Arkansas Tech University
MATH 2934: Calculus III
Dr. Marcel B. Finan

21 Constraint Optimization: Lagrange's Multipliers

Most optimization problems encountered in the real world are constrained by external circumstances. For example, a city wanting to improve its roads has only limited number of tax dollars it can spend on the project.
Constrained optimization is the maximization or minimization of an objective function subject to constraints on the possible values of the independent variable(s). Constraints can be either equality constraints or inequality constraints. In this section, we see how to find an optimum value of a function of two variables subject to some constraints using a graphical approach and an analytical approach that employs the so-called **Lagrange's Multipliers**.

Graphical Approach

We consider the following example. Let $f(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ represent the production of a product as a function of the quantities of two raw materials specified by x and y. The quantities of these raw materials are constrained by the budget available to purchase them. If x and y cost \$1000 per unit and \$3780 is the budget available to purchase them, then what is the maximum production that can be obtained under these circumstances?
Mathematically, we are asked to maximize the function f subject to the constraint $g(x,y) \leq 3.78$ where $g(x,y) = x + y$.
Graphically, the line $x + y = 3.78$ represents all the combinations of raw materials that just exhaust the budget but are still affordable. Points below this line, do not exhaust the budget but are still affordable. Points above the line are unaffordable.
The maximum production can be located graphically by plotting contours of f on a plot containing the line $x + y = 3.78$ as shown in Figure 21.1.
To maximize f we find the point which lies on the level curve with the largest possible value of f and which lies on (or below) the line $x + y = 3.78$. This figure shows that at the maximum, f must be tangent to the constraint line

since if we are on a contour to the left of the point of tangency, we can increase the value of f by moving to the right along the budget constraint curve until we reach the point of tangency. Likewise if we are to the right of the point of tangency, we can move left to the point of tangency and increase f. Also, note that the maximum value of f is about 2.
In general, the maximum of $f(x,y)$ is located at a point where the constraint curve $g(x,y)$ is tangent to a level curve of f.

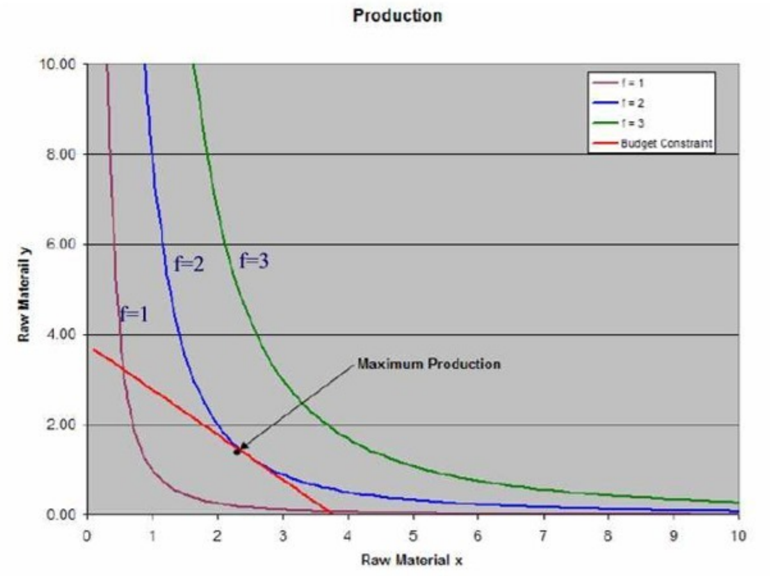

Figure 21.1

Analytical Approach: Lagrange Multipliers
As noted above, the maximum production is achieved at the point where the constraint is tangent to a level curve of the production function. The method of Lagrange Multipliers uses this fact in algebraic form to calculate the maximum.
From Figure 21.1, we see that $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel so that $\nabla f(x,y) = \lambda \nabla g(x,y)$ for some λ which we call the **Lagrange multiplier.** We therefore have

$$\frac{2}{3}\sqrt[3]{\frac{y}{x}}\vec{i} + \frac{1}{3}\sqrt[3]{\left(\frac{x}{y}\right)^2}\vec{j} = \lambda\vec{i} + \lambda\vec{j}.$$

Hence,

$$\frac{2}{3}\sqrt[3]{\frac{y}{x}} = \lambda \quad \text{and} \quad \frac{1}{3}\sqrt[3]{\left(\frac{x}{y}\right)^2} = \lambda$$

Eliminating λ gives

$$\frac{2}{3}\sqrt[3]{\frac{y}{x}} = \frac{1}{3}\sqrt[3]{\left(\frac{x}{y}\right)^2} \text{ which leads to } 2y = x$$

Substituting this into the equation $x+y = 3.78$ we find $x = 2.52$ and $y = 1.26$. Hence,

$$f(2.52, 1.26) = (2.52)^{\frac{2}{3}}(1.26)^{\frac{1}{3}} \approx 2.$$

As before, we see that the maximum value of f is approximately 2. Also, note that $\lambda \approx 0.53$

Generalization

We are given a constraint equation

$$g(x, y) = c$$

and an objective function $f(x, y)$. The goal is to find the maximum and minimum values of f among the values taken on by f along the constraint curve; i.e., the set of points for which $g(x, y) = c$. Moreover, we would like to find all of the points (x, y) at which these maxima and minima are attained. This is provided by the following theorem. Before, stating the result, we need to define what we mean by a maximum or a minimum: A point P satisfying the constraint is called a **global maximum** (respectively a **global minimum**) of f if $f(P) \geq f(Q)$ (respectively $f(P) \leq f(Q)$ for all points Q on the constraint.

Theorem 21.1

If there is a maximum or a minimum of the values that the function $f(x, y)$ assumes on the constraint curve $g(x, y) = c$, then it occurs at a point at P which

$$\nabla f(P) = \lambda \nabla g(P) \text{ and } g(P) = c$$

or at an endpoint of the constraint, or at points where ∇g is the zero vector on the constraint. Compare the values of f at these points. The largest value is the global maximum and the smallest value is the global minimum.

If the set of points satisfying the constraint is closed and bounded (such as a circle or a line segment) then there must be a global maximum and a global minimum of f subject to the constraint. If the constraint is not closed and bounded, such as a line or a hyperbola, then there may or may not be a global maximum or a minimum.

Example 21.1
Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution.
Since $\nabla f(x,y) = 5\vec{i} - 3\vec{j}$ and $\nabla g(x,y) = 2x\vec{i} + 2y\vec{j}$, we must have $2\lambda x = 5$ and $2\lambda y = -3$. Eliminating λ we find $\frac{5}{2x} = -\frac{3}{2y}$ and this leads to $y = -\frac{3}{5}x$. Substituting into the constraint equation we find $x^2 + \frac{9}{25}x^2 = 136$. Solving this equation for x we find $x = \pm 10$. If $x = -10$ then $y = 6$ and if $x = 10$ then $y = -6$. Note that the constraints has no endpoints and that $\nabla g \neq \vec{0}$ on the circle. Since, $f(-10,6) = -68$ and $f(10,-6) = 68$, the maximum of f occurs at the point $(10,-6)$ and the minimum occurs at the point $(-10,6)$ ■

Optimization with Inequality Constraints
To this point we have only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we have been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality $g(x,y) < c$ in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.
Let's work an example to see how these kinds of problems work.

Example 21.2
Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Solution.
The first step is to find all the critical points that are inside the disk (i.e. satisfy the constraint $x^2 + y^2 < 4$.). We have

$$\begin{aligned} f_x(x,y) =& 8x = 0 \\ f_y(x,y) =& 20y = 0 \end{aligned}$$

So $(0,0)$ is the only critical point of f satisfying $x^2+y^2<4$.
Next, we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. Since $\nabla f(x,y)=8x\vec{i}+20y\vec{j}$ and $\nabla g(x,y)=2x\vec{i}+2y\vec{j}$, we must have $2\lambda x=8x$ and $2\lambda y=20y$. These equations imply

$$\begin{aligned}2x(\lambda-4)=&0\\2y(\lambda-10)=&0.\end{aligned}$$

If $x=0$ we find $y=\pm2$. If $x\neq0$ then $\lambda=4$ and so the second equation gives $y=0$ and so $x=\pm2$. If we had performed a similar analysis on the second equation we would arrive at the same points.
So, Lagrange Multipliers gives us four points to check : $(0,-2),(0,2),(-2,0),(2,0)$.
Now, since

$$\begin{aligned}f(0,0)=&0\\f(2,0)=&f(-2,0)=16\\f(0,-2)=&f(0,2)=40\end{aligned}$$

the maximum of f occur at the points $(0,-2)$ and $(0,2)$ and the minimum occurs at $(0,0)$ ■

Practical Interpretation of λ

Let (x^*,y^*) be an optimum value. Then its location depends on c where $g(x,y)=c$. Thus, $x^*=x^*(c)$ and $y^*=y^*(c)$. Moreover, we can look to f as a composite function $f(x^*(c),y^*(c))$. Using the chain rule we can write

$$\frac{df}{dc}=\frac{\partial f}{\partial x}\frac{dx^*}{dc}+\frac{\partial f}{\partial y}\frac{dy^*}{dc}$$

However, we have

$$\tfrac{\partial f}{\partial x}(x^*,y^*)=\lambda\tfrac{\partial g}{\partial x}(x^*,y^*)\text{ and }\tfrac{\partial f}{\partial y}(x^*,y^*)=\lambda\tfrac{\partial g}{\partial y}(x^*,y^*)$$

so that

$$\frac{df}{dc}=\lambda\frac{\partial g}{\partial x}\frac{dx^*}{dc}+\lambda\frac{\partial g}{\partial y}\frac{dy^*}{dc}=\lambda\frac{dg}{dc}.$$

Since $g(x^*(c),y^*(c))=c$ we must have $\frac{dg}{dc}=1$. Thus,

$$\frac{df}{dc}=\lambda$$

This says that λ is the rate of change of the optimum value of f as c increases. For example, in the budget function discussed earlier, an increase of \$1000 in the budget will lead to an increase of about 0.53 unit in production.