

6.5 Present and Future Value of a Continuous Income Stream

When an income stream flows into an investment, the investment grows because of the continuous flows of money and the interest compounded on the money invested. Thus, two functions are required: a function defining the flow of money, and a function defining a function multiplier.

Discrete Income Stream

Many business deals involve payments in the future. For example, when a car or a home is bought on credits, payments are made over a period of time. The **future value**, FV , of a payment P is the amount to which P would have grown if deposited today in an interest bearing bank account. The **present value**, PV , of a future payment FV , is the amount that would have to be deposited in a bank account today to produce exactly FV in the account at the relevant time future.

If interest is compounded n times a year at an annual rate r for t years, then the relationship between FV and PV is given by the formula

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}.$$

In the case of continuous compound interest, the formula is given by

$$FV = PVe^{rt}.$$

Example 6.5.1

You need \$10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

Solution.

We have $FV = \$10,000$, $r = 0.08$, $t = 3$ and we want to find PV . Solving the formula $FV = PVe^{rt}$ for PV we find $PV = FVe^{-rt}$. Substituting to obtain, $PV = 10,000e^{-0.24} \approx \$7,866.28$. ■

Continuous Income Stream

In the above discussion we introduced the present and future value of a single payment. Next, we want to calculate the present and future value of a continuous stream of payments.

The revenues earned by a huge corporation (such as an electricity company), for example, come in essentially all the time, and therefore they can be represented by a continuous income stream.

Since the rate at which revenue is earned may vary from time to time, the income stream is described by a function $S(t)$ which represents the flow rate in dollars per year. Note that the rate depends on time t , usually measured in years from the present.

To find the present value of a continuous income stream over a period of M years we divide the interval $[0, M]$ into n equal subintervals each of length $\Delta t = \frac{M}{n}$ and with division points $0 = t_0 < t_1 < \dots < t_n = M$. That is, over each time interval we are assuming a single payment is made. Assuming interest r is compounded continuously, the present value of the total money deposited is approximated by the following Riemann sum:

$$PV \approx S(t_1)e^{-rt_1} \Delta t + S(t_2)e^{-rt_2} \Delta t + \dots + S(t_n)e^{-rt_n} \Delta t = \sum_{i=1}^n S(t_i)e^{-rt_i} \Delta t.$$

Letting $\Delta t \rightarrow 0$, i.e. $n \rightarrow \infty$, we obtain

$$PV = \int_0^M S(t)e^{-rt} dt.$$

The future value is given by

$$FV = e^{rM} \int_0^M S(t)e^{-rt} dt.$$

Example 6.5.2

An investor is investing \$3.3 million a year in an account returning 9.4% APR. Assuming a continuous income stream and continuous compounding of interest, how much will these investments be worth 10 years from now?

Solution.

Using the formula for the future value defined above we find

$$FV = e^{.94} \int_0^{10} 3.3e^{-0.094t} dt \approx \$54.8 \text{ million.} \blacksquare$$

Example 6.5.3

At what constant, continuous rate must money be deposited into an account if the account contain \$20,000 in 5 years? The account earns 6% interest compounded continuously.

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Solution.

Given $FV = \$20,000$, $M = 5$, $r = 0.06$. Since S is assumed to be constant then we have

$$20,000 = S \int_0^5 e^{-0.06t} dt.$$

Solving for S we find

$$S = \frac{20,000}{\int_0^5 e^{-0.06t} dt} \approx \$4,630 \text{ per year.} \blacksquare$$