5.5 The Fundamental Theorem of Calculus

The following result is considered among the most important result in calculus.

The Fundamental Theorem of Calculus
If $f(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

We call the function $F(x)$ an **antiderivative** of $f(x)$.

Proof.
Partition the interval $[a, b]$ into n subintervals each of length $\Delta x = \frac{b-a}{n}$ and let $a = x_0 < x_1 < \cdots < x_n = b$ be the partition points. Applying the Mean Value Theorem on the interval $[x_0, x_1]$ we can find a number $x_0 < c_1 < x_1$ such that

$$F(x_1) - F(x_0) = F'(c_1)\Delta x.$$

Continuing this process on the remaining intervals we find

$$F(x_2) - F(x_1) = F'(c_1)\Delta x$$
$$\vdots$$
$$F(x_n) - F(x_{n-1}) = F'(c_n)\Delta x$$

Adding these equalities we find

$$F(x_n) - F(x_0) = \sum_{i=1}^{n} f(c_i)\Delta x.$$

Letting $n \to \infty$ to obtain

$$F(b) - F(a) = \int_a^b f(x)dx \blacksquare$$

Example 5.5.1
Use FTC to compute $\int_1^2 2xdx$. Use a calculator to find the answer to the integral and compare.

Solution.
Since the derivative of x^2 is $2x$, $F(x) = x^2$. Thus, by the FTC we have

$$\int_1^2 2xdx = F(2) - F(1) = 4 - 1 = 3.$$

Using a calculator we find $\int_1^2 2xdx = 3$ ∎

Example 5.5.2
Let $F(t)$ represent a bacteria population which is 5 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Solution.
Since total change is the definite integral of $F'(t) = 2^t$ from $t = 0$ to $t = 1$, we have

$$\text{Change in population} = F(1) - F(0) = \int_0^1 2^t dt \approx 1.44 \text{ million bacteria.}$$

Since $F(0) = 5$, we find

$$F(1) = F(0) + \int_0^1 2^t dt \approx 5 + 1.44 = 6.44 \text{ million.}$$

If $C(q)$ is the total cost to produce a quantity q of a certain comodity then we can use the Fundamental Theorem of Calculus and compute the total cost of producing b units as follows

$$C(b) - C(0) = \int_0^b C'(q)dq$$

or

$$C(b) = C(0) + \int_0^b C'(q)dq.$$

We call the quantity $\int_0^b C'(q)dq$ the **total variable cost** ∎

Example 5.5.3
The marginal cost function for a company is given by

$$C'(q) = q^2 - 16q + 70 \; dollars/unit,$$

where q is the quantity produced. If $C(0) = 500$, find the total cost of producing 20 units. What is the fixed cost and what is the total variable cost for this quantity?

Solution.
We find $C(20)$ as follows:

$$C(20) = C(0) + \int_0^{20} C'(q)dq = 500 + \int_0^{20} (q^2 - 16q + 70)dq$$

where $C(0) = 500$ is the fixed cost.
Using a calculator we find total variable cost to be

$$\int_0^{20} (q^2 - 16q + 70)dq \approx 866.7.$$

Thus, the total cost of producing 20 units is

$$C(20) \approx 500 + 866.7 = 1366.7 \blacksquare$$