$. Hence, f is Riemann integrable.

Practice Problems

Exercise 6.1.9
Suppose that $f(x) = x$ for $x \in [1, 2]$.
(a) Find $U(f, P)$ and $L(f, P)$. Hint: Consider a partition with equal subintervals.
(b) Show that f is Riemann integrable. Hint: Exercise 6.1.7.
(c) Show that $U(f, P) \geq \frac{3}{2}$ and $L(f, P) \leq \frac{3}{2}$.
(d) Find $\int_1^2 x dx$.

Exercise 6.1.10
Let $f : [a, b] \to \mathbb{R}$ be bounded. Let P and Q be any two partitions of $[a, b]$. Prove that $L(f, P) \leq U(f, Q)$.

Exercise 6.1.11
Let $f : [a, b] \to \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Prove that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq (M - m)(b - a).$$

Exercise 6.1.12
Let $f : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove the following:
(a) $\overline{\int_a^b} f(x)dx \leq \overline{\int_a^b} g(x)dx$;
(b) $\underline{\int_a^b} f(x)dx \leq \underline{\int_a^b} g(x)dx$.

Exercise 6.1.13
Let $f : [a, b] \to \mathbb{R}$ be bounded functions. Let P be any partition of $[a, b]$. Prove

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Exercise 6.1.14
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Prove that there is a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \int_a^b f(x)dx$.

Exercise 6.1.15
Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = ax+b$ where $a > 0$ and $b > 0$. Assume that this function is Riemann integrable. For each positive

integer n consider the partition $P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ with equal subintervals.
(a) Compute $L(f, P_n)$ and $U(f, P_n)$.
(b) Show that $\int_0^1 f(x)dx = \frac{a}{2} + b$.

6.2 Classes of Riemann Integrable Functions

In this section we discuss some families of Riemann integrable functions, namely, monotone and continuous functions.

Exercise 6.2.1
Let $f : [a, b] \to \mathbb{R}$ be an increasing function on $[a, b]$.
(a) Show that f is bounded on $[a, b]$.
(b) Let $\epsilon > 0$ be given. Choose a positive integer N such that $\frac{f(b)-f(a)}{N} < \epsilon$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \frac{1}{N}$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, express $M_i(f)$ and $m_i(f)$ in terms of $f(x)$.
(c) Show that $U(f, P) - L(f, P) < \epsilon$. Thus, conclude that f is Riemann integrable.

Exercise 6.2.2
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$.
(a) Show that there exist numbers m and M such that $m \leq f(x) \leq M$ for all $a \leq x \leq b$. That is, f is bounded on $[a, b]$.
(b) Show that f is uniformly continuous on $[a, b]$.
(c) Let $\epsilon > 0$. Show that there is a positive number $\delta > 0$ such that if $|u - v| < \delta$ then $|f(u) - f(v)| < \frac{\epsilon}{b-a}$.
(d) Choose a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that $x_i - x_{i-1} < \delta$ for all $1 \leq i \leq n$. Show that for each interval $[x_i, x_{i-1}]$ there exist $s_i, t_i \in [x_i, x_{i-1}]$ such that $M_i(f) = f(t_i)$ and $m_i(f) = f(s_i)$. Hint: Exercise 5.2.4.
(e) Show that $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$ for each $1 \leq i \leq n$.
(f) Using (e), show that $U(f, P) - L(f, P) < \epsilon$. Hence, conclude that f is Riemann integrable.

We have seen that a continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ (Exercise 6.2.2.) We have also seen that a function $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ except at one single point is still Riemann integrable (Exercise 6.1.4.) This results extends to a function with a finite number of discontinuity. That is, a bounded function $f : [a, b] \to \mathbb{R}$ that is continuous except at the points $c_1, c_2, \cdots, c_n \in [a, b]$ is Riemann integrable. In the next two problems we will establish such a result.
Let $f : [a, b] \to \mathbb{R}$ be bounded with $|f(x)| \leq M$ for all $x \in [a, b]$.

Exercise 6.2.3
Suppose f is continuous except at a point c in $[a, b]$. Let $\epsilon > 0$ be given and consider a partition $Q = \{a = x_0 < x_1 < \cdots < x_{k-1} < c < x_{k+1} < \cdots < x_n = b\}$ such that $\mu(Q) < \frac{\epsilon}{12M}$.
(a) Prove that $|x_{k-1} - xk + 1| < \frac{\epsilon}{6M}$.
(b) Show that there exist $\delta' > 0$ and $\delta'' > 0$ such that for all $x, y \in [a, x_{k-1}]$ with $|x - y| < \delta'$ we have $|f(x) - f(y)| < \frac{\epsilon}{3(b-a)}$ and for all $x, y \in [x_{k+1}, b]$ with $|x - y| < \delta''$ we have $|f(x) - f(y)| < \frac{\epsilon}{3(b-a)}$.
(c) Let P_1 be a refinement of Q on $[a, x_{k-1}]$ such that $\mu(P_1) < \delta'$ and P_2 be a refinement of P on $[x_{k+1}, b]$ such that $\mu(P_2) < \delta''$. Let $P = P_1 \cup P_2$. Then we have

$$U(f, P) - L(f, P) = \sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) + (M_k - m_k)(c - x_{k-1})$$
$$+(M_{k+1} - m_{k+1}(x_{k+1} - c) + \sum_{i=k+2}^{n}(M_i - m_i)(x_i - x_{i-1}).$$

Show that
$$\sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3}$$
$$(M_k - m_k)(c - x_{k-1}) + (M_{k+1} - m_{k+1}(x_{k+1} - c) < \frac{\epsilon}{3}$$
and
$$\sum_{i=k+2}^{n}(M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3}.$$
(d) Conclude that $U(f, P) - L(f, P) < \epsilon$ and therefore f is Riemann integrable.

Exercise 6.2.4
Suppose f is continuous except at points $c_1, c_2, \cdots, c_n$ in $[a, b]$. We want to show that f is Riemann integrable on $[a, b]$. The proof is by induction on n. For $n = 1$ the result holds by the previous exercise. Suppose that the result holds for $c_1, c_2, \cdots, c_n$. Suppose that f is continuous except at $c_1 < c_2 < \cdots < c_n < c_{n+1}$. Let $\epsilon > 0$. Choose $\delta > 0$ small enough so that $\delta < \frac{\epsilon}{8M}$ and $(c_{n+1} - \delta, c_{n+1} + \delta) \subset [c_n, b]$.
(a) Show that there is a partition P_1 of $[a, c_{n+1} - \delta]$ such that $U(f, P_1) -$

$L(f, P_1) < \frac{\epsilon}{4}$ and a partition P_2 of $[c_{n+1}, b]$ such that $U(f, P_2) - L(f, P_2) < \frac{\epsilon}{4}$.
(b) Let $P = P_1 \cup P_2$. Show that $U(f, P) - L(f, P) < \epsilon$. Hence, f is Riemann integrable on $[a, b]$.

Practice Problems

Exercise 6.2.5
Let $f : [a,b] \to \mathbb{R}$ be a increasing function on $[a,b]$.
(a) Show that f is bounded on $[a,b]$.
(b) Let $\epsilon > 0$ be given. Choose a positive integer N such that $\frac{f(a)-f(b)}{N} < \epsilon$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ such that $x_i - x_{i-1} < \frac{1}{N}$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, express $M_i(f)$ and $m_i(f)$ in terms of $f(x)$.
(c) Show that $U(f,P) - L(f,P) < \epsilon$. Thus, conclude that f is Riemann integrable.

Exercise 6.2.6
Suppose $f : [a,b] \to \mathbb{R}$ is continuous and $f \geq 0$ on $[a,b]$. Let $[c,d] \subset [a,b]$. Prove that $\int_a^b f(x)dx \geq \int_c^d f(x)dx$.

Exercise 6.2.7
(a) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $f \geq 0$ on $[0,1]$. Let $a \in [0,1]$ be such that $f(a) > 0$. Show that $\int_0^1 f(x)dx > 0$.
(b) Construct a nonnegative function f on $[0,1]$ such that $f(0.5) > 0$ but $\int_0^1 f(x)dx = 0$.

Exercise 6.2.8
Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$. Prove that f is Riemann integrable on $[a,b]$.

Exercise 6.2.9
Let $f : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Prove that f is not Riemann integrable on $[a,b]$. Hint: Show that the lower Riemann integral is different from the upper Riemann integral.
(b) Prove that $|f|$ is Riemann integrable.

Exercise 6.2.10
Suppose f is a continuous function on $[a,b]$ and that $f(x) \geq 0$ for all $x \in [a,b]$. Show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$. Hint: Assume the contrary and get a contradiction.

6.3 Riemann Sums

The Riemann sum approach is another common method for defining Riemann integrals.

Definition 6.3.1
Let $f : [a, b] \to \mathbb{R}$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each $1 \leq i \leq n$, let $t_i \in [x_{i-1}, x_i]$. The sum

$$S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

is called a **Riemann sum** for f.

Remark 6.3.1
(a) Since $m_i(f) \leq f(t_i) \leq M_i(f)$ for all $1 \leq i \leq n$, one can easily see that $L(f, P) \leq S(f, P) \leq U(f, P)$.
(b) Note also that in the definition, the function f need not be bounded.
(c) Note that $S(f, P)$ depends on the choice of the $t_i's$.
(d) If the function f is positive on $[a, b]$, a Riemann Sum geometrically corresponds to a summation of areas of rectangles with length $x_i - x_{i-1}$ and height $f(t_i)$.
(e) Riemann sums have the practical disadvantage that we do not know which point to take inside each subinterval. To remedy that one could agree to always take the left endpoint (resulting in what is called the **left Riemann sum**) or always the right one (resulting in the **right Riemann sum**). Another two are the **upper Riemann sum** and the **lower Riemann sum** as defined before.

Definition 6.3.2
Let $f : [a, b] \to \mathbb{R}$. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define the **norm** of P to be the length of the largest interval in the partition, that is,

$$\mu(P) = \max_{1 \leq i \leq n}(x_i - x_{i-1}).$$

Exercise 6.3.1
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that $\lim_{\mu(P) \to 0} S(f, P) = A$.

(a) Let $\epsilon > 0$. Show that there is a $\delta > 0$ such that for any partition P of $[a,b]$ such that $\mu(P) < \delta$ we must have $|S(f,P) - A| < \frac{\epsilon}{4}$.
(b) Let $Q = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ such that $\mu(Q) < \delta$, that is, $x_i - x_{i-1} < \delta$ for all $1 \le i \le n$. Fix $1 \le i \le n$. Show that if $f(u_i) \ge m_i(f) + \frac{\epsilon}{4(b-a)}$ for all $x_{i-1} \le u_i \le x_i$ then this contradicts the definition of $m_i(f)$.
(c) With Q as above, show that if $f(v_i) \le M_i(f) - \frac{\epsilon}{4(b-a)}$ for all $x_{i-1} \le v_i \le x_i$ then this contradicts the definition of $M_i(f)$.
(d) Show that for every $1 \le i \le n$, there exists $u_i, v_i \in [x_{i-1}, x_i]$ such that $f(u_i) < m_i(f) + \frac{\epsilon}{4(b-a)}$ and $f(v_i) > M_i(f) - \frac{\epsilon}{4(b-a)}$.
(e) Show that $\sum_{i=1}^n f(u_i)(x_i - x_{i-1}) < L(f,Q) + \frac{\epsilon}{4}$ and $\sum_{i=1}^n f(v_i)(x_i - x_{i-1}) > U(f,Q) - \frac{\epsilon}{4}$.
(f) Show that

$$\begin{gathered} A - \tfrac{\epsilon}{4} < \textstyle\sum_{i=1}^n f(u_i)(x_i - x_{i-1}) < A + \tfrac{\epsilon}{4} \text{ and} \\ A - \tfrac{\epsilon}{4} < \textstyle\sum_{i=1}^n f(v_i)(x_i - x_{i-1}) < A + \tfrac{\epsilon}{4}. \end{gathered}$$

(g) Use (f) to show that

$$A - \frac{\epsilon}{2} < L(f,Q) \le U(f,Q) < A + \frac{\epsilon}{2}.$$

(h) Show that $U(f,Q) - L(f,Q) < \epsilon$. That is, f is Riemann integrable.
(i) Show that

$$\left| \int_a^b f(x)dx - A \right| < \epsilon.$$

(k) Use the Squeeze rule to show that $\int_a^b f(x)dx = A$.
Conclusion: Suppose that there is a number A such that $\lim_{\mu(P) \to 0} S(f,P) = A$. Then f is Riemann integrable with $\int_a^b f(x)dx = A$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded Riemann integrable function with $|f(x)| \le M$ for all $x \in [a,b]$. Let $\int_a^b f(x)dx = A$. The goal of the next four problems is to show that

$$\lim_{\mu(P) \to 0} S(f,P) = A.$$

Let $\epsilon > 0$. If $U(f,P) \ge \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}$ for all partitions of $[a,b]$, then $\overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}$ is a lower bound of S_U. But $\overline{\int_a^b} f(x)dx$ is the largest lower bound of S_U. Thus, we must have $\overline{\int_a^b} f(x)dx + \frac{\epsilon}{2} < \overline{\int_a^b} f(x)dx$, a contradiction.

Hence, there is a partition P_1 such that $U(f,P_1) < \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2} = A + \frac{\epsilon}{2}$. Similarly, there is partition P_2 such that $L(f,P_2) > \underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} = A - \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2 = \{a = x_0 < x_1 < \cdots < x_n = b\}$. From Exercise 6.1.2 we have $U(f,P) < U(f,P_1) < A + \frac{\epsilon}{2}$ and $L(f,P) > L(f,P_2) > A - \frac{\epsilon}{2}$.
Let $\delta = \frac{\epsilon}{4Mn}$ and $Q = \{a = z_0 < z_1 < \cdots < z_m = b\}$ be a partition of $[a,b]$ such that $\mu(Q) < \delta$. Consider the partition $R = P \cup Q$.

Exercise 6.3.2
Prove that $A - L(f,R) < \frac{\epsilon}{2}$ and $U(f,R) - A < \frac{\epsilon}{2}$.

Because R is a refinment of Q, for each $i = 1, 2, \cdots, m$ we let R_i denote the partition of $[z_{i-1}, z_i]$ induced by R. Clearly, we have

$$L(f,R) - L(f,Q) = \sum_{i=1}^{m} [L(f,R_i) - m_i(z_i - z_{i-1})]$$

and

$$U(f,Q) - U(f,R) = \sum_{i=1}^{m} [M_i(z_i - z_{i-1}) - U(f,R_i)].$$

Because P has at most $n-1$ partition points that are not partition points of Q, there are at most $n-1$ subintervals $[z_{i-1}, z_i]$of Q such that (z_{i-1}, z_i) contains at least one point from P. For the remaining subintervals the terms in the above sums are zero.

Exercise 6.3.3
(a) For $1 \le i \le m$ such that $L(f,R_i) - m_i(z_i - z_{i-1}) \ne 0$ and $M_i(z_i - z_{i-1}) - U(f,R_i)$ prove that

$$L(f,R_i) - m_i(z_i - z_{i-1}) < 2M\delta \text{ and } M_i(z_i - z_{i-1}) - U(f,R_i) < 2M\delta.$$

(b) Use (a) and the sums above to show that

$$L(f,R) - L(f,Q) < \frac{\epsilon}{2} \text{ and } U(f,Q) - U(f,R) < \frac{\epsilon}{2}.$$

Exercise 6.3.4
Use Exercise 6.3.2 and 6.3.3 to prove that

$$U(f,Q) < A + \epsilon \text{ and } L(f,Q) > A - \epsilon.$$

Exercise 6.3.5
Using the previous problem, show that

$$A - \epsilon < S(f, Q) < A + \epsilon.$$

That is,

$$|S(f, Q) - A| < \epsilon.$$

Exercise 6.3.6
Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable. The goal of this problem is to show that for any sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ such that $\lim_{n\to\infty} \mu(P_n) = 0$ we have $\lim_{n\to\infty} S(f, P_n) = \int_a^b f(x)dx$.
(a) Let $\epsilon > 0$. Show that there is a $\delta > 0$ such that if P is a partition of $[a, b]$ with $\mu(P) < \delta$ we have

$$\left| S(f, P) - \int_a^b f(x)dx \right| < \epsilon.$$

(b) Show that there is a positive integer N such that if $n \geq N$ then $\mu(P_n) < \delta$.
(c) Use (a) and (b) to conclude that for $n \geq N$ we have

$$\left| S(f, P_n) - \int_a^b f(x)dx \right| < \epsilon.$$

Hence,

$$\lim_{n\to\infty} S(f, P_n) = \int_a^b f(x)dx.$$

Practice Problems

Exercise 6.3.7
Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Let $\epsilon > 0$ be given. Show that there is a $\delta > 0$ such that for any partition P of $[a, b]$ with $\mu(P) < \delta$ we have

$$U(f, P) - L(f, P) < \epsilon.$$

Exercise 6.3.8
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ and that $f' : [a, b] \to \mathbb{R}$ is Riemann integrable. Let $P_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $x_i - x_{i-1} = \frac{b-a}{n}$.
(a) For each $1 \le i \le n$, show that there exists $x_{i-1} < t_i < x_i$ such that $f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1})$.
(b) Show that $S(f', P_n) = \sum_{i=1}^n f'(t_i)(x_i - x_{i-1}) = f(b) - f(a)$.
(c) Show that $\lim_{n\to\infty} \mu(P_n) = 0$.
(d) Show that $\lim_{n\to\infty} S(f', P_n) = \int_a^b f'(x)dx$.
(e) Show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Exercise 6.3.9 *(Fundamental Theorem of Calculus)*
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and let $F : [a, b] \to \mathbb{R}$ be a differentiable function such that $F'(x) = f(x)$ for all $a \le x \le b$. Show that

$$\int_a^b f(x)dx = F(b) - F(a).$$

The function $F(x)$ is called an **antiderivative** of f.

6.4 The Algebra of Riemann Integrals

In this section we discuss the various properties of Riemann integrals.

Exercise 6.4.1
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions and α, β be real numbers. Let $\epsilon > 0$.
(a) Show that there is a $\delta_1 > 0$ such that if $\mu(P) < \delta_1$ then $\left|S(f, P) - \int_a^b f(x)dx\right| < \frac{\epsilon}{|\alpha|+|\beta|}$.
(b) Show that there is a $\delta_2 > 0$ such that if $\mu(P) < \delta_2$ then $\left|S(g, P) - \int_a^b g(x)dx\right| < \frac{\epsilon}{|\alpha|+|\beta|}$.
(c) Show that there is a $\delta > 0$ such that if $\mu(P) < \delta$ then

$$\left|S(\alpha f + \beta g, P) - \left[\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx\right]\right| < \epsilon.$$

We conclude that $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Exercise 6.4.2
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that $f(x) \le g(x)$ for all $x \in [a, b]$.
(a) Show that for any partition P of $[a, b]$ we have $L(f, P) \le L(g, P)$.
(b) Show that $\underline{\int_a^b} f(x)dx \le \underline{\int_a^b} g(x)dx$.
(c) Show that $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

Exercise 6.4.3
Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $m \le f(x) \le M$ for all $x \in [a, b]$.
(a) Show that $m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$ for any partition P of $[a, b]$.
(b) Show that $\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \le M(b - a)$.
(c) Show that $m(b - a) \le \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$.
Conclusion: $m(b - a) \le \int_a^b f(x)dx \le M(b - a)$.

Exercise 6.4.4
Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and $a < c < b$.
(a) Let $\epsilon > 0$. Show that there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.
(b) Let $Q = P \cup \{c\}$, $Q_1 = Q \cap [a, c]$, and $Q_2 = Q \cap [c, b]$. That is, Q is partition of $[a, b]$, Q_1 is a partition of $[a, c]$, and Q_2 is a partition of $[c, b]$. Show that

$$[U(f, Q_1) - L(f, Q_1)] + [U(f, Q_2) - L(f, Q_2)] < \epsilon.$$

(c) Show that $U(f, Q_1) - L(f, Q_1) < \epsilon$. Thus, by Exercise 6.2.1, $\int_a^c f(x)dx$ exists and is finite.
(d) Show that $U(f, Q_2) - L(f, Q_2) < \epsilon$. Thus, by Exercise 6.2.1, $\int_c^b f(x)dx$ exists and is finite.

Exercise 6.4.5
Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and $a < c < b$. Let $\epsilon > 0$.
(a) Show that there is a $\delta_1 > 0$ such that if P_1 is a partition of $[a, c]$ such that $\mu(P_1) < \delta_1$ then $\left|S(f, P_1) - \int_a^c f(x)dx\right| < \frac{\epsilon}{2}$.
(b) Show that there is a $\delta_2 > 0$ such that if P_2 is a partition of $[c, b]$ such that $\mu(P_2) < \delta_2$ then $\left|S(f, P_2) - \int_c^b f(x)dx\right| < \frac{\epsilon}{2}$.
(c) Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$. Show that there is $\delta > 0$ such that $\mu(P) < \delta$ and

$$\left|S(f, P) - \left[\int_a^c f(x)dx + \int_c^b f(x)dx\right]\right| < \epsilon.$$

That is,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Practice Problems

Exercise 6.4.6
Let $f : [a, b] \to \mathbb{R}$ be continuous. Use the Intermediate Value Theorem to prove the existence of a number $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

The number $f(c)$ is called the **average value** of f on $[a, b]$.

Exercise 6.4.7
Suppose that f and g are continuous function on $[a, b]$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove there is a $c \in [a, b]$ such that $f(c) = g(c)$.

Exercise 6.4.8
(a) For any set S, one can see that $M(f, S) - m(f, S) = \sup_{s,t \in S} |f(s) - f(t)|$. Let f be a function defined on a set S. Show that $M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)$.
(b) Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that $|f|$ is also Riemann integrable.

Exercise 6.4.9
Let $f : [a, b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Compute $\underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx$.
(b) Is f Riemann integrable?
(c) Show that $|f|$ is Riemann integrable.

Exercise 6.4.10
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable with $|f(x)| \leq M$ for all $x \in [a, b]$.
(a) Prove that $|f^2(x) - f^2(y)| \leq 2M|f(x) - f(y)|$ for all $x, y \in [a, b]$ where $f^2(x) = (f(x))^2$.
(b) Let $\epsilon > 0$. Show that there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \frac{\epsilon}{2M}.$$

(c) Prove that $U(f^2, P) - L(f^2, P) < \epsilon$. That is, f^2 is Riemann integrable.

Exercise 6.4.11
Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions.
(a) Show that

$$f \cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

(b) Prove that $f \cdot g$ is Riemann integrable.

6.5 Composition of Riemann Integrable Functions and its Applications

We have seen that the composition of two continuous functions is continuous and the composition of two differentiable functions is differentiable. This property does not hold in general for Riemann integrable functions. That is, the composition of two integrable functions is not necessarily integrable (See Exercises 6.5.5 - 6.5.7).
So under what conditions the composition of two functions is Riemann integrable?

Exercise 6.5.1
Suppose that $f : [a, b] \to [c, d]$ is a Riemann integrable function on $[a, b]$ and that $g : [c, d] \to \mathbb{R}$ is continuous (and hence integrable by Exercise 6.2.2).
(a) Show that the set $\{|g(x)| : x \in [c, d]\}$ is bounded. Hence, by the Completeness Axiom of $\mathbb{R}$ there exists $K > 0$ such that $K = \sup\{|g(x)| : x \in [c, d]\}$.
(b) Let $\epsilon > 0$. Chosse ϵ' so that $\epsilon' < \frac{\epsilon}{b-a+2K}$. Show that there is a $\delta < \epsilon'$ such that if $|s - t| < \delta$, where $s, t \in [c, d]$, then $|g(s) - g(t)| < \epsilon'$.
(c) Show that there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \delta^2$.
(d) Let $A = \{1 \le i \le n : M_i(f) - m_i(f) < \delta\}$. Show that if $i \in A$ then $M_i(g \circ f) - m_i(g \circ f)| < \epsilon'$.
(e) Let $B = \{1 \le i \le n : M_i(f) - m_i(f) \ge \delta\}$. Show that $\delta \sum_{i \in B}(x_i - x_{i-1}) < \delta^2$ and hence $\sum_{i \in B}(x_i - x_{i-1}) < \delta$.
(f) Show that for all $1 \le i \le n$ we have $M_i(g \circ f) - m_i(g \circ f) < 2K$. Hint: Use Exercise 4.5.8 and the triangle inequality.
(g) Use (d) (e) and (f) to show that $U(g \circ f, P) - L(g \circ f, P) < \epsilon$. Hence, by Exercise 6.1.7, $g \circ f : [a, b] \to \mathbb{R}$ is Riemann integrable.

We next discuss few applications of composition.

Exercise 6.5.2
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable and bounded such that $|f(x)| \le M_1$ and $|g(x)| \le M_2$ for all $x \in [a, b]$.
(a) Find a positive constant M such that $|f(x)| \le M$ and $|g(x)| \le M$. Thus, $f([a, b]) \subseteq [-M, M]$ and $g([a, b]) \subseteq [-M, M]$
(b) Consider the continuous function $h : [-2M, 2M] \to \mathbb{R}$ given by $h(x) = x^2$.

Show that $(f+g)^2$ and $(f-g)^2$ are Riemann integrable on $[a,b]$. Hint: Note that $h \circ (f+g) = (f+g)^2$ and $h \circ (f-g) = (f-g)^2$.
(c) Show that $f \cdot g$ is Riemann integrable on $[a,b]$.

Exercise 6.5.3
Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and bounded such that $|f(x)| \leq M$ for all $x \in [a,b]$.
(a) Consider the continuous function $g : [-M,M] \to \mathbb{R}$ defined by $g(x) = |x|$. Show that $|f|$ is Riemann integrable on $[a,b]$.
(b) Using the fact that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a,b]$, show that

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

Hence, show that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Exercise 6.5.4 *(Integration by Parts)*
Let $f, g : [a,b] \to \mathbb{R}$ be continuous and $f', g' : [a,b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that f and g are Riemann integrable on $[a,b]$.
(b) Show that $f' \cdot g$ and $f \cdot g'$ are Riemann integrable on $[a,b]$.
(c) Show that $\int_a^b f'g dx + \int_a^b fg'dx = (fg)(b) - (fg)(a)$. Hint: Use product rule and Exercise 6.3.8.

Practice Problems

Exercise 6.5.5
Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is Riemann integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)dx$?

Exercise 6.5.6
Consider the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = 1 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational with } p \text{ and } q > 0 \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Let $\epsilon > 0$ and $\epsilon' = \min\{0.5, \epsilon\}$. Thus, $0 < \epsilon' \le 0.5$ and $0 < \epsilon' \le \epsilon$. Show that there is a finite number of rationals in $[0, 1]$ such that $g(x) \ge \frac{\epsilon'}{2}$. Denote the rationals by $\{r_0, r_1, \cdots, r_n\}$ where $r_0 = 0$ and $r_n = 1$.
(b) Define the partition $Q = \{0 = x_0 < x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1\}$ where $x_0 = 0; x_1 < r_1$ with $x_1 < \frac{\epsilon'}{2(n+1)}; x_1 < x_2 < r_1 < x_3$ with $x_3 - x_2 < \frac{\epsilon'}{2(n+1)}; \cdots ; x_{2n-2} < r_{n-1} < x_{2n-1}$ with $x_{2n-1} - x_{2n-2} < \frac{\epsilon'}{2(n+1)}; x_{2n-1} < x_{2n} < 1$ with $1 - x_{2n} < \frac{\epsilon'}{2(n+1)}$ and $x_{2n+1} = 1$. Show that $U(g, Q) < \epsilon'$. Hint: Note that the sum involves intervals containing $r_i's$ and those that do not.
(c) Show that $L(g, Q) = 0$. Hint: Exercise 1.2.6.
(d) Using (b) and (c) show that $U(g, Q) - L(g, Q) < \epsilon$. Thus, g is Riemann integrable.
(e) What is the value of the integral $\int_0^1 g(x)dx$?

Exercise 6.5.7
Consider the functions f and g introduced in the previous two exercises. Let $h(x) = (f \circ g)(x)$.
(a) Write explicitly the formula of $h(x)$ as a piecewise defined function.
(b) Show that h is not Riemann integrable on $[0, 1]$.

Exercise 6.5.8
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that $\max\{f(x), g(x)\} = \frac{|f(x)-g(x)|+f(x)+g(x)}{2}$.
(b) Show that the function $\max\{f(x), g(x)\}$ is Riemann integrable.

Exercise 6.5.9
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that $\min\{f(x), g(x)\} = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$.
(b) Show that the function $\min\{f(x), g(x)\}$ is Riemann integrable.

6.6 The Derivative of an Integral

In this section we introduce functions that are represented by integrals.

Definition 6.6.1
Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable on $[a, b]$. We define the function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

We also define

$$\textstyle\int_c^c f(x)dx = 0 \text{ and } \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Exercise 6.6.1
Let f and F as defined in Definition 6.6.1. Let M be such that $|f(x)| \le M$ for all $x \in [a, b]$. Fix c in $[a, b]$.
(a) Show that for any $x \in [a, b]$ we have

$$-M(x - c) \le \int_c^x f(t)dt \le M(x - c).$$

Hence, we can write

$$\left|\int_c^x f(t)dt\right| \le M|x - c|.$$

Hint: Exercise 6.4.2.
(b) Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Show that for any $x \in [a, b]$ such that $|x - c| < \delta$ we must have $|F(x) - F(c)| < \epsilon$. Hence, F is continuous at c. Since c was arbitrary in $[a, b]$, we conclude that F is continuous on $[a, b]$.

Exercise 6.6.2
Let f and F as above. Suppose furthermore that f is continuous at $c \in [a, b]$.
(a) Show that

$$\frac{F(c + h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt.$$

(b) Show that $F'(c)$ exists and is equal to $f(c)$.

Exercise 6.6.3
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' continuous on $[a, b]$.
(a) Show that f' is Riemann integrable on $[a, b]$.
(b) Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f'(t)dt$. Show that $F'(x) = f'(x)$ for all $x \in [a, b]$.
(c) Show that $F(x) = f(x) - f(a)$ for all $x \in [a, b]$.
(d) Use (c) to show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Exercise 6.6.4
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $g : [c, d] \to [a, b]$ is differentiable on $[a, b]$. Define $F : [c, d] \to \mathbb{R}$ by

$$F(x) = \int_a^{g(x)} f(t)dt.$$

(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $G : [a, b] \to \mathbb{R}$ by $G(x) = \int_a^x f(t)dt$. Show that G is differentiable and $G'(x) = f(x)$ for all $x \in [a, b]$.
(c) Write F in terms of G and g. Show that F is differentiable on $[c, d]$ with

$$F'(x) = f(g(x)) \cdot g'(x).$$

Exercise 6.6.5 *(Mean Value Theorem for Integrals)*
Let $f : [a, b] \to \mathbb{R}$ be continuous.
(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

Show that F is differentiable with $F'(x) = f(x)$.
(c) Show that there is $a < c < b$ such that $F(b) - F(a) = F'(c)(b - a)$.
(d) Use (c) to show that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Exercise 6.6.6 *(Change of Variables Formula)*
Let $\phi : [a,b] \to [c,d]$ be differentiable with continuous derivative and such that $\phi(a) = c, \phi(b) = d$. Let $f : [c,d] \to \mathbb{R}$ be continuous.
(a) Show that the functions f and $(f \circ \phi) \cdot \phi'$ are Riemann integrable.
(b) Define $F(x) = \int_c^x f(t)dt$. Show that F is differentiable with $F'(x) = f(x)$ for all $x \in [c,d]$.
(c) Define $G(x) = \int_a^x f(\phi(t))\phi'(t)dt$. Show that G is differentiable with $G'(x) = f(\phi(x))\phi'(x)$ for all $x \in [a,b]$.
(d) Show that $F \circ \phi$ is differentiable on $[a,b]$ with $(F \circ \phi)'(x) = G'(x)$ for all $x \in [a,b]$. Hint:Exercise 5.3.1.
(e) Use (d) and Exercise 5.3.9 to show that $(F \circ \phi)(x) = G(x)$ for all $x \in [a,b]$.
(f) Use (e) to show that

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_c^d f(x)dx.$$

Practice Problems

Exercise 6.6.7
Find the derivative of
$$F(x) = \int_1^{\sqrt{x}} \cos(t^2)dt.$$

Exercise 6.6.8 *(Mean Value Theorem for Monotone Functions)*
Let $f : [a, b] \to \mathbb{R}$ be increasing on $[a, b]$.
(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(a)(x - a) + f(b)(b - x)$. Show that g is continuous on $[a, b]$.
(c) Show that $g(b) \leq \int_a^b f(x)dx \leq g(a)$.
(d) Show that there is $c \in [a, b]$ such that
$$\int_a^b f(x)dx = f(a)(c - a) + f(b)(c - b).$$

Exercise 6.6.9
Use change of variables to evaluate $\int_1^3 (3x + 1)^{100} dx$.

Exercise 6.6.10
Find the smallest positive critical point of
$$F(x) = \int_0^x \cos(t^{\frac{3}{2}})dt.$$

Exercise 6.6.11
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$. Find
$$\lim_{x \to a} \frac{1}{x - a} \int_a^x f(t)dt.$$

Exercise 6.6.12
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A, B : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define $g : \mathbb{R} \to \mathbb{R}$ by
$$g(x) = \int_{A(x)}^{B(x)} f(t)dt.$$
Prove that g is differentiable and find a formula for $g'(x)$.

Exercise 6.6.13
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 2 and $f(2) = 4$. Find

$$\lim_{x \to 2} \frac{1}{x-2} \int_2^x x f(t) dt.$$

Exercise 6.6.14
Use a definite integral to define a function $F(x)$ having derivative $\frac{\cos 2x^3}{\sqrt{1+x^4}}$ for all x and satisfying $F(\sqrt[3]{2}) = 0$.

Chapter 7

Series

7.1 Series and Convergence

In this section we introduce the general definition of a series and study its convergence.

Definition 7.1.1
Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence. The sum of the term of the sequence is called a **series**, denoted by

$$\Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

To determine whether this series converges or not we consider the sequence of **partial sums** defined as follows:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n \\ &\vdots \end{aligned}$$

We say that a series $\Sigma_{n=1}^{\infty} a_n$ **converges** to a number L if and only if the sequence $\{S_n\}_{n=1}^{\infty}$ converges to L and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

A series which is not convergent is said to **diverge.**

Exercise 7.1.1
Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. Hint: Show that for each $n \geq 1$ we have $S_n = 1 - \frac{1}{n+1}$.

Exercise 7.1.2
Is the series $\Sigma_{n=1}^{\infty}(-1)^n$ convergent or divergent?

The following result provides a procedure for testing the divergence of a series. This is known as the the n^{th} **term** test for convergence.

Exercise 7.1.3
Suppose that $\sum_{i=1}^{\infty} a_n = L$. Show that $\lim_{n\to\infty} a_n = 0$. Hint: Note that $S_{n+1} - S_n = a_n$.

The test states that if we know the series is convergent then $\lim_{n\to\infty} a_n = 0$. The converse is not true in general. That is, the condition $\lim_{n\to\infty} a_n = 0$ does not necessarily imply that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Exercise 7.1.4
Consider the series $\sum_{i=1}^{n} \log\left(\frac{n+1}{n}\right)$.
(a) Show that $\lim_{n\to\infty} a_n = 0$.
(b) Show that $\lim_{n\to\infty} S_n = \infty$. Hence, the series is divergent.

Exercise 7.1.5
Consider the sequence $\{r^n\}_{n=1}^{\infty}$.
(a) Show that if $r = -1$ the sequence is divergent.
(b) Show that if $|r| > 1$, i.e. $r < -1$ or $r > 1$, the sequence is divergent.
(c) Show that if $|r| < 1$, the sequence is convergent.

Exercise 7.1.6
The series $\sum_{n=1}^{\infty} ar^{n-1}$ is called a **geometric series** with ratio r.
(a) Show that

$$S_n = a\frac{1-r^{n+1}}{1-r} \text{ for } r \neq 1.$$

Hint: Calculate $S_n - rS_n$.
(b) Show that the series converges to $\frac{a}{1-r}$ for $|r| < 1$ and diverges for $|r| \geq 1$.

Exercise 7.1.7 *(Harmonic Series)*
Consider the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(a) Let $n = 2^m$ where m is a positive integer. Then

$$\begin{aligned} S_n =& 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^m} \\ =& 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &+ \cdots + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right). \end{aligned}$$

Show that $S_n \geq 1 + \frac{m}{2}$.
(b) Use (a) to show that $\lim_{n\to\infty} S_n = \infty$. Thus, the Harmonic series is divergent.

Exercise 7.1.8
Show that if $\sum_{n=1}^{\infty} a_n = L_1$ and $\sum_{n=1}^{\infty} b_n = L_2$ then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha L_1 + \beta L_2$ for all $\alpha, \beta \in \mathbb{R}$.

Practice Problems

Exercise 7.1.9
Find the value of the infinite sum $\sum_{n=1}^{\infty}\left(\frac{3}{4^n}+\frac{5}{n(n+1)}\right)$.

Exercise 7.1.10
Show that the sequence $\{\sqrt{n^2-1}-n\}_{n=1}^{\infty}$ is convergent and find its limit.

Exercise 7.1.11
Let $\sum_{n=1}^{\infty} a_n$ be a **conditionally convergent** series[1]. Define $b_n=\frac{1}{2}(a_n+|a_n|)$ and $c_n=\frac{1}{2}(a_n-|a_n|)$. Prove that the two series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ are divergent.

Exercise 7.1.12
Let S_n be the n-th partial sum of the series $\sum_{n=1}^{\infty}\frac{n-2}{n(n+1)(n+2)}$.
(a) Show that $S_n=\frac{3}{n+1}-\frac{2}{n+1}-\frac{2}{n+2}$. Hint: Partial fractions.
(b) Find the value of the series $\sum_{n=1}^{\infty}\frac{n-2}{n(n+1)(n+2)}$.

Exercise 7.1.13
Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $\sum_{n=1}^{\infty} a_n$ is convergent.
(a) Show that $a_n\geq 0$ for all $n\in\mathbb{N}$.
(b) Let $\epsilon>0$. Show that there is a positive integer N such that if $n>m\geq N$ we have

$$|a_{m+1}+a_{m+2}+\cdots+a_n|<\epsilon.$$

(c) Show that $(n-N)a_n<\epsilon$.
(d) Let $n>2N$. Show that $\frac{n}{2}<n-N$.
(e) Show that $\frac{na_n}{2}<\epsilon$.
(f) Show that $\lim_{n\to\infty} na_n=0$.

Exercise 7.1.14
Let N be a positive integer. Suppose that $a_n=b_n$ for all $n\geq N$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. Thus, adding or deleting a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

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[1] $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty}|a_n|$ is divergent.

7.2 Series with Non-negative Terms

In this section we consider the question of convergence of series with non-negative terms.

Exercise 7.2.1 *(Comparison test)*
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two series such that $0 \le a_n \le b_n$ for all $n \ge 1$. Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ that of $\{b_n\}_{n=1}^{\infty}$.
(a) Show that the sequences $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are increasing.
(b) Show that $S_n \le T_n$ for all $n \ge 1$.
(c) Show that if $\{b_n\}_{n=1}^{\infty}$ is convergent then $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are bounded.
(d) Show that if $\{b_n\}_{n=1}^{\infty}$ is convergent then $\{a_n\}_{n=1}^{\infty}$ is also convergent.
(e) Show that if $\{a_n\}_{n=1}^{\infty}$ is divergent then $\{b_n\}_{n=1}^{\infty}$ is also divergent.

Exercise 7.2.2
(a) Show that for $n \ge 1$ we have $\frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}$.
(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent.

Exercise 7.2.3
Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n+1}}$ is divergent.

The difficulty with the comparison test is that when the nth term of a series $\sum_{n=1}^{\infty} a_n$ is complicated then it might be difficult to figure out the series $\sum_{n=1}^{\infty} b_n$ that need to be compared with. The following comparison test is often easier to apply, because after deciding on $\sum_{n=1}^{\infty} b_n$ we need only take a limit of the quotient $\frac{a_n}{b_n}$ as $n \to \infty$.

Exercise 7.2.4 *(Limit Comparison Test)*
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0.$$

(a) Let $\epsilon = \frac{L}{2}$. Show that there exists a positive integer N such that

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \text{ for all } n \ge N.$$

(b) Use (a) to establish

$$\frac{L}{2}b_n < a_n < \frac{3}{2}Lb_n \text{ for all } n \geq N.$$

(c) Show that $\sum_{n=1}^{\infty} a_n$ is divergent if and only if $\sum_{n=1}^{\infty} b_n$ is divergent.

Exercise 7.2.5
Determine whether the series $\sum_{n=1}^{\infty} \frac{3n+1}{4n^3+n^2-2}$ converges or diverges.

Practice Problems

Exercise 7.2.6
Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of nonnegative terms. Show that if the series $\sum_{n=1}^{\infty} a_n$ is divergent so does the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$. Hint: Comparison test.

Exercise 7.2.7
Use the limit comparison test to show that the series $\sum_{i=1}^{\infty} \frac{1}{2n+\ln n}$ is divergent.

Exercise 7.2.8
Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and that the series $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that $\{a_n\}_{n=1}^{\infty}$ is unbounded. Show that $\lim_{n\to\infty} \frac{a_n}{1+a_n} \neq 0$. Hint: assume the contrary and get a contradction. Conclude that the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is divergent.

Exercise 7.2.9
Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and that the series $\sum_{n=1}^{\infty} a_n$ converges.
(a) Show that there is a positive integer N such that $a_n < 1$ for all $n \geq N$.
(b) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Exercise 7.2.10
Use the comparison test to show that the series $\sum_{n=1}^{\infty}(\sqrt{n^2+1} - n)$ is divergent.

7.3 Alternating Series

By an **alternating series** we mean a series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n > 0$. For instance, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Here $a_n = \frac{1}{n}$. The following result provides a way for testing alternating series for convergence.

Exercise 7.3.1 *(Alternating Series Test)*
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that
(i) $a_n \geq a_{n+1}$, that is the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing.
(ii) $\lim_{n\to\infty} a_n = 0$.
Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. That is, $S_n = \sum_{k=1}^{n}(-1)^{k-1}a_k$.
(a) Show that for each $n \geq 1$ we have $S_{2n} \leq S_{2n+2}$. That is, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is increasing. Hint: Show that $S_{2n+2} - S_{2n} \geq 0$.
(b) Show that the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is decreasing.
(c) Show that for all $n \geq 1$, we have $S_{2n} \leq a_1$. Hence, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is bounded from above. Conclude that the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is convergent, say to L_1.
(d) Show that for all $n \geq 1$, we have $S_{2n+1} \geq (a_1 - a_2)$. Hence, the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is bounded from below. Conclude that the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is convergent, say to L_2.
(e) Show that $L_1 = L_2$. Hint: $S_{2n+1} = S_{2n} + a_{2n+1}$.
(f) Let $L = L_1 = L_2$. Show that $\lim_{n\to\infty} S_n = L$. We conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent. Hint: Look at the sequence $\{c_n\}_{n=1}^{\infty}$ in Exercise 3.3.4.

Exercise 7.3.2
Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 7.3.3
Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}$ is convergent.

It is imporatant to keep in mind that the tests used so far are basically used to test for convergence. However, when a series is convergent these tests do not provide a value for the sum.

Practice Problems

Exercise 7.3.4
Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}$ converges or diverges.

Exercise 7.3.5
Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln{(4n)}}{n}$ converges or diverges.

Exercise 7.3.6
(a) Show that $\frac{n^n}{n!} \geq 1$ for all $n \geq 1$.
(b) Show that ther series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^n}{n!}$ is divergent.

Exercise 7.3.7
Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{n+1}+2^{n+1}}{3^n-n}$ diverges.

7.4 Absolute and Conditional Convergence

In this section we consider types of convergence for series with positive and negative terms.

Definition 7.4.1
Consider a series $\sum_{n=1}^{\infty} a_n$ which has both positive and negative terms. We say that this series is **absolutely convergent** if and only if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Exercise 7.4.1
Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ is absolutely convergent.

The following result provides a test of convergence for series of the above type.

Exercise 7.4.2
Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Define the sequence $\sum_{n=1}^{\infty} b_n$ by $b_n = |a_n|$ and note that $a_n \leq b_n$. Show that the sequence $\sum_{n=1}^{\infty} a_n$ is convergent. That is, absolute convergence implies convergence.

The converse of the above result is not true in general. That is, it is possible to have a series that is convergent but not absolutely convergent.

Exercise 7.4.3
Give an example of a series that is convergent but not absolutely convergent.

Definition 7.4.2
Consider a series $\sum_{n=1}^{\infty} a_n$ which has both positive and negative terms. We say that this series is **conditionally convergent** if and only if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is divergent whereas the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Exercise 7.4.4
Give an example of a series that is conditionally convergent.

Practice Problems

Exercise 7.4.5
Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(a) Show that $0 \leq \frac{|a_n|+a_n}{2} \leq |a_n|$ and $0 \leq \frac{|a_n|-a_n}{2} \leq |a_n|$
(b) Show that the series $\sum_{n=1}^{\infty}\left(\frac{|a_n|+a_n}{2}\right)$ and $\sum_{n=1}^{\infty}\left(\frac{|a_n|-a_n}{2}\right)$ are convergent.

Exercise 7.4.6
(a) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then the series $\sum_{n=1}^{\infty} a_n^2$ is also absolutely convergent.
(b) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} a_n^2$ is divergent.

Exercise 7.4.7
Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\{b_n\}_{n=1}^{\infty}$ is bounded. Show that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent (and thus convergent).

Exercise 7.4.8
Test the following series for absolute convergence, conditional convergence, or divergence.
(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n2^n}$.
(b) $\sum_{n=1}^{\infty}(-1)^n \frac{5n}{n^2+2n}$.
(c) $\sum_{n=1}^{\infty}(-1)^n \frac{2^n-2^{-n}}{2^n+2^{-n}}$.

Exercise 7.4.9
Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln 4n}{n}$ is absolutely convergent.

Exercise 7.4.10
Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence such that $|b_n| \leq a_n - a_{n+1}$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

7.5 The Integral Test for Convergence

The integral test for convergence is a method used to test infinite series of positive terms for convergence.

Exercise 7.5.1 *(Integral Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose that there is a function $f : [1, \infty) \to \mathbb{R}$ such that f is decreasing and positive with $f(n) = a_n$ for all $n \geq 1$.
(a) Show that $\{S_n\}_{n=1}^{\infty}$ is increasing.
(b) Define $F : [1, \infty) \to \mathbb{R}$ by $F(x) = \int_1^x f(t)dt$. Show that F is increasing.
(c) For $n \geq 2$ and $x \in [n-1, n]$, show that $a_n \leq f(x) \leq a_{n-1}$ and $a_n \leq \int_{n-1}^{n} f(x)dx \leq a_{n-1}$.
(d) Show that $S_n - a_1 \leq F(n) \leq S_{n-1}$.
(e) Suppose that $\int_1^{\infty} f(x)dx = L$. Since F is increasing we can write $F(n) \leq L$ for all $n \geq 1$. Show that $\{S_n\}_{n=1}^{\infty}$ is bounded. Hint: Use (d).
(f) Show that $\{S_n\}_{n=1}^{\infty}$ is convergent. Hence, $\sum_{n=1}^{\infty} a_n$ is convergent.
(g) Conversely, suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to a number S. Show that for any positive integer $n \geq 2$ we have

$$F(n) \leq S.$$

(h) Show that for all $R \geq 1$ we have $F(R) \leq S$. Thus, $\int_1^{\infty} f(x)dx = \lim_{R\to\infty} \int_1^R f(x)dx$ is convergent. Hint: For any $R \geq 1$ we have $R \leq [R] + 1$ with $[R] + 1 \geq 2$.

Exercise 7.5.2 *(p-series)*
(a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.
(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent for $p \leq 1$.

Practice Problems

Exercise 7.5.3
Show that the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)(\ln(n^2+1))^a}$ is convergent for all $a > 1$. Hint: The integral test.

Exercise 7.5.4
Use the integral test to test the convergence of the series $\sum_{n=4}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$.

Exercise 7.5.5
Use the Integral Test to show that $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent.

Exercise 7.5.6
Use the integral test to show that the series $\sum_{n=1}^{\infty} e^{-n^2}$ is convergent.

Exercise 7.5.7
Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ is divergent.

7.6 The Ratio Test and the n^{th} Root Test

The integral test is hard to apply when the integrand involves factorials or complicated expressions. In this section we introduce two tests that can be used to help determine convergence or divergence of series when the previously discussed tests are not applicable.

Exercise 7.6.1 *(Ratio Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero terms and suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$.
(a) Suppose $0 \leq L < 1$. Let $\epsilon = \frac{1-L}{2}$. Show that there is a positive integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1+L}{2} \text{ for all } n \geq N.$$

Hint: Use definition of convergence and Exercise 1.1.18.
(b) Let $r = \frac{1+L}{2}$. Show that $0 < r < 1$ and $|a_{N+k}| < r^k|a_N|$ for all $k = 1, 2, \cdots$.
(c) Find the value of the sum

$$\sum_{n=1}^{\infty} r^n |a_N|.$$

(d) Let $b_n = \sum_{k=1}^{n} |a_k|$. Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.
(e) Let $M = b_N + \frac{r|a_N|}{1-r}$. Show that $|b_n| \leq M$ for all $n \geq 1$.
(f) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent. Conclude that the series $\{a_n\}_{n=1}^{\infty}$ is absolutely convergent and hence convergent.

Exercise 7.6.2 *(Ratio Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero terms and suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$.
(a) Suppose $L > 1$. Let $\epsilon = L - 1$. Show that there is a positive integer N such that

$$L - \left|\frac{a_{n+1}}{a_n}\right| < \epsilon \text{ for all } n \geq N.$$

(b) Show that $|a_{n+1}| > |a_N| > 0$ for all $n \geq N$.
(c) Show that the series $\sum_{n=1}^{\infty} a_n$ is divergent. Hint: The nth term test.

What about the case $L = 1$? Unfortunately, the test is inconclusive for this case. That is, for $L = 1$ it is possible to have a convergent sequence as well as a divergent sequence. We will illustrate this in the next two examples.

Exercise 7.6.3
Consider the harmoninc series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know it is divergent. Find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Exercise 7.6.4
Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
(a) Show that this series is convergent.
(b) Find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Exercise 7.6.5
Use the ratio test to determine the convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{100^n}{n!}$.

Exercise 7.6.6
Use the ratio test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$. Hint: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Remark 7.6.1
When testing a series for convergence, normally concentrate on the nth term test and the ratio test. Use the comparison test, the limit comparison test or the integral test only when both tests fail.

Practice Problems

Exercise 7.6.7
Find $\lim_{n\to\infty} \frac{n!}{n^2}$.

Exercise 7.6.8 *(n^{th} root test)*
Consider a series $\sum_{n=1}^{\infty} a_n$. Define $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.
(a) Suppose first that $0 \geq L < 1$. Let $\epsilon = \frac{1-L}{2}$. Show that there is a positive integer N such that

$$|a_n|^{\frac{1}{n}} < \frac{1+L}{2} \text{ for all } n \geq N.$$

(b) Let $r = \frac{1+L}{2}$. Show that $0 < r < 1$ and $|a_n| < r^n$ for all $n \geq N$.
(c) Use (b) to conclude that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence convergent.

Exercise 7.6.9
Suppose that $L > 1$ in the previous exercise. Prove that the series $\sum_{n=1}^{\infty} a_n$ is divergent. Hint: nth term test.

Exercise 7.6.10
The n^{th} root test is inconclusive if $L = 1$.
(a) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent. Show that $L = 1$.
(b) We know that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is conditionally convergent. Show that $L = 1$.
(c) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Show that $L = 1$.

Exercise 7.6.11
Use the n^{th} root test to show that the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$ is divergent.

Exercise 7.6.12
Use the n^{th} root test to show that the series $\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$ is absolutely convergent.

Chapter 8

Series of Functions

8.1 Sequences of Functions: Pointwise and Uniform Convergence

Earlier in the course, we have studied sequences of real numbers. Now we discuss the topic of sequences of real valued functions.
A sequence of functions $\{f_n\}_{n=1}^{\infty}$ is a list of functions $\{f_1, f_2, \cdots\}$ such that each f_n maps a given subset D of $\mathbb{R}$ into $\mathbb{R}$.
For sequences of functions one considers two types of convergenve: Pointwise convergence and uniform convergence.

Definition 8.1.1
Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** on D to a function $f : D \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there is a positive integer $N = N(x, \epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$.

Exercise 8.1.1
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 0$ for all $x \geq 0$.

Exercise 8.1.2
For each positive integer n let $f_n : (0, \infty) \to \infty$ be given by $f_n(x) = nx$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise on D.

For pointwise convergence, the positive integer N depends on both the given x and ϵ. A stronger convergence concept can be defined where N depends only on ϵ.

Definition 8.1.2
Let D be a subset of $\mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on D. We say that $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** on D to a function $f : D \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

Exercise 8.1.3
For each positive integer n let $f_n : [0, 1] \to \infty$ be given by $f_n(x) = \frac{x}{n}$. Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function. Hint: For a given ϵ, choose N such that $N > \frac{1}{\epsilon}$.

Clearly, uniform convergence implies pointwise convergence. However, the converse is not true in general.

Exercise 8.1.4
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. By Exercise 8.1.1, this sequence converges pointwise to $f(x) = 0$. Let $\epsilon = \frac{1}{3}$. Show that there is no positive integer N with the property $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 0$. Hence, the given sequence does not converge uniformly to $f(x)$.

We showed earlier in the course that a function that is continuous on a closed interval is automatically uniformly continuous. Is that true also for pointwise and uniform convergence, i.e. is a sequence that converges pointwise on a closed interval automatically uniformly convergent?

Exercise 8.1.5
Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(a) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f. Hint: Suppose otherwise. Let $\epsilon = 0.5$ and get a contradiction by using a point $(0.5)^{\frac{1}{N}} < x < 1$.

Exercise 8.1.6
Give an example of a sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to a discontinuous function f.

It follows from the previous exercise that pointwise convergence does not preserve the property of continuity. One of the interesting features of uniform convergence is that it preserves continuity as shown in the next exercise.

Exercise 8.1.7
Suppose that for each $n \geq 1$ the function $f_n : D \to \mathbb{R}$ is continuous in D. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Let $a \in D$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.
(b) Show that there is a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$
(c) Using (a) and (b) show that for $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence, f is continuous in D since a was arbitrary. Symbolically we write

$$\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x).$$

We have seen above that pointwise convergence does not preserve continuity. What about integrability? That is, if a sequence of Riemann integrable functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function f, does it follow automatically that f is also Riemann integrable? The answer is no as seen in the next exercise.

Exercise 8.1.8
Consider the interval $[0, 1]$ and let the rationals in this interval be labeled $r_1, r_2, \cdots$ arranged in increasing order. For each positive integer n we define the function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f_n is Riemann integrable on $[0, 1]$. Hint: Remark 3.
(b) Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(c) Show that f is not Riemann integrable.

It is possible that a sequence of Riemann integrable functions converges pointwise to a Riemann integrable function. Does it automatically follow that

$$\lim_{n\to\infty}\int_D f_n(x)dx = \int_D \lim_{n\to\infty} f_n(x)dx = \int_D f(x)dx? \tag{8.1.1}$$

That is, can we interchange limit and integration? The answer is no as seen in the next exercise.

Exercise 8.1.9
Consider the functions $f_n : [0,1] \to \infty$ defined by $f_n(x) = n^2xe^{-nx}$.
(a) Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$. Hint: L'Hôpital's rule.
(b) Find $\lim_{n\to\infty}\int_0^1 f_n(x)dx$. Hint: Integration by parts.
(c) Show that $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

Contrary to pointwise convergence, uniform convergence preserves integration as seen in the next exercise. Moreover, limits and integration can be interchanged as given in (8.1.1).

Exercise 8.1.10
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a,b]$ that converges uniformly to a f defined on $[a,b]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $n \ge N$ we have

$$|f_n(x) - f(x)| < \tfrac{\epsilon}{4(b-a)} \text{ for all } x \in [a,b].$$

(b) Let $n \ge N$. Show that there is a partition P of $[a,b]$ such that

$$U(f_n,P) - L(f_n,P) < \frac{\epsilon}{2}.$$

(c) Suppose $n \ge N$ and P as in (b). Show that

$$U(f,P) \le U(f_n,P) + \frac{\epsilon}{4}$$

and therefore

$$L(f,P) \ge L(f_n,P) - \frac{\epsilon}{4}.$$

Hint: $|f(x)| \le |f_n(x)| + \frac{\epsilon}{4(b-a)}$ and $|f_n(x)| \le |f(x)| + \frac{\epsilon}{4(b-a)}$
(d) Conclude that $U(f,P) - L(f,P) < \epsilon$ and therefore f is Riemann integrable on $[a,b]$.

Exercise 8.1.11
Let $\{f_n\}_{n=1}^{\infty}$ and f be as in the previous exercise.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{b-a} \text{ for all } x \in [a, b].$$

(b) Show that for every $n \geq N$ we have

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| < \epsilon.$$

Thus, (8.1.1) holds. Hint: Exercise 6.4.1 and Exercise 6.5.3

Exercise 8.1.12
Give an example of a sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to a non-differentiable function f.

It follows from the previous exercise that pointwise convergence does not preserve the property of differentiablity. What about uniform convergence? The answer is still no as seen in the next exercise.

Exercise 8.1.13
Consider the family of functions $f_n : [-1, 1]$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
(a) Show that f_n is differentiable for each $n \geq 1$.
(b) Show that for all $x \in [-1, 1]$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

where $f(x) = |x|$. Hint: Note that $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \frac{1}{\sqrt{n}}$.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in [-1, 1].$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the non-differentiable function $f(x) = |x|$.

Exercise 8.1.14
Give an example of a sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ that converges uniformly to a a differentiable function f such that $\lim_{n\to\infty} f_n'(x) \neq f'(x) = [\lim_{n\to\infty} f_n(x)]'$. That is, one cannot, in general, interchange limits and derivatives. Hint: Exercise 8.1.3

Pointwise convergence was not enough to preserve differentiability, and neither was uniform convergence by itself. Even with uniform convergence the process of interchanging limits with derivatives is not true in general. However, if we combine pointwise convergence with uniform convergence we can indeed preserve differentiability and also switch the limit process with the process of differentiation. In order to prove such a result we need the following

Definition 8.1.3
A sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on a set D is said to be **uniformly Cauchy** if and only if for every $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that for all $m, n \geq N$ we have

$$|f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in D.$$

Note that this is the version of Cauchy sequences for functions. Recall that every Cauchy sequence of numbers is convergent. The following results shows that every uniform Cauchy sequence is uniformly convergent.

Exercise 8.1.15
Let $\{f_n\}_{n=1}^{\infty}$ defined on a set D be uniformly Cauchy.
(a) Show that for each $x \in D$, the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ is convergent. Call the limit $f(x)$. Thus, we can define a function $f : D \to \mathbb{R}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$. Hint: Exercise 2.5.7
(b) Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $m, n \geq N$ we have

$$|f_m(x) - f_n(x)| < \tfrac{\epsilon}{2} \text{ for all } x \in D.$$

(d) Fix $x \in D$. Show that there is a positive integer $m \geq N$ such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$.
(e) For the fixed x in (d), let $n \geq N$. Show that $|f_n(x) - f(x)| < \epsilon$.
(f) Conclude that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f.

Exercise 8.1.16
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $[a, b]$ such that $\{f_n(c)\}_{n=1}^{\infty}$ converges for some $c \in [a, b]$. Assume also that $\{f'_n\}_{n=1}^{\infty}$ converges uniformly to g in $[a, b]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N_1 such that for all $m, n \geq N_1$ we have

$$|f'_m(x) - f'_n(x)| < \tfrac{\epsilon}{2(b-a)} \text{ for all } x \in [a, b].$$

(b) Show that there is a positive integer N_2 such that for all $m, n \geq N_2$ we have

$$|f_m(c) - f_n(c)| < \frac{\epsilon}{2}.$$

Hint: Exercise 2.5.3
(c) Show that for all $x \in [a, b]$ there is a d between c and x such that

$$f_m(x) - f_n(x) = f_m(c) - f_n(c) + (x - c)[f'_m(d) - f_n(d)].$$

Hint: Apply the Mean Value theorem to the function $f_m - f_n$ restricted to the interval $[c, x]$.
(d) Let $N = N_1 + N_2$. Use (a) - (c) to show that for $n \geq N$ we have

$$|f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in [a, b].$$

That is, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(e) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a a function f.

Exercise 8.1.17
In this exercise we want to show that f of the previous exercise is differentiable in $[a, b]$ and $f' = g$.
(a) Show that there is a positive integer N_1 such that for all $n \geq N_1$ we have

$$|f'_m(x) - f'_n(x)| < \tfrac{\epsilon}{3} \text{ for all } x \in [a, b].$$

(b) Let $x_0 \in [a, b]$. Use the MVT to the function $f_m - f_n$ to show the existence of a point d between x_0 and x such that

$$f_m(x) - f_n(x) = f_m(x) - f_n(x_0) + (x - x_0)[f'_m(d) - f'_n(d)].$$

(c) Use (a) and (b) to show that

$$\left| \frac{f_m(x) - f_n(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \frac{\epsilon}{3}.$$

(d) Show that

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - \frac{f_n(x)-f_n(x_0)}{x-x_0}\right| \leq \frac{\epsilon}{3}.$$

(e) Show that there is a positive integer N_2 such that for all $n \geq N_2$ we have

$$|f_n'(x_0) - g(x_0)| < \frac{\epsilon}{3}.$$

(f) Let $N = N_1 + N_2$. Show that there is a $\delta > 0$ such that

$$\text{If } 0 < |x - x_0| < \delta \text{ then } \left|\tfrac{f_N(x)-f_N(x_0)}{x-x_0} - f_N'(x_0)\right| < \tfrac{\epsilon}{3}.$$

(g) Use (d) - (f) to conclude that

$$\text{If } 0 < |x - x_0| < \delta \text{ then } \left|\tfrac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right| < \epsilon.$$

That is, f is differentiable at x_0 with $f'(x_0) = g(x_0)$.

Practice Problems

Exercise 8.1.18
Consider the sequence of functions $f_n(x) = x - \frac{x^n}{n}$ defined on $[0, 1)$.
(a) Does $\{f_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.
(b) Does $\{f'_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

Exercise 8.1.19
Suppose that each f_n is uniformly continuous on D and that $f_n \to f$ uniformly on D. Prove that f is uniformly continuous on D.

Exercise 8.1.20
Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$.
(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ on $[0, 2]$.
(b) Does $f_n \to f$ uniformly on $[0, 2]$?

Exercise 8.1.21
Prove that if $f_n \to f$ and $g_n \to g$ uniformly on a set D then $f_n + g_n \to f + g$ uniformly on D.

Exercise 8.1.22
Prove that if $f_n \to f$ uniformly on a set D then $\{f_n\}_{n=1}^{\infty}$ uniformly Cauchy on D.

Exercise 8.1.23
Suppose that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on a set D where each f_n is bounded on D, that is $|f_n(x)| \leq M_n$ for all $x \in D$. Show that there is a positive constant M such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in D$.

Exercise 8.1.24
Suppose that $f_n \to f$ and $g_n \to g$ uniformly on D. Moreover, suppose that $|f_n(x)| \leq M_n$ and $|g_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and all $x \in D$. Prove that $f_n g_n \to fg$ uniformly on D.

Exercise 8.1.25
Let $f_n(x) = x + \frac{1}{n}$ for all $x \in \mathbb{R}$ and $g_n(x) = \left(x + \frac{1}{n}\right)^2$.
(a) Show that $f_n \to f$ uniformly where $f(x) = x$.
(b) Show that g_n does not converge uniformly to the function $g(x) = x^2$.

Exercise 8.1.26
Give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ and a function f such that $f_n \to f$ uniformly but f_n^2 does not converge uniformly to f^2.

Exercise 8.1.27
Give an example of two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ such that $f_n \to f$ and $g_n \to g$ uniformly but $f_n g_n$ does not converge uniformly to fg. Thus, the condition of boundedness in Exercise 8.1.24 is crucial.

8.2 Power Series and their Convergence

Power series are example of series of functions where the terms of the series are power funtions.
Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of numbers. Then a **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Example
1. A polynomial of degree m is a power series about $x = 0$ since

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m.$$

Note that $a_n = 0$ for $n \geq m+1$.
2. The geometric series $1 + x + x^2 + \cdots$ is a power series about $x = 0$ with $a_n = 1$ for all n.
3. The series $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$ is not a power series since it has negative powers of x.
4. The series $1 + x + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$ is not a power series since each term is a power of a different quantity.

To study the convergence of a power series about $x = a$ one starts by fixing x and then constructing the partial sums

$$\begin{aligned}
S_0(x) =& a_0,\\
S_1(x) =& a_0 + a_1(x-a),\\
S_2(x) =& a_0 + a_1(x-a) + a_2(x-a)^2,\\
&\vdots\\
S_n(x) =& a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n.\\
&\vdots
\end{aligned}$$

Thus obtaining the sequence $\{S_n(x)\}_{n=0}^{\infty}$. If this sequence converges (pointwise) to a number L, i.e. $\lim_{n\to\infty} S_n(x) = L$, then we say that the power series **converges** to L for the specific value of x. Otherwise, we say that the power series **diverges**.

Power series may converge for some values of x and diverge for other values. We next discuss results that provide a tool for determining the values of x for which a power series converges and those for which it diverges. Note that a power series about $x = a$ always converges at $x = a$ with sum equals to a_0.

Exercise 8.2.1

Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series that converges for $x = c$. Note that the series converges to a_0 if $c = a$. So we will assume that $c \neq a$.

(a) What is the value of the limit $\lim_{n\to\infty} a_n(c-a)^n$?

(b) Show that there is a positive integer N such that $|a_n(c-a)^n| < 1$ for all $n \geq N$.

(c) Let $M = \sum_{n=0}^{N-1} |a_n(c-a)^n| + 1$. Show that $|a_n(c-a)^n| \leq M$ for all $n \geq 0$.

(d) Let x be such that $|x-a| < |c-a|$. Show that for any $n \geq 0$ we have

$$|a_n(x-a)^n| \leq M \left|\frac{x-a}{c-a}\right|^n.$$

(e) Show that the series $\sum_{n=0}^{\infty} M \left|\frac{x-a}{c}\right|^n$ is convergent.

(f) Show that the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is absolutely convergent and hence convergent.

We conclude that if a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x = c$ it is convergent for any x satisfying $|x-a| < |c-a|$.

Exercise 8.2.2

Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series that diverges for $x = d$. Let x be a number satisfying $|x-a| > |d-a|$. Show that the assumption $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges at x leads to a contradiction. Hence, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ must be divergent. Hint: Use Exercise 8.2.1.

Exercise 8.2.3

Consider a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$. Let C be the collection of all real numbers at which the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges. That is,

$$C = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges}\}.$$

(a) Show that $C \neq \emptyset$.

(b) Explain in words the meaning that $C = \{a\}$.

(c) Explain in words the meaning that $C = (-\infty, \infty) = \mathbb{R}$.
(d) Suppose that $C \neq \{a\}$ and $C \neq \mathbb{R}$. That is, there is a real number $d \neq a$ such that $\sum_{n=0}^{\infty} a_n(d-a)^n$ diverges. Show that if $x \in C$ then $|x-a| \leq |d-a|$. Conclude that $\{|x-a| : x \in C\}$ is bounded from above with an upper bound M. What is the value of M?
(e) Show that there is a finite number R such that R is the least upper bound of $\{|x-a| : x \in C\}$. Thus, $|x-a| \leq R$ for all $x \in C$. Show that $R > 0$.
(f) Show that for any real number x such that $|x-a| > R$, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is divergent.
(g) Show that for any real number x such that $|x-a| < R$, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent. Hint: Let $\epsilon = R - |x-a|$ and use the definition of supremum to show that there exist an $x_0 \in C$ such that $R - \epsilon < |x_0 - a| \leq R$.

The above results states the following: For any given power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, one and only one of the following holds:
(i) The series converges only at $x = a$;
(ii) the series converges for all x;
(iii) There is some positive number R such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. The series may or may not converge for $|x-a| = R$. That is for the values $x = a-R$ and $x = a+R$.

Definition 8.2.1
The number R is called the **radius of convergence** of the power series. In (i), $R = 0$ and in (ii) $R = \infty$. The interval $(-R, R)$ along with neither, one, or both endpoints is called the **interval of convergence** of a power series.

Exercise 8.2.4
Find the radius of convergence of each of the following series:
(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
(b) $\sum_{n=0}^{\infty} n!x^n$.
(c) $\sum_{n=0}^{\infty} x^n$.

The next result gives a method for computing the radius of convergence of many power series.

Exercise 8.2.5 *(Absolute Ratio Test)*
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series with $a_n \neq 0$ for all $n \geq 0$.

Suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geq 0.$$

(a) Find $\lim_{n\to\infty}\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right|$.
(b) Suppose that $L=0$. Show that $R=\infty$. That is, a power series converges for all $x\in\mathbb{R}$.
(c) Suppose that $L>0$. Show that $R=\frac{1}{L}$.
(d) Suppose that $L=\infty$. Show that $R=0$, that is, the series diverges for all $x\neq a$.

It follows from the previous result that the redius of convergence satisfies

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Exercise 8.2.6
Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{n}$.

Practice Problems

Exercise 8.2.7
Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{n^2+2} x^n$.

Exercise 8.2.8
Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^2+1}$.

Exercise 8.2.9
Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{e}{2}\right)^n \frac{(x-1)^n}{n}$.

Exercise 8.2.10
Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = -3$ and diverges if $x = 7$. Indicate which of the following statements must be true, cannot be true, or may be true.
(a) The power series converges if $x = -10$.
(b) The power series diverges if $x = 3$.
(c) The power series converges if $x = 6$.
(d) The power series diverges if $x = 2$.
(e) The power series diverges if $x = -7$.
(f) The power series converges if $x = -4$.

Exercise 8.2.11
Give an example of a power series that converges on the interval $[-11, -3)$.

Exercise 8.2.12
Determine all the values of the real number x for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n (\log(3n))^3}$$

converges.

8.3 Taylor Series Approximations

In this section we study a special family of power series known as Taylor series. Taylor series is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point.
Let $f(x)$ be a function with derivatives of any order at $x = a$, that is, f is an **infinitely differentiable** function. Fix a value of x near a and consider the sequence of **Taylor polynomials** $\{P_n(x)\}_{n=0}^{\infty}$ where

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $\lim_{n\to\infty} P_n(x)$ exists and is equal to $f(x)$ then we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{8.3.1}$$

Definition 8.3.1
The right-hand series is called the **Taylor series expansion of** $f(x)$ **about** $x = a$. We call $\frac{f^{(n)}(a)}{n!}(x-a)^n$ the **general term** of the series. It is a formula that gives any term in the series. If $a = 0$ the Taylor series in known as the **MacLaurin series.**

Exercise 8.3.1
Find the Taylor series of $f(x) = \frac{1}{1-x}$, where $-1 < x < 1$.

For a given function f at a given x, it is possible that the Taylor series converges to a value different from $f(x)$. However, the Taylor series of most of the functions discussed in this section do converge to the original function.

Exercise 8.3.2
Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{2}{x^2}} & \text{if } x \neq 0. \end{cases}$$

(a) Find the Taylor polynomial of order n of f at $x = 0$.
(b) Show that $f(x) \neq \lim_{n\to\infty} P_n(x)$ for all x near 0. That is, the Taylor series of f about $x = 0$ does not converge to $f(x)$ for number very close to 0.

When does a Taylor series converge to its generating function? To answer this question we need the following result known as **Taylor's Theorem**: Let $f : [a, a+h] \to \mathbb{R}$ be a function such that all derivatives of f up to order $n+1$ exist and are continuous on $[a, a+h]$. For any $x \in [a, a+h]$ there exists a point $c \in [a, x]$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

We prove the above result by induction on n.

Exercise 8.3.3
(a) Show that the above result holds for $n = 0$. Hint: Apply the Fundamental Theorem of Caculus on the interval $[a, x]$.
(b) Suppose that the result holds for up to n. That is, for any $x \in [a, a+h]$ we can estimate $f(x)$ by $P_n(x)$ for x near a :

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

Suppose that f has continuous derivatives up to order $n+2$. Use integration by parts to show that

$$R_{n+1}(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R_{n+2}(x).$$

Hence,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R_{n+2}(x).$$

Exercise 8.3.4 *(Lagrange's Form of Remainder)*
(a) Show that there exist $x_1, x_2 \in [a, x]$ such that $f^{(n+1)}(x_1) \le f^{(n+1)}(t) \le f^{(n+1)}(x_2)$ for all $t \in [a, x]$.
(b) Use (a) to show that

$$\frac{f^{(n+1)}(x_1)}{(n+1)!}(x-a)^{n+1} \le R_{n+1}(x) \le \frac{f^{(n+1)}(x_2)}{(n+1)!}(x-a)^{n+1}$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

(c) Show that

$$f^{(n+1)}(x_1) \le R_{n+1}(x)\frac{(n+1)!}{(x-a)^{n+1}} \le f^{(n+1)}(x_2).$$

(d) Show that there is a $c \in [a, x]$ such that

$$f^{(n+1)}(c) = R_{n+1}(x)\frac{(n+1)!}{(x-a)^{n+1}}$$

and therefore

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Thus, we can write

$$f(x) = P_n(x) + R_{n+1}(x).$$

Exercise 8.3.5 *(Estimating $R_{n+1}(x)$)*
Suppose that there is $M > 0$ such that $|f^{(n+1)}(x)| \le M$ for all $x \in [a, a+h]$.
(a) Show that for all $x \in [a, a+h]$ we have

$$|R_{n+1}(x)| \le \frac{M}{(n+1)!|}|x-a|^{n+1}.$$

(b) Show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hint: Exercise 1.1.14 and Squeeze rulw.

Suppose that $f : [a, a+h] \to \mathbb{R}$ is infinitely differentiable on $[a, a+h]$. By Exercise 8.2.3 there is $R > 0$ such that $\lim_{n\to\infty} P_n(x)$ converges (absolutely) for $|x-a| < R$ and diverges for $|x-a| > R$. Thus, if $\lim_{n\to\infty} R_{n+1}(x) = 0$ for all $x \in [a, a+h]$ such that $|x-a| < R$ then the Taylor series will converge to the generating function.

Practice Problems

Exercise 8.3.6
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f, f', f'' exist and are continuous. Furthermore, $f \geq 0$ and $f'' \leq 0$. Show that f is a constant function.

Exercise 8.3.7
Find the Taylor polynomial of order n about 0 for $f(x) = e^x$, and write down the corresponding remainder term.

Exercise 8.3.8
Find the Taylor Polynomial of order 3 for the function $f(x) = \cos x$ centered at $x = \frac{\pi}{6}$.

Exercise 8.3.9
Find the Lagrange form of the remainder $R_n(x)$ for the function $f(x) = \frac{1}{1+x}$.

Exercise 8.3.10
Let $g(x)$ be a function such that $g(5) = 3, g'(5) = -1, g''(5) = 1$ and $g'''(5) = -3$.
(a) What is the Taylor polynomial of degree 3 for $g(x)$ near 5?
(b) Use (a) to approximate $g(4.9)$.

Exercise 8.3.11
Suppose that the function $f(x)$ is approximated near $x = 0$ by a sixth degree Taylor polynomial

$$P_6(x) = 3x - 4x^3 + 5x^6.$$

Find the value of the following:
(a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$ (d) $f^{(5)}(0)$ (e) $f^{(6)}(0)$

Exercise 8.3.12
Find the third degree Taylor polynomial approximating

$$f(x) = \arctan x,$$

near $a = 0$.

Exercise 8.3.13
Find the fifth degree Taylor polynomial approximating

$$f(x) = \ln(1 + x),$$

near $a = 0$.

8.4 Taylor Series of Some Special Functions

In this section we introduce the Taylor series of some special functions that are encountered very frequently in analysis. The first one is the function $f(x) = \frac{1}{1-x}$ whose Taylor series was discussed in Exercise 8.3.1 and is given by

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

which is valid only for $-1 < x < 1$.

Exercise 8.4.1
Let $f(x) = \cos x$.
(a) Using successive differentiation find a formula for $f^{(n)}(0)$.
(b) Show that

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

(c) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(d) Show that

$$|R_{n+1}(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that $\lim_{n\to\infty} R_{n+1}(x) = 0$. Hence, conclude that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

Exercise 8.4.2
Let $f(x) = e^x$.
(a) Find $f^{(n)}(0)$ for all $n \ge 0$.
(b) Find an expression for $P_n(x)$.
(c) Consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots.$$

Find the radius of convergence.
(d) Find an expression for $R_{n+1}(x)$ and show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence, conclude that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Exercise 8.4.3
Let $f(x) = \ln(1+x)$.
(a) Find $f^{(n)}(0)$ for all $n \geq 0$.
(b) Find an expression for $P_n(x)$.
(c) Consider the series

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots.$$

Find the radius of convergence.
(d) Show that

$$|R_{n+1}(x)| \leq \frac{1}{|1+c|^{n+1}} \cdot \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence, conclude that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x \leq 1.$$

New series can be found using prviously known series.

Exercise 8.4.4
Find the Taylor series of $\frac{x}{e^x}$ about $x = 0$.

Exercise 8.4.5
Find the Taylor series of $f(x) = \frac{1}{1+x^2}$ about $x = 0$.

Practice Problems

Exercise 8.4.6
Let $f(x) = \sin x$.
(a) Using successive differentiation find a formula for $f^{(n)}(0)$.
(b) Show that

$$P_{2n}(x) = P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

(c) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

(d) Show that

$$|R_{n+1}(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that $\lim_{n\to\infty} R_{n+1}(x) = 0$. Hence, conclude that

$$\sin x = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Exercise 8.4.7
Find the MacLaurin series of $\frac{x}{1-2x}$.

Exercise 8.4.8
Find the coefficient of $(x-2)^2$ in the Taylor series expansion of $f(x) = \frac{1}{x}$ about $x = 2$.

Exercise 8.4.9
Find the Maclaurin series for the function $f(x) = x^6 e^{-x^2}$. Give your answer in sigma notation.

Exercise 8.4.10
Compute each of the following sums in terms of known functions:
(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{n!}$
(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}$
(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n+2)!}$

Exercise 8.4.11
The hyperbolic cosine of x is defined to be the function $\cosh x = \frac{e^x+e^{-x}}{2}$. Find the MacLaurin series of $\cosh x$.

Exercise 8.4.12
The hyperbolic sine of x is defined to be the function $\sinh x = \frac{e^x-e^{-x}}{2}$. Find the MacLaurin series of $\sinh x$.

Exercise 8.4.13 *(Binomial Series)*
Consider the function $f(x) = (1+x)^n$ where $n \in \mathbb{R}$.
(a) Using successive differentiation show that $f^{(k)}(0) = k(k-1)\cdots(k-n+1)$. Thus, $\frac{f^{(k)}(0)}{k!} = C(n,k)$ where

$$C(n,k) = \frac{n!}{k!(n-k)!} \text{ and } C(n,0) = 1.$$

(b) Find the interval of convergence of the binomial series $(1+x)^n = \sum_{k=0}^{\infty} C(n,k)x^k$.

Exercise 8.4.14
Find the MacLaurin series of $f(x) = \frac{1}{\sqrt{x+1}}$.

8.5 Uniform Convergence of Series of Functions: Weierstrass M Test

The Weierstrass M Test is a test used to show uniform convergence of series of functions. It has many applications that will be discussed in next section.

Definition 8.5.1
For each positive integer $n \geq 1$ we let $f_n : D \to \mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly** in D if and only if the sequence of partial sums

$$S_n(x) = \sum_{k=1}^{n} f_n(x)$$

converges uniformly in D.

Exercise 8.5.1
Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D. For each $x \in D$ let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. That is, $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ we have

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \frac{\epsilon}{2}$$

for all $x \in D$.
(b) Show that for $n > m \geq N$ we have

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| = \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x) \right| < \epsilon$$

for all $x \in D$.

Exercise 8.5.2 *(Weierstrass)*
For each positive integer $n \geq 1$, let $f_n : D \to \mathbb{R}$ be a continuous function that is bounded on D with $|f_n(x)| \leq M_n$ for all $x \in D$. Suppose that the series of numbers $\sum_{n=1}^{\infty} M_n$ is convergent. For each positive integer n define the partial sum

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $m, n \geq N$ we have
$$\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| < \epsilon.$$
Hint: The sequence $\{\sum_{k=1}^{n} M_k\}_{n=1}^{\infty}$ is Cauchy.
(b) Suppose that $n > m \geq N$. By (a) we have $|\sum_{k=m+1}^{n} M_k| < \epsilon$. Show that for all $x \in D$ we have
$$|S_n(x) - S_m(x)| < \epsilon$$
Hence, the sequence $\{S_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(c) Conclude that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Hint: Exercise 8.1.15

Exercise 8.5.3
Use Weierstrass M test to show that the series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converges uniformly on $[-2, 2]$.

An important application of the Weierstrass M test is the following result.

Exercise 8.5.4
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $0 < c < R$ and $D = [-c, c]$.
(a) Define $f_n(x) = a_n x^n$ and $M_n = |a_n c^n|$. Clearly, f_n is continuous in D and $M_n > 0$ for all integer $n \geq 0$. Show that $\sum_{n=0}^{\infty} M_n$ converges. Hint: Exercise 8.2.1(f)
(b) Let $x \in D$. Show that if $x \in [0, c]$ then $|g_n(x)| \leq M_n$. Hint: x^n is increasing for $x \geq 0$.
(c) Answer the same question if $x \in [-c, 0]$. (d) Conclude that the series is uniformly convergent on D.

Remark 8.5.1
(1) For a series about $x = a$, $D = [a - c, a + c]$.
(2) The previous result says that the series converges uniformly on any closed interval centered at a and contained in $(a - R, a + R)$.

Practice Problems

Exercise 8.5.5
Show that the following series converges uniformly.

$$\sum_{n=0}^{\infty} \frac{x^2}{3^n(x^2+1)}.$$

Exercise 8.5.6
Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence with $|a_n| \leq M$ for all $n \in \mathbb{N}$. Show that ther series $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges uniformly for all $x \geq c > 1$.

Exercise 8.5.7
Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly for all $x \in \mathbb{R}$.

Exercise 8.5.8
Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on a set D such that $|f_{n+1}(x) - f_n(x)| \leq M_n$ for all $x \in D$ and $n \in \mathbb{N}$. Assume that $\sum_{n=1}^{\infty} M_n$ is convergent. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on D.

Exercise 8.5.9
Show that the series $\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}$ converges uniformly on $[1, 2]$.

Exercise 8.5.10
Prove that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly on any bounded interval $[a, b]$.

Exercise 8.5.11
Show that the series $\sum_{n=1}^{\infty} \frac{1}{3^n} \cos\left(\frac{x}{3^n}\right)$ converges uniformly on $\mathbb{R}$.

8.6 Continuity, Integration and Differentiation of Power Series

Since a power series is a function, it is natural to ask if the function is continuous, differentiable or integrable. In this section we answer these questions. Let R be the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$. Thus, for each x in $D = \{x \in \mathbb{R} : |x-a| < R\}$ there is a unique number $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$. Hence, we can define a function $f : D \to \mathbb{R}$.
The next result shows that this function f is a continuous function.

Exercise 8.6.1
Let $c \in D$. Let $R_0 > 0$ be a number such that $|c-a| < R_0 < R$. By Exercise 8.5.4, the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges uniformly on the interval $[a-R_0, a+R_0]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $n > m \geq N$ we have

$$\left|\sum_{k=0}^{n} a_k(x-a)^k - \sum_{k=0}^{n} a_k(x-a)^k\right| = \left|\sum_{k=m+1}^{n} a_k(x-a)^k\right| < \tfrac{\epsilon}{3} \text{ for all } x \in [a-R_0, a+R_0].$$

Hint: Exercise 8.5.1
(b) Show that there is a $\delta_1 > 0$ such that if $|x-a| < \delta_1$ then

$$\left|\sum_{k=0}^{N} a_k(x-a)^k - \sum_{k=0}^{N} a_k(c-a)^k\right| < \frac{\epsilon}{3}.$$

(c) Let $\delta = \min\{\delta_1, R_0 - |c-a|\}$. Show that for $|x-a| < \delta$ we have

$$|f(x) - f(c)| < \epsilon.$$

Hence, the function $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous on D.

Our next result concerns integrating term-by-term a given power series to yield a new power series with the same radius of convergence.

Exercise 8.6.2
Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ where the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Let $F(x) = \int_a^x f(t)dt$. Suppose that $a - R < x \leq a$. A similar result holds for $a \leq x < a + R$. (a) Show that $\{S_n\}_{n=1}^{\infty}$

converges uniformly to f on $[x, a]$.
(b) Evaluate $\int_x^a S_n(t)dt$.
(c) Show that the power series $\sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$ has radius of convergence R.
(d) Show that $F(x) = \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$. Hint: Exercise 8.1.11

We conclude this section by showing that integration term-by-term of a power series yields a new power series with the same radius of convergence.

Exercise 8.6.3
Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ where the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$.
(a) Show that the power series $g(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ has radius of convergence R.
(b) Let $G(x) = \int_a^x g(t)dt$. Show that $G(x) = f(x) - a_0$ for $|x-a| < R$. Hint: Exercise 8.6.2.
(c) Show that $g(x) = f'(x)$ for all $|x-a| < R$. Hint: Exercise 6.6.2

Practice Problems

Exercise 8.6.4
Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and show that the series converges uniformly to a continuous function on $[-2, 2]$.

Exercise 8.6.5
Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin(3x)}{3^n}$. Prove that the series converges for all $x \in \mathbb{R}$ and that $g(x)$ is continuous everywhere.

Exercise 8.6.6
Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges to a continuous function for all $x \in \mathbb{R}$.

Exercise 8.6.7
Find the Taylor series about $x = 0$ of $\cos x$ from the series of $\sin x$.

Exercise 8.6.8
Find the Taylor's series about $x = 0$ for $\arctan x$ from the series for $\frac{1}{1+x^2}$.

Exercise 8.6.9
Use the first 500 terms of series of $\arctan x$ and a calculator to estimate the numerical value of π.

Exercise 8.6.10
Estimate the value of $\int_0^1 \sin(x^2) dx$.

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