# 5.3 The Definite Integral as Area

In this section, we will see how definite integrals are used to find areas.

**Case 1:**$f(x) \geq 0$

Looking closely to either the left Riemann sum or the right Riemann sum we see that if $f(x) \geq 0$ then a term of the form $f(x)\Delta x$ represents the area of a rectangle. As $n$ increases without bound, that is, the width $\Delta x$ of the rectangles approaches zero, the rectangles fit the curve of the graph more exactly, and the sum of their areas gets closer and closer to the area under the graph, bounded by the vertical lines $x = a$ and $x = b$ and the $x-$axis. Thus,

$$\int_a^b f(x)dx = \text{Area under graph of f between a and b.}$$

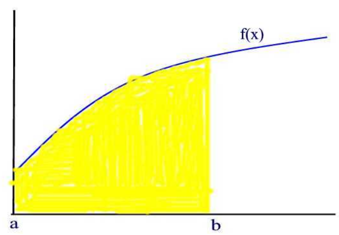


Figure 5.3.1

**Example 5.3.1**

Consider the integral $\int_{-1}^{1}\sqrt{1-x^2}dx$.
(a) Interpret the integral as an area, and find its exact value.
(b) Estimate the integral using a calculator.

**Solution.**

(a) Note that the equation of a circle centered at the origin and with radius 1 is given by $x^2 + y^2 = 1$. Solving for $y$ we find $y = \pm\sqrt{1-x^2}$. The function $y = \sqrt{1-x^2}$ corresponds to the upper semicircle and the function $y = -\sqrt{1-x^2}$ corresponds to the lower semicircle. See Figure 5.3.2.

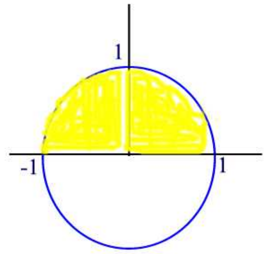


Figure 5.3.2

It follows that the given integral represents the area of the upper semicircle and therefore is equal to $\frac{\pi}{2}$. That is,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2}.$$

(b) Using a TI-83 calculator we find

$$fnInt(\sqrt{1-x^2}, x, -1, 1) \approx 1.571 \blacksquare$$

<u>**Case 2:**$f(x) \leq 0$</u>
In this case, since each product of the form $f(x)\Delta x$ is less than or equal to zero, the area gets counted negatively. That is, the absolute value of the integral gives the area above the curve between $x = a$ and $x = b$.

**Example 5.3.2**
Find the area above the graph of $y = x^2 - 1$ from $x = -1$ to $x = 1$.

**Solution.**
The graph of $y = x^2 - 1$ is shown in Figure 5.3.3.

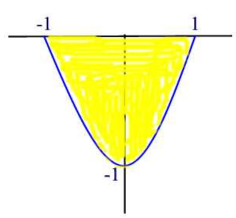


Figure 5.3.3

The area is given by $|\int_{-1}^{1}(x^2-1)dx| \approx |-1.33| = 1.33$ ■

**<u>Case 3:</u>** $f$ changes sign
In this case, the integral is the sum of the areas above the $x-$axis, counted positively, and the areas below the $x-$axis, counted negatively. If the integral is positive then the region above the $x-$axis has larger area than the region below the $x-$axis. If the integral is negative then the region below the $x-$axis has a larger area then the region above the $x-$axis.

**Example 5.3.3**
Find the area between the graph of $y = x^3$ and the x-axis from $x = -1$ to $x = 1$.

**Solution.**
The area is shown in Figure 5.3.4.

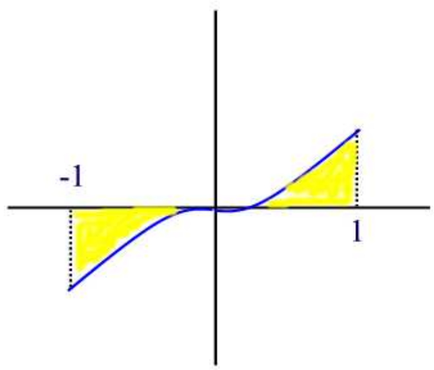


Figure 5.3.4

It follows that the area is given by

$$\left|\int_{-1}^{0} x^3 dx\right| + \int_{0}^{1} x^3 dx = 0.5 \blacksquare$$

**<u>Area Between Two Curves</u>**
Consider the problem of finding the area between two curves as shown in

Figure 5.3.5.

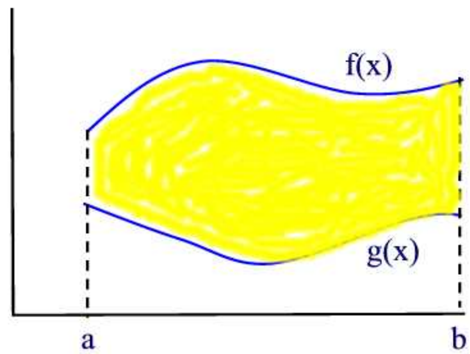


Figure 5.3.5

Then, the area between the two curves is the area under $f$ minus the area under $g$. That is,

$$\text{Area between f and g} = \int_a^b (f(x) - g(x))dx.$$

**Example 5.3.4**
Find the area between the graphs of $f(x) = x$ and $g(x) = x^2$.

**Solution.**
The area is shown in Figure 5.3.6.

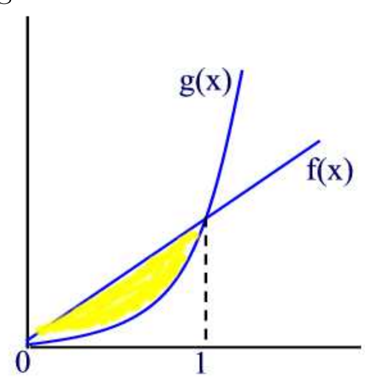


Figure 5.3.6

Thus, the area is given by the integral

$$\int_0^1 (x - x^2)dx = \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{1}{6} \blacksquare$$