6.2 Constructing Antiderivatives Analytically

In this section we will find analytical expressions of antiderivatives. Recall that a function F is an **antiderivative** of a function f if $F'(x) = f(x)$. However, for any constant C, $F(x) + C$ is also an antiderivative of f. That is, there are infinitely many antiderivatives of a given function $f(x)$. They all differ by a constant and the family of antiderivatives is represented by $F(x) + C$. The notation of the general antiderivative is called an **indefinite integral** and is written

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is the symbol of integration, $f(x)$ is called the **integrand** and C is called the **constant of integration**. Keep in mind the relationship between $f(x)$ and $F(x)$ which is given by $F'(x) = f(x)$.

Warning: The indefinite integral is a short-hand notation for a family of functions $F(x) + C$ with the property $F'(x) = f(x)$ for all x. It is not to be confused with the definite integral $\int_a^b f(x)dx$ which is a real number.

Example 6.2.1
Show that $\int 0dx = C$.

Solution.
Since the derivative of a constant function is always zero, we have

$$\int 0dx = C \blacksquare$$

Example 6.2.2
Show that $\int kdx = kx + C$ where k is a constant.

Solution.
Since the derivative of kx is just k,

$$\int kdx = kx + C \blacksquare$$

Example 6.2.3
Show that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.

Solution.
By the power rule, if $F(x) = \frac{x^{n+1}}{n+1}$ then $F'(x) = x^n$. Thus,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \blacksquare$$

Note that this formula is valid only if $n \neq -1$ for if $n = -1$ we would have $\frac{x^0}{0}$ which doesn't make sense. The case $n = -1$ is treated in the next problem.

Example 6.2.4
Show that

$$\int \frac{dx}{x} = \ln|x| + C.$$

Solution.
Suppose first that $x > 0$ so that $\ln|x| = \ln x$. Then $(\ln|x|)' = (\ln x)' = \frac{1}{x}$. Now, if $x < 0$ then $\ln|x| = \ln(-x)$ and by the chain rule $(\ln|x|)' = (\ln(-x))' = \frac{-1}{-x} = \frac{1}{x}$. Thus, in both cases $(\ln|x|)' = \frac{1}{x}$ $\blacksquare$

Example 6.2.5
Show that for $a \neq 0$, $\int e^{ax}dx = \frac{e^{ax}}{a} + C$.

Solution.
If a is a nonzero constant and $F(x) = \frac{e^{ax}}{a}$ then $F'(x) = e^{ax}$. This shows that

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C \blacksquare$$

Properties of Indefinite Integrals

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$$

To see why this property is true, let $F(x)$ be an antiderivative of $f(x)$ and $G(x)$ be an antiderivative of $g(x)$. The result follows from the fact that $\frac{d}{dx}[F(x) \pm G(x)] = f(x) \pm g(x)$.

$$\int cf(x)dx = c\int f(x)dx.$$

To see this, suppose that $F(x)$ is an antiderivative of $f(x)$. Then $\int f(x)dx = F(x) + C$. But $\frac{d}{dx}(cF(x)) = cf(x)$ so that $cF(x)$ is an antiderivative of $cf(x)$, that is, $\int cf(x)dx = cF(x) + C'$. This implies

$$\int cf(x)dx = cF(x)+C' = c(\int f(x)dx-C)+C' = c\int f(x)dx-cC+C' = c\int f(x)dx.$$

Note that the constant $-cC + C'$ is ignored since a constant of integration will result from $\int f(x)dx$.

Example 6.2.6
Find

$$\int (e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx.$$

Solution.
Using the linearity property of indefinite integrals together with the formulas of integration obtained above we have

$$\begin{aligned}\int (e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx &= \int e^{-3x}dx + 3\int \frac{dx}{x} - 5\int x^{-3}dx \\ &= -\frac{e^{-3x}}{3} + 3\ln|x| + \frac{5}{2x^2} + C \blacksquare\end{aligned}$$

Once we have found an antiderivative of $f(x)$, computing definite integrals is easy by the Fundamental Theorem of Calculus.

Example 6.2.7
Compute $\int_1^2 3x^2dx$.

Solution.
Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, by FTC we have

$$\int_1^2 3x^2dx = x^3|_1^2 = 2^3 - 1^3 = 7 \blacksquare$$