5.4 The Definite Integral of a Derivative

We start this section by showing that the definite integral of a rate of change gives the total change of the function. We define the total change of a function $F(t)$ from $t = a$ to $t = b$ to be the difference $F(b) - F(a)$. Suppose that $F(t)$ is continuous in $[a, b]$ and differentiable in (a, b). Divide the interval $[a, b]$ into n equal subintervals each of length $\Delta t = \frac{b-a}{n}$. Let $a = t_0 < t_1 < \cdots < t_n = b$ be the partition points of the subdivision. Then on the interval $[t_0, t_1]$ the change in F can be estimated by the formula

$$F'(t_0) \approx \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}$$

or

$$F(t_0 + \Delta t) - F(t_0) \approx F'(t_0)\Delta t.$$

That is

$$F(t_1) - F(t_0) \approx F'(t_0)\Delta t.$$

On the interval $[t_1, t_2]$ we get the estimation

$$F(t_2) - F(t_1) = F'(t_1)\Delta t.$$

Continuing in this fashion we find that on the interval $[t_{n-1}, t_n]$ we have

$$F(t_{n-1}) - F(t_n) \approx F'(t_{n-1})\Delta t.$$

Adding all these approximations we find that

$$F(t_n) - F(t_0) \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t.$$

Letting $n \to \infty$ we see that

$$F(b) - F(a) = \int_a^b F'(t)dt.$$

Example 5.4.1
The amount of waste a company produces, W, in metric tons per week, is approximated by $W = 3.75e^{-0.008t}$, where t is in weeks since January 1, 2000. Waste removal for the company costs \$15/ton. How much does the company pay for waste removal during the year 2000?

Solution.
The amount of tons produced during the year 2000 is just the definite integral $\int_0^{52} W(t)dt$. Using a calculator we find that

$$\text{Total waste during the year} = \int_0^{52} 3.75e^{-0.008t}dt \approx 159 \; tons$$

The cost to remove this quantity is $159 \times 15 = \$2385$ ■

Remark 5.4.1
When using $\int_a^b f(t)dt$ in applications then its units is the product of the units of $f(t)$ with the units of t.