Solutions to Practice Problems

Exercise 16.9
Prove that there exists a number $c \in \left[0, \frac{\pi}{2}\right]$ such that $2c - 1 = \sin\left(c^2 + \frac{\pi}{4}\right)$.

Solution.
Let $f(x) = 2x - 1 - \sin\left(x^2 + \frac{\pi}{4}\right)$. Then $f(0) = -1 - \frac{1}{\sqrt{2}} < 0$ and $f\left(\frac{\pi}{2}\right) = \pi - 1 - \sin\left(\frac{\pi^2}{2} + \frac{\pi}{4}\right) > 0$. By the Intermediate Value Theorem, there is a $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = 0$ or $2c - 1 = \sin\left(c^2 + \frac{\pi}{4}\right)$ ■

Exercise 16.10
Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there is $c \in [a, b]$ such that $f(c) = c$. We call c a **fixed point** of f. Hint: Intermediate Value Theorem applied to a specific function F (to be found) defined on $[a, b]$.

Solution.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) = x - f(x)$. Then F is continuous on $[a, b]$. Since $a \le f(a) \le b$ and $a \le f(b) \le b$ we find $F(a) = a - f(a) \le 0$ and $F(b) = b - f(b) \ge 0$. By the Intermediate Value Theorem, there is a $c \in [a, b]$ such that $F(c) = 0$ or $c - f(c) = 0$. Thus, $f(c) = c$ ■

Exercise 16.11
Using the Intermediate Value Theorem, show that
(a) the equation $3 \tan x = 2 + \sin x$ has a solution in the interval $[0, \frac{\pi}{4}]$.
(b) the polynomial $p(x) = -x^4 + 2x^3 + 2$ has at least two real roots.

Solution.
(a) Let $f(x) = 3 \tan x - \sin x - 2$. Then f is continuous on $[0, \frac{\pi}{4}]$ and $f(0) = -2 < 0$, $f\left(\frac{\pi}{4}\right) = 1 - \frac{1}{\sqrt{2}} > 0$. By IVT, there is a $c \in [0, \frac{\pi}{4}]$ such that $f(c) = 0$. This means, the given equation has at least one solution in the interval $[0, \frac{\pi}{4}]$.
(b) Since $p(-1) = -1 < 0$, $p(0) = 2 > 0$, and $p(3) = -25 < 0$ there exist at least two numbers $-1 \le c_1 \le 0 \le c_2 \le 3$ such that $f(c_1) = f(c_2) = 0$. Since $p(0) \ne 0$, we must have $c_1 \ne c_2$ ■

Exercise 16.12
Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(a) \le g(a)$ and $f(b) \ge g(b)$. Show that there is a $c \in [a, b]$ such that $f(c) = g(c)$.

Solution.
Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$ with $h(a) \leq 0 \leq h(b)$. By the IVT, there is a $c \in [a, b]$ such that $h(c) = 0$ or $f(c) = g(c)$ ■

Exercise 16.13
Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) \leq a$ and $f(b) \geq b$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$.

Solution.
Let $g(x) = f(x) - x$. Then g is continuous on $[a, b]$ with $g(a) \leq 0$ and $g(b) \geq 0$. By IVT, there is a $c \in [a, b]$ such that $g(c) = 0$. That is, $f(c) = c$ ■

Exercise 16.14
Let $f : [a, b] \to \mathbb{R} \backslash \mathbb{Q}$ be continuous. Prove that f must be a constant function. Hint: Exercise 3.6(c).

Solution.
Suppose for a contradiction that f is not constant. Then, we can find $x, y \in [a, b]$ with $x < y$ and such that $f(x) \neq f(y)$. Choose a rational number m lying between $f(x)$ and $f(y)$. Then, by the Intermediate Value Theorem, there exists $z \in [x, y]$ with $f(z) = m$. Hence, f takes a rational value, contradicting the hypotheses ■

Exercise 16.15
Prove that a polynomial of odd degree considered as a function from the reals to the reals has at least one real root.

Solution.
Let $f(x)$ be a polynomial of odd degree. Then $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$ (or $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$ depending on whether the leading coefficient is positive or negative, respectively). Hence, there exist two real numbers a and b such that $a < b$ with $f(a) < 0$ and $f(b) > 0$. Now the Intermediate Value Theorem applies to give an $x \in [a, b]$ such that $f(x) = 0$ ■

Exercise 16.16
Suppose $f(x)$ is continuous on the interval $[0, 2]$ and $f(0) = f(2)$. Show that there is a number c between 0 and 1 so that $f(c+1) = f(c)$. Hint: Consider the function $g(x) = f(x+1) - f(x)$ on $[0, 1]$.

Solution.
We let $g(x) = f(x+1) - f(x)$. Then $g(x)$ is continuous on $[0,1]$. Furthermore,

$$g(0) = f(1) - f(0)$$

and

$$g(1) = f(2) - f(1) = f(0) - f(1) = -(f(1) - f(0)).$$

If $f(1) = f(0)$ we have obtained the desired conclusion upon taking $c = 0$. We therefore assume $f(0) \neq f(1)$. But then $g(0)$ and $g(1)$ have opposite signs. The Intermediate Value Theorem therefore guarantees the existence of a number c in the interval $[0,1]$ satisfying $g(c) = 0$. But by definition of $g(x)$, this means $f(c+1) = f(c)$ ■

Exercise 16.17
Let $f : [a,b] \to \mathbb{R}$ be continuous and one-to-one. We want to show that f is monotone,i.e. either f is always increasing on $[a,b]$ or always decreasing. Let's assume the contrary, then one of the following cases applies:
(i) There are $x, y, z \in [a,b]$ such that $x < y < z$ and $f(x) < f(y), f(z) < f(y)$. That is, the graph of f is increasing on $[x,y]$ and decreasing on $[y,z]$.
(ii) There are $x, y, z \in [a,b]$ such that $x < y < z$ and $f(x) > f(y), f(y) < f(z)$. That is, the graph of f is decreasing on $[x,y]$ and increasing on $[y,z]$.
Consider Case (i). We have either $f(y) < f(x) < f(z)$ or $f(z) < f(x) < f(y)$.
(a) Suppose that $f(z) < f(x) < f(y)$. Use the Intermediate Value theorem restricted to $[y,z]$ to show that such a double inequality can not occur.
(b) Suppose that $f(x) < f(z) < f(y)$. Use the Intermediate Value theorem restricted to $[x,y]$ to show that such a double inequality can not occur.
We conclude that Case (i) does not hold.
(c) Answer (a) and (b) for case (ii). Hence, we conclude that f must be monotone.

Solution.
(a) If $f(z) < f(x) < f(y)$, we can apply the Intermediate Value Theorem to $[y,z]$ to find $y < w < z$ such that $f(w) = f(x)$. Since f is one-to-one we must have $w = x < y$ which contradicts the inequality $y < w$.
(b) If $f(x) < f(z) < f(y)$, we can apply the Intermediate Value Theorem to $[x,y]$ to find $x < w < y$ such that $f(w) = f(z)$. Since f is one-to-one we must have $w = z < y$ which contradicts the inequality $y < z$.
(c) If $f(y) < f(x) < f(z)$, we can apply the Intermediate Value Theorem

to $[y, z]$ to find $y < w < z$ such that $f(w) = f(x)$. Since f is one-to-one we must have $w = x < y$ which contradicts the inequality $y < w$.
If $f(y) < f(z) < f(x)$, we can apply the Intermediate Value Theorem to $[x, y]$ to find $x < w < y$ such that $f(w) = f(z)$. Since f is one-to-one we must have $w = z < y$ which contradicts the inequality $y < z$ ■