6.3 Using the Fundamental Theorem to Find Definite Integrals

Recall the Fundamental Theorem of Calculus (abbreviated by FTC): If $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$. In particular, we have

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Once we have found an antiderivative of $f(x)$, computing definite integrals is easy by the Fundamental Theorem of Calculus.

Example 6.3.1
Compute $\int_1^2 3x^2dx$.

Solution.
Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, then by FTC

$$\int_1^2 3x^2dx = x^3|_1^2 = 2^3 - 1^3 = 7 \blacksquare$$

Example 6.3.2
Write a definite integral to represent the area under the graph of $f(t) = e^{0.5t}$ between $t = 0$ and $t = 4$. Use the Fundamental Theorem of Calculus to calculate the area.

Solution.
An antiderivative of $f(t)$ is $2e^{0.5t}$. Thus,

$$Area = \int_0^4 e^{0.5t}dt = 2e^{0.5t}\Big|_0^4 = 2e^2 - 2 \approx 12.778 \blacksquare$$

Next, we discuss the evaluation of a definite integral using the technique of substitution. From the Chain Rule of differnetiation, we have that $f(g(x))$ is an antiderivative of the function $f'(g(x))g'(x)$. Applying the Fundamental Theorem of Calculus we can write

$$\int_a^b f'(g(x))g'(x)dx = f(g(x))|_a^b = f(g(b)) - f(g(a)).$$

If we let $u = g(x)$ then the previous formula reduces to

$$\int_a^b f'(g(x))g'(x)dx = f(g(b)) - f(g(a)) = \int_{g(a)}^{g(b)} f'(u)du.$$

Warning: When evaluating definite integrals, there is no constant of integration in the final answer.

Example 6.3.3
Compute $\int_0^2 xe^{x^2}dx$.

Solution.
Let $u(x) = x^2$. Then $du = 2xdx, u(0) = 0$, and $u(2) = 4$. Thus,

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}\int_0^4 e^u du = \frac{e^u}{2}|_0^4 = \frac{1}{2}(e^4 - 1) \blacksquare$$