5.1 Accumulated Change

We have seen that the velocity of an object moving along the curve $s(t)$ is obtained by taking the average rate of change on smaller and samller intervals, that is finding the derivative of s, i.e.,

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}. \tag{5.1.1}$$

In this and the following sections we want to go the opposite direction. That is, given the velocity function $v(t)$ we want to find the position function $s(t)$. Now, from equation (5.1.1) the distance traveled from time t to time $t+h$ can be approximated by the product

$$s(t+h) - s(t) \approx v(t)h.$$

Thus, the product $v(t)h$ measures the change in the function $s(t)$ between t and $t+h$. Geometrically, this product can be viewed as the area of the rectangle of length $v(t)$ and width h.

Example 5.1.1
Suppose that the velocity of the car is given every two seconds as shown in the table below. Find the lower and upper estimates of the total distance traveled after 10 seconds? What is the difference between the lower and upper estimates?

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

Solution.
We first find and underestimate of the total distance traveled. For the first two seconds, the velocity is at least 20 feet per second so that the distance traveled is at least $20 \times 2 = 40$ feet. Likewise, at least $30 \times 2 = 60$ feet has been traveled the next two seconds and so on. Thus, we obtain a lower estimate to the exact distance traveled

$$20 \times 2 + 30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 = 360 \text{ feet}.$$

However, we can reason differently and get an overestimate to the total distance traveled as follows: For the first two seconds the car's velocity is at most 30 feet so that the car travels at most $30 \times 2 = 60$ feet. In the next two seconds, it travels $38 \times 2 = 76$ feet and so on. So an upper estimate of the total distance traveled is

$$30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 + 50 \times 2 = 420 \text{ feet}.$$

Hence,

$$360 \text{ feet} \leq \text{ Total distance traveled} \leq 420 \text{ feet}.$$

Notice that the difference between the upper and lower estimates is 60 feet ■
Figure 5.1.1 shows both the lower estimate and the upper estimate. The graph of the velocity is obtained by plotting the points given in the above table and then connect them with a smooth curve. The area of the lower rectangles represent the lower estimate and the larger rectangles represent the upper estimate.

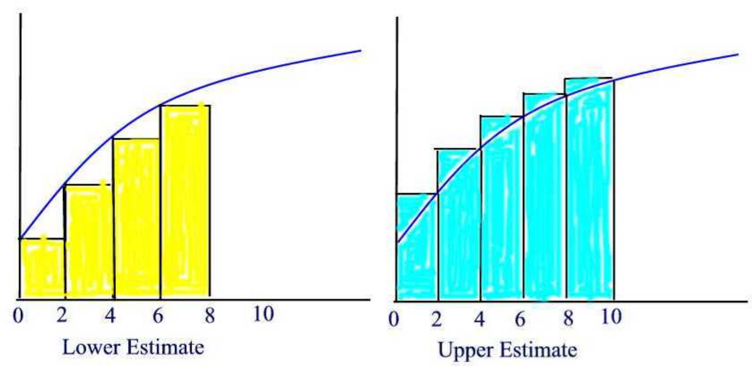

Figure 5.1.1

Remark 5.1.1
Note that the understimate and the oversestimate are basically sums of changes in the value of $s(t)$ and therefore are referred to as **accumulated changes**.

Example 5.1.2
Suppose that the velocity of the car is given every second instead as shown in the table below. Find the lower and upper estimates of the total distance traveled. What is the difference between the lower and upper estimates? Do you think that knowing the velocity at every second is a better estimate than

knowing the velocity every two seconds?

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50

Solution.
The lower estimate is

$$(20)(1) + (26)(1) + \cdots + (48)(1) + (49)(1) = 378 \text{ feet}$$

and the upper estimate is

$$(26)(1) + (30)(1) + \cdots + (49)(1) + (50)(1) = 408 \text{ feet.}$$

Hence,

$$378 \text{ feet} \leq \text{ Total distance traveled} \leq 408 \text{ feet.}$$

So the difference between the upper and lower estimates is $408-378 = 30$ feet. This shows that by increasing the partition points we get better and better estimate ■

Remark 5.1.2
Once the upper estimate and the lower estimate are found then one can get an even better estimate by taking the average of the two estimates.

The use of the rate of change of the distance leads to estimating the total distance traveled. This same method can be used to estimate the total change from the rate of change of other quantities.

Example 5.1.3
The following table gives world oil consumptions, in billions of barrels per year. Estimate the total oil consumption during this $20-$year period.

Year	1980	1985	1990	1995	2000
Oil (barrels/yr)	22.3	23.0	23.9	24.9	27.0

Solution.
The underestimate of the total oil consumption is:

$$22.3 \times 5 + 23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 = 470.5 \text{ billion barrels.}$$

The overestimate is

$$23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 + 27.0 \times 5 = 494 \text{ billion barrels.}$$

A good single estimate of the total oil consumption is the average of the above estimates. That is

$$\text{Total oil consumption} \approx \frac{470.5 + 494}{2} = 482.25 \text{ billion barrels} \blacksquare$$