Arkansas Tech University
MATH 2934: Calculus III
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12.3 Double Integrals in Polar Coordinates

There are regions in the plane that are not easily used as domains of iterated integrals in rectangular coordinates. For instance, regions such as a disk, ring, or a portion of a disk or ring.
We start by recalling the relationship between Cartesian and polar coordinates. The Cartesian system consists of two rectangular axes. A point P in this system is uniquely determined by two numbers x and y as shown in Figure 12.3.1(a). The polar coordinate system consists of a point O, called the **pole,** and a half-axis starting at O and pointing to the right, known as the **polar axis**. A point P in this system is determined by two numbers: the distance r between P and O and an angle θ between the ray OP and the polar axis as shown in Figure 12.3.1(b).

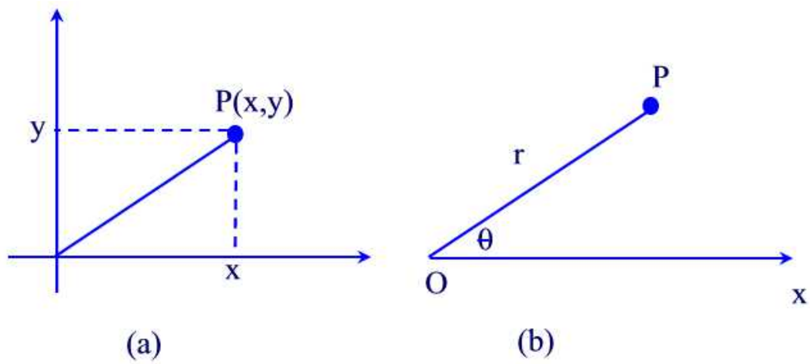

Figure 12.3.1

The Cartesian and polar coordinates can be combined into one figure as shown in Figure 12.3.2.
Figure 12.3.2 reveals the relationship between the Cartesian and polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta \qquad y = r\sin\theta \quad \tan\theta = \frac{y}{x}.$$

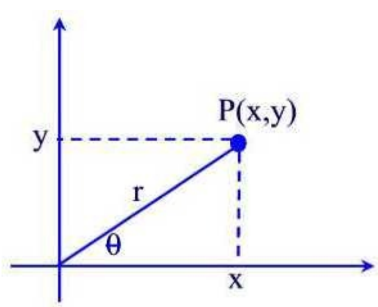

Figure 12.3.2

A double integral in polar coordinates can be defined as follows. Suppose we have a region

$$R = \{(r, \theta) : a \le r \le b, \alpha \le \theta \le \beta\}$$

as shown in Figure 12.3.3(a).

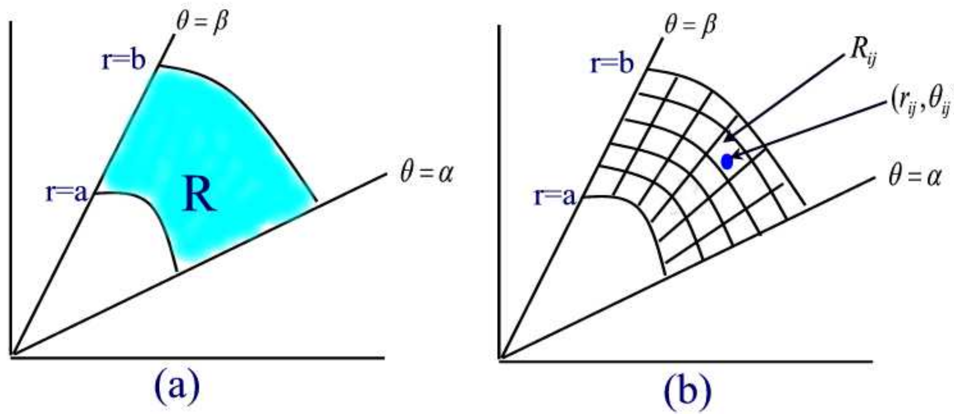

Figure 12.3.3

Partition the interval $\alpha \le \theta \le \beta$ into m equal subintervals, and the interval $a \le r \le b$ into n equal subintervals, thus obtaining mn subrectangles as shown in Figure 12.3.3(b). Choose a sample point (r_i, θ_j) in the subrectangle R_{ij} defined by $r_{i-1} \le r \le r_i$ and $\theta_{j-1} \le \theta \le \theta_j$. Then

$$\int\int_R f(x,y)dxdy \approx \sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i, \theta_j)\Delta A_{ij}$$

where ΔA_{ij} is the area of the subrectangle R_{ij}.
To calculate the area of R_{ij}, look at Figure 12.3.4. If Δr and $\Delta\theta$ are small then R_{ij} is approximately a rectangle with area $r_i \Delta r \Delta\theta$ so

$$\Delta R_{ij} \approx r_i \Delta r \Delta\theta.$$

Thus, the double integral can be approximated by a Riemann sum

$$\int\int_R f(x,y)dxdy \approx \sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i, \theta_j) r_i \Delta r \Delta\theta$$

Taking the limit as $m, n \to \infty$ we obtain

$$\int\int_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r,\theta) r dr d\theta.$$

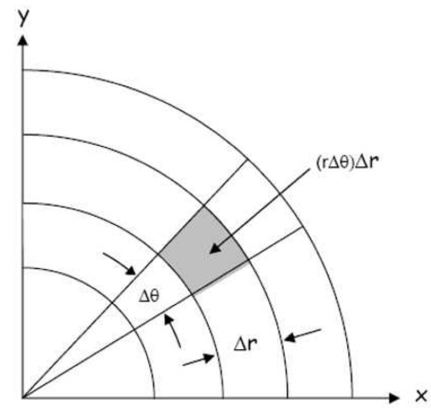

Figure 12.3.4

Example 12.3.1
Evaluate $\int\int_R e^{x^2+y^2}dxdy$ where $R : x^2 + y^2 \leq 1$.

Solution.
We have

$$\begin{aligned}\int\int_R e^{x^2+y^2}dxdy =& \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}dydx\\ =& \int_0^{2\pi}\int_0^1 e^{r^2}rdrd\theta = \left(\int_0^{2\pi}d\theta\right)\left(\int_0^1 e^{r^2}rdr\right)\\ =& [\theta]_0^{2\pi}\left[\frac{1}{2}e^{r^2}\right|_0^1\\ =& \pi(e-1) \blacksquare\end{aligned}$$

Example 12.3.2
Compute the area enclosed by the unit circle.

Solution.
The area is given by

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dydx =& \int_0^{2\pi}\int_0^1 rdrd\theta\\ =& \left(\int_0^{2\pi}d\theta\right)\left(\int_0^1 rdr\right) = [\theta]_0^{2\pi}\left[\frac{r^2}{2}\right|_0^1\\ =& \pi \blacksquare\end{aligned}$$

Example 12.3.3
Evaluate the double integral of $f(x,y) = \frac{1}{x^2+y^2}$ over the region D shown in

Figure 12.3.5.

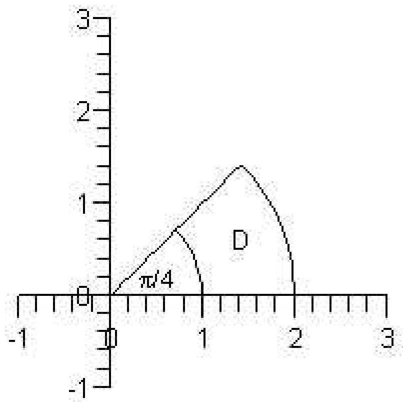

Figure 12.3.5

Solution.
We have

$$\int_0^{\frac{\pi}{4}}\int_1^2 \frac{1}{r^2} r dr d\theta = \left(\int_0^{\frac{\pi}{4}} d\theta\right)\left(\int_1^2 \frac{1}{r^2} r dr\right)$$
$$= [\theta]_0^{\pi/4}\,[\ln r]_1^2$$
$$= \frac{\pi}{4}\ln 2 \ \blacksquare$$

Example 12.3.4
For each of the regions shown in Figure 12.3.6, decide whether to integrate using rectangular or polar coordinates. In each case write an iterated integral of an arbitrary function $f(x,y)$ over the region.

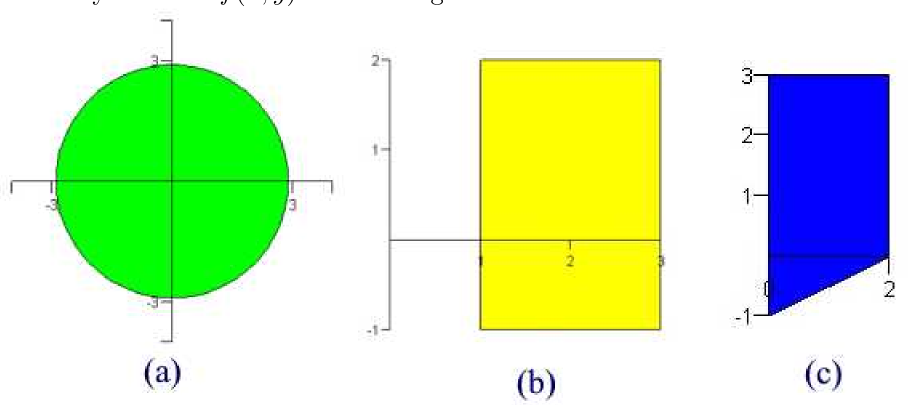

Figure 12.3.6

Solution.
(a) $\int_0^{2\pi}\int_0^3 f(r,\theta) r dr d\theta$

(b) $\int_1^3 \int_{-1}^2 f(x,y)dydx$
(c) $\int_0^2 \int_{\frac{1}{2}x-1}^3 f(x,y)dydx$ ■

What we have done so far can be extended to the more complicated type of region as shown in Figure 12.3.7.

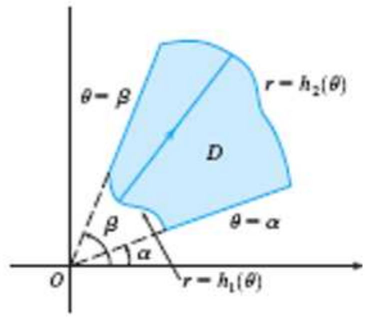

Figure 12.3.7

In an argument similar to the one for type I region of Section 12.2, one can show that

$$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example 12.3.5
Find the area enclosed by one loop of the rose $r = \cos(3\theta)$.

Solution.
One loop is given by the region $R = \{(r,\theta) : -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6},\ 0 \leq r \leq \cos(3\theta)\}$ as shown in the Figure 12.3.8.

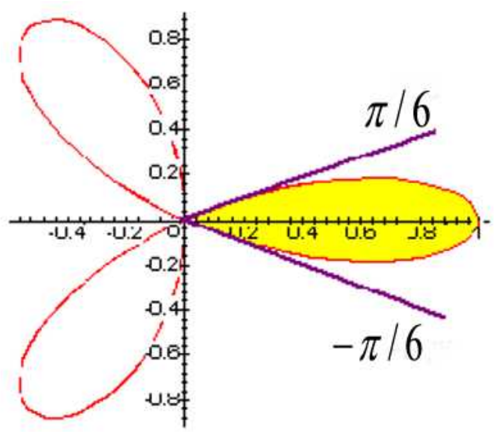

Figure 12.3.8

We have

$$
\begin{aligned}
A &= \int\int_R dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos(3\theta)} r dr d\theta \\
&= \int_{-\pi/6}^{\pi/6} \left[\frac{r^2}{2}\right]_{r=0}^{r=\cos(3\theta)} d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2(3\theta) d\theta = 2\int_0^{\pi/6} \frac{1}{2}\left(\frac{1+\cos(6\theta)}{2}\right) d\theta = \quad \frac{1}{2}\left[\frac{\theta}{2} + \frac{1}{6}\sin(6\theta)\right]_0^{\pi/6} = \frac{\pi}{12} \blacksquare
\end{aligned}
$$