

4.4 Applications of Optimization to Marginality

Management of most businesses always aim to maximizing profit. In this section we will use the derivative to optimize profit and revenue functions.

Optimizing Profit

Recall that the profit resulting from producing and selling q items is defined by

$$P(q) = R(q) - C(q)$$

where $C(q)$ is the total cost of producing a quantity q and $R(q)$ is the total revenue from selling a quantity q of some good.

To maximize or minimize profit over a closed interval, we optimize the profit function P . We know that global extrema occur at the critical numbers of P or at the endpoints of the interval. Thus, the process of optimization requires finding the critical numbers which are the zeros of the marginal profit function

$$P'(q) = R'(q) - C'(q) = 0$$

where $R'(q)$ is the marginal revenue function and $C'(q)$ is the marginal cost function. Thus, the global maximum or the global minimum of P occurs when

$$MR(q) = MC(q)$$

or at the endpoints of the interval.

Example 4.4.1

Find the quantity q which maximizes profit given the total revenue and cost functions

$$\begin{aligned} R(q) &= 5q - 0.003q^2 \\ C(q) &= 300 + 1.1q. \end{aligned}$$

where $0 \leq q \leq 800$ units. What production level gives the minimum profit?

Solution.

The profit function is given by

$$P(q) = R(q) - C(q) = -0.003q^2 + 3.9q - 300.$$

The critical numbers of P are the solutions to the equation $P'(q) = 0$. That is,

$$3.9 - 0.006q = 0$$

or $q = 650$ units. Since $P(0) = -\$300$, $P(800) = \$900$ and $P(650) = \$967.50$, the maximum profit occurs when $q = 800$ units and the minimum profit (or loss) occurs when $q = 0$, i.e. when there is no production ■

Example 4.4.2

The total revenue and total cost curves for a product are given in Figure 4.4.1.

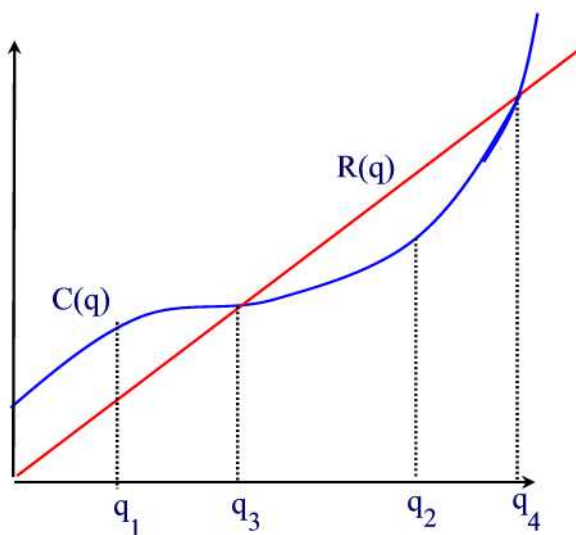


Figure 4.4.1

- (a) Sketch the curves for the marginal revenue and marginal cost on the same axes. Show on this graph the quantities where marginal revenue equals marginal cost. What is the significance of these two quantities? At which quantity is profit maximum?
- (b) Graph the profit function $P(q)$.

Solution.

(a) Since R is a straight line with positive slope, its derivative is a positive constant. That is, the graph of the marginal revenue is a horizontal line at some value $a > 0$. Since C is always increasing, its derivative MC is always positive. For $0 < q < q_3$ the curve is concave down so that MC is decreasing.

For $q > q_3$ the graph of C is concave up and so MC is increasing. Thus, the graphs of MC and MR are shown in Figure 4.4.2.

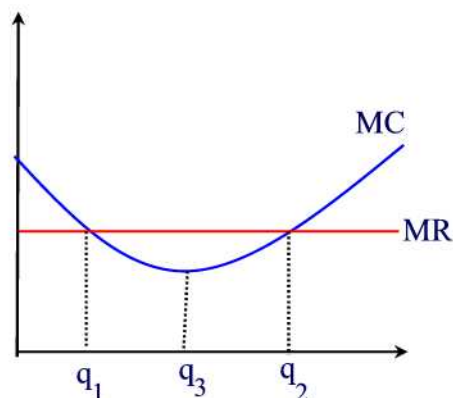


Figure 4.4.2

According to the graph, marginal revenue equals marginal cost at the values $q = q_1$ and $q = q_2$. So maximum profit occurs either at q_1, q_2 or at the endpoints. Notice that the production levels q_1 and q_2 correspond to the two points where the tangent line to C is parallel to the tangent line to R . Now, for $0 < q < q_1$ we have $MR < MC$ so that $P' = MR - MC < 0$ and this shows that P is decreasing. For $q_1 < q < q_2$, $MR > MC$ so that $P' > 0$ and hence P is increasing. So P changes from decreasing to increasing at q_1 which means that P has a minimum at q_1 . Now, for $q > q_2$ we have that $MR < MC$ so that $P' < 0$ and P is decreasing. Thus, P changes from increasing to decreasing at q_2 so q_2 is a local maximum for P . So maximum profit occurs either at the endpoints or at q_2 . Since profit is negative for $q < q_3$ and $q > q_4$, the profit is maximum for $q = q_2$.

(b) The graph of P is shown in Figure 4.4.3.

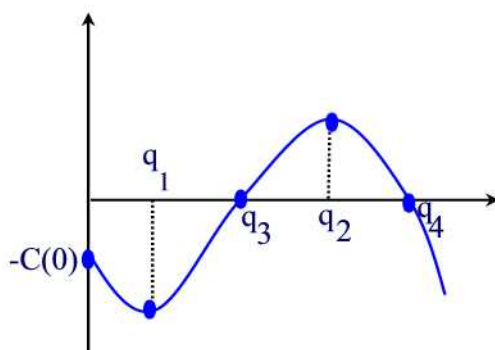


Figure 4.4.3

Optimizing Revenue

Example 4.4.3

The demand equation for a product is $p = 45 - 0.01q$. Write the revenue function as a function of q and find the quantity that maximizes revenue. What price corresponds to this quantity? What is the total revenue at this price?

Solution.

The revenue function is given by $R(q) = pq = 45q - 0.01q^2$. This is a parabola that opens down so its vertex is the global maximum. The maximum then occurs at the critical number of $R(q)$. That is, at the solution of $R'(q) = 0$ or $45 - 0.02q = 0$. Solving for q we find $q = 2250$ units. The maximum revenue is $R(2250) = \$50,625$. The unit price in this case is $p = 45 - 0.01(2250) = \$22.50$ ■