Solutions to Practice Problems

Exercise 19.7
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that $f$ is a constant function on $[a, b]$ if and only if there is a constant $C$ such that $f(x) = C$ for all $a \leq x \leq b$. Suppose that $f'(x) = 0$ for all $a < x < b$.

Let $x_1$ and $x_2$ be any two numbers in the interval $[a, b]$ with $x_1 < x_2$. Suppose that $f(x_1) \neq f(x_2)$. Show that by applying the Mean Value Theorem on the interval $[x_1, x_2]$ we obtain the contradiction $f(x_1) = f(x_2)$.

Solution.
Applying the MVT on the interval $x_1 \leq x \leq x_2$, we can find a number $c$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) = 0$ we obtain $f(x_1) = f(x_2)$, a contradiction. Since $x_1$ and $x_2$ were arbitrary, we have $f(x) = C$ for all $x \in [a, b]$.

Exercise 19.8
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Suppose that $f'(x) = g'(x)$ for all $a < x < b$. Show that $f(x) = g(x) + C$ for all $a \leq x \leq b$, where $C$ is a constant. Hint: Exercise 19.7

Solution.
Let $h(x) = f(x) - g(x)$. Then $h(x)$ is continuous in $[a, b]$ being the difference of two continuous functions and $h'(x) = 0$ for all $a < x < b$. By Exercise 19.7, there is $C$ such that $h(x) = C$ for all $a \leq x \leq b$ or equivalently $f(x) = g(x) + C$ for all $a \leq x \leq b$.

Exercise 19.9
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that $f$ is decreasing in $[a, b]$ if and only if for every $x_1$ and $x_2$ in $[a, b]$, if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$. Show that if $f'(x) \leq 0$ for all $a < x < b$ then $f(x)$ is decreasing in $[a, b]$. Hint: Use the MVT restricted to the interval $[x_1, x_2]$.

Solution.
Let $x_1, x_2 \in [a, b]$. Clearly, if $x_1 = x_2$ then $f(x_1) = f(x_2)$. So assume that
\( x_1 < x_2 \). By the MVT there is a \( x_1 < c < x_2 \) such that \( f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \leq 0 \) which implies that \( f(x_1) \geq f(x_2) \). Thus, we have shown that \( x_1 \leq x_2 \) implies \( f(x_1) \geq f(x_2) \). That is, \( f \) is decreasing in \([a, b]\) \( \blacksquare \)

**Exercise 19.10**

Consider the function \( f(x) = (1 + x)^p \) where \( 0 < p < 1 \). Let \( h > 0 \).

(a) Apply the MVT to the interval \([0, h]\) to show that \( f(h) = p(1 + t)^{p-1}h + 1 \) for some \( 0 < t < h \).

(b) Use (a) to show that \((1 + h)^p < 1 + ph\).

In annuity theory, \((1 + h)^p\) may represent compound interest and \(1 + ph\) represent simple interest. A result in annuity theory says that for time \( p \) less than a year, the compound interest formula can be estimated by the simple interest formula.

**Solution.**

(a) Applying the mean value theorem to the interval \([0, h]\), we can find a \( 0 < t < h \) such that \( f(h) - f(0) = f'(t)h \) or \( f(h) - 1 = p(1 + t)^{p-1}h \). Hence, \( f(h) = (1 + h)^p = p(1 + t)^{p-1}h + 1 \).

(b) Since \( t > 0 \), we have \( 1 + t > 1 \rightarrow (1 + t)^{1-p} > 1 \rightarrow (1 + t)^{p-1} < 1 \rightarrow p(1 + t)^{p-1}h < ph \rightarrow 1 + p(1 + t)^{p-1}h < 1 + ph \). Hence, \((1 + h)^p < 1 + ph\) \( \blacksquare \)

**Exercise 19.11**

Suppose that \( f : [a, b] \rightarrow \mathbb{R} \) is differentiable in \([a, b]\). Let \( \lambda \) be a real number such that either \( f'(a) < \lambda < f'(b) \) or \( f'(b) < \lambda < f'(a) \).

(a) Define \( g(x) = f(x) - \lambda x \). Show that if \( f'(a) < \lambda < f'(b) \) then \( g'(x) \) changes sign between \( a \) and \( b \).

(b) Establish the same result for \( f'(b) < \lambda < f'(a) \).

(c) Show that the condition \( g'(c) \neq 0 \) for all \( c \in [a, b] \) leads to a contradiction. Hint: Exercise 19.6. Conclude that there must be a \( a < c < b \) such that \( f'(c) = \lambda \).

**Solution.**

(a) Note that \( g \) is continuous in \([a, b]\) and differentiable there with derivative \( g'(x) = f'(x) - \lambda \). Since \( f'(a) < \lambda < f'(b) \), we find \( g'(a) = f'(a) - \lambda < 0 < g'(b) = f'(b) - \lambda \). So \( g' \) changes sign from negative to positive.

(b) Since \( f'(b) < \lambda < f'(a) \), we find \( g'(b) = f'(b) - \lambda < 0 < g'(a) = f'(a) - \lambda \). So \( g' \) changes sign from positive to negative.

(c) If \( g'(c) \neq 0 \) for all \( c \in [a, b] \) then by Exercise 19.6 either \( g' \) is always
nonnegative in \([a, b]\) or always nonpositive which contradict (a) and (b). We conclude that there must be a \(a < c < b\) such that \(g'(c) = 0\) which is the same as \(f'(c) = \lambda\) □

**Exercise 19.12**

Let \(f, g : [a, b] \to \mathbb{R}\) be two differentiable functions on \([a, b]\) such that \(f(a) = g(a)\). Show that if \(f'(x) = g'(x)\) for all \(x \in (a, b)\) then \(f(x) = f(x)\) for all \(x \in [a, b]\). Hint: Exercise 19.7.

**Solution.**

Let \(F : [a, b] \to \mathbb{R}\) be given by \(F(x) = f(x) - g(x)\). Then \(F'\) is differentiable on \([a, b]\) and \(F'(x) = 0\) for all \(x \in (a, b)\). By Exercise 19.7, there is a constant \(C\) such that \(F(x) = C\) for all \(x \in [a, b]\). But \(F(a) = 0\) so that \(C = 0\). Thus, \(F(x) = 0\) for all \(x \in [a, b]\). This is equivalent to \(f(x) = g(x)\) for all \(x \in [a, b]\) □

**Exercise 19.13**

Let \(f : \mathbb{R} \to \mathbb{R}\) be differentiable such that \(|f'(x)| < 1\) for all \(x \in \mathbb{R}\). Show that \(f\) can have at most one fixed point. That is, There is at most one \(c \in \mathbb{R}\) such that \(f(c) = c\). Hint: Mean Value Theorem.

**Solution.**

Suppose the contrary. Let \(a, b \in \mathbb{R}\) such that \(a < b\), \(f(a) = a\), and \(f(b) = b\). We have that \(f\) is continuous in \([a, b]\) and differentiable in \((a, b)\). By the MVT, there is a \(c \in (a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.
\]

This is impossible since \(|f'(x)| < 1\) for all \(x \in \mathbb{R}\). We conclude that \(f\) has at most one fixed point □

**Exercise 19.14**

Let \(f : \mathbb{R} \to \mathbb{R}\) be differentiable everywhere and that \(f'(a) < 0\) and \(f'(b) > 0\) for some \(a < b\). Prove that there is a \(c \in (a, b)\) such that \(f'(c) = 0\).

**Solution.**

This is just Exercise 19.11 with \(\lambda = 0\) □
Exercise 19.15
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| \leq K < 1$ for all $x \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$. Define the numbers $a_n = f(a_{n-1})$.
(a) Show that $|a_{n+1} - a_n| \leq K^n |a_1 - a_0|$ for all $n \in \mathbb{N}$.
(b) Show that for all $m, n \in \mathbb{N}$ such that $m > n$ we have

$$|a_m - a_n| \leq \frac{K^n}{1 - K}.$$

Solution.
(a) By the MVT there is a $c_1 \in (a_1, a_0)$ such that $f(a_1) - f(a_0) = f'(c_1)(a_1 - a_0)$. Thus, $|a_2 - a_1| \leq K|a_1 - a_0|$ since $|f'(c_1)| \leq K$. Likewise, we can write $|a_3 - a_2| \leq K|a_2 - a_1| \leq K^2|a_1 - a_0|$. Now, suppose that $|a_n - a_{n-1}| \leq K^n|a_1 - a_0|$. Then $|a_{n+1} - a_n| \leq K|a_n - a_{n-1}| \leq K^{n+1}|a_1 - a_0|$.

(b) Let $m, n \in \mathbb{N}$ such that $m > n$. Then we have $|a_m - a_n| \leq |a_m - a_{m-1}| + \cdots + |a_m - a_{m-1}| \leq |a_1 - a_0| \sum_{i=n}^{m} K^i = \frac{K^n}{1 - K} |a_1 - a_0|$. 

Exercise 19.16
Show that if $0 < a < b$ then $1 - \frac{a}{b} < \ln \left( \frac{b}{a} \right) < \frac{b}{a} - 1$. Hint: Apply the MVT for the function $f(x) = \ln x$.

Solution.
The function $f(x) = \ln x$ is continuous on $[a, b]$ and differentiable in $(a, b)$. By the Mean value theorem there is a $c \in (a, b)$ such that $f'(c) = \frac{\ln b - \ln a}{b-a}$. Thus,

$$\frac{1}{b} < \frac{1}{c} = \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$$

or

$$1 - \frac{a}{b} < \ln \left( \frac{b}{a} \right) < \frac{b}{a} - 1 \quad \blacksquare$$

Exercise 19.17
Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable with continuous derivative. Suppose there are four distinct points $w, x, y, z$ such that $f(w) = f(x), f(y) = y$ and $f(z) = z$. Prove that there is a point $u$ where $f'(u) = \frac{1}{2}$.

Solution.
Since $f(w) = f(x)$, we can apply Rolle’s theorem to find a point $c$ between $w$ and $x$ where $f'(c) = 0$. Similarly, since $f(y) = y$, we can apply Rolle’s theorem to find a point $d$ between $y$ and $z$ where $f'(d) = 0$. Since $f(w) = f(x)$ and $f(y) = y$, we have $f'(c) = 0$ and $f'(d) = 0$. Therefore, $f'(u) = \frac{f'(c) + f'(d)}{2} = \frac{0 + 0}{2} = 0$. Thus, there is a point $u$ where $f'(u) = \frac{1}{2} \quad \blacksquare$
and \( c \) such that \( f'(c) = 0 \). Now, if we apply the Mean Value Theorem to the interval \([y, z]\) (or \([z, y]\) we can find a point \( d \) between \( y \) and \( z \) such that
\[
f'(d) = \frac{f(y) - f(z)}{y - z} = \frac{y - z}{y - z} = 1.
\]
Since \( 0 \leq 1 \leq 1 \) we can apply IVT to find a point \( u \) between \( c \) and \( d \) such that \( f'(u) = \frac{1}{2} \).

**Exercise 19.18**
Suppose \( f : [a, b] \to \mathbb{R} \) is differentiable and \( f'(x) \geq M \) for all \( x \in [a, b] \). Prove that \( f(b) \geq f(a) + M(b - a) \).

**Solution.**
By the Mean Value Theorem, there is a \( a < c < b \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]
But \( f'(x) \geq M \) so that \( \frac{f(b) - f(a)}{b - a} \geq M \) which implies \( f(b) \geq f(a) + M(b - a) \).

**Exercise 19.19**
Suppose \( f \) is differentiable everywhere on \((a, b)\), and that there is a number \( M \) with \(|f'(x)| \leq M\) for all \( x \in (a, b) \). Prove that \( f \) is uniformly continuous on \((a, b)\) using the Mean Value Theorem.

**Solution.**
Let \( x \) and \( y \) be two numbers in \((a, b)\). By the MVT there is a \( z \) between \( x \) and \( y \) such that
\[
f(x) - f(y) = f'(z)(x - y).
\]
Thus,
\[
|f(x) - f(y)| = |f'(z)||x - y| \leq M|x - y|.
\]
Let \( \epsilon > 0 \) be given. Let \( \delta < \frac{\epsilon}{M} \). For all \( x, y \in (a, b) \) such that \(|x - y| < \delta \) we have \(|f(x) - f(y)| < M \cdot \frac{\epsilon}{M} = \epsilon \). This shows that \( f \) is uniformly continuous on \((a, b)\).

**Exercise 19.20**
Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable such that \(|f(x) - f(y)| \leq |x - y|^2\) for all \( x, y \in \mathbb{R} \). Show that \( f'(x) = 0 \) for all \( x \in \mathbb{R} \) and therefore \( f \) is a constant function.
Solution.
Let \( x \in \mathbb{R} \). Then \( |f(x + h) - f(x)| \leq |h|^2 \). Thus, \( 0 \leq \left| \frac{f(x+h)-f(x)}{h} \right| \leq |h| \).
Letting \( h \to 0 \) and using the squeeze rule we find \( |f'(x)| = 0 \) which implies that \( f'(x) = 0 \). Since \( x \) was arbitrary, we conclude that \( f'(x) = 0 \) for all \( x \in \mathbb{R} \) and hence \( f \) is a constant function.

Exercise 19.21
Use the Mean Value Theorem to show that for all \( x > 1 \) we have
\[
\frac{x - 1}{x} < \ln x.
\]
Solution.
Let \( f(x) = \ln x \) on \([1, x]\). This function satisfies the condition of the Mean Value Theorem. Thus, we can find a \( c \in (1, x) \) such that \( f(x) - f(1) = f'(c)(x-1) \) or \( \ln x = \frac{x-1}{c} \). But \( 1 < c < x \) implies \( \frac{1}{x} < \frac{1}{c} < 1 \). Hence, \( \frac{x-1}{x} < \frac{x-1}{c} = \ln x \).

Exercise 19.22
Prove that \( 1 + x \leq e^x \) for all \( x \in \mathbb{R} \).
Solution.
The equality is true for \( x = 0 \). Let \( x > 0 \) and define \( f : [0, x] \to \mathbb{R} \) by \( f(t) = e^t - 1 - t \). \( f \) satisfies the conditions of the MVT so that there is a \( 0 < c < x \) such that \( f(x) - f(0) = f'(c)(x-0) \) which implies that \( e^x - 1 - x = (e^c - 1)x \). But \( (e^c - 1)x > 0 \) so that \( e^x > 1 + x \). If \( x < 0 \) we apply the above argument on \([x, 0]\) to obtain \( e^x > 1 + x \).

Exercise 19.23
Let \( f : [a, b] \to \mathbb{R} \) be a function with continuous derivative on \([a, b]\). Moreover, suppose that \( |f'(x)| < 1 \) for all \( x \in [a, b] \).
(a) Show that there exits two numbers \( c, d \in [a, b] \) such that \( f'(c) \leq f'(x) \leq f'(d) \) for all \( x \in [a, b] \).
(b) How that \( |f'(x)| \leq \max\{|f'(c)|, |f'(d)|\} = s \leq 1 \) for all \( x \in [a, b] \).
(c) Show that \( |f(x_1) - f(x_2)| \leq s|x_1 - x_2| \) for all \( x_1, x_2 \in [a, b] \).
Solution.
(a) Since \( f' \) is continuous on the closed interval \([a, b]\), Exercise 16.8 asserts the existence of two numbers \( c, d \in [a, b] \) such that \( f'(c) \leq f'(x) \leq f'(d) \).
(b) Let \( s = \max\{|f'(c)|, |f'(d)|\} < 1 \) since \(|f'(x)| < 1 \) for all \( x \in [a, b] \). We have \(-s \leq -|f'(c)| \leq f'(c) \leq f'(x) \leq f'(d) \leq |f'(d)| \leq s\).

(c) Let \( x_1, x_2 \in [a, b] \). By the MVT, there exists an \( x_3 \) between \( x_1 \) and \( x_2 \) such that \( f(x_1) - f(x_2) = f'(x_3)(x_1 - x_2) \). Hence, \(|f(x_1) - f(x_2)| = |f'(x_3)||x_1 - x_2| \leq s|x_1 - x_2| \).