4.2 Concavity and Points of Inflection

We have seen that a local extremum is a point where the first derivative changes sign. In this section we will discuss points where the second derivative changes sign. That is, the points where the graph of the function changes concavity. We call such points **points of inflection**.

How do you find the points of inflection? Well, since f'' changes sign on the two sides of an inflection point then it makes sense to say that points of inflection occur at points where either the second derivative is 0 or undefined.

Example 4.2.1

Find the point(s) of inflection of the function $f(x) = xe^{-x}$.

Solution.

Using the product rule to obtain $f'(x) = e^{-x} - xe^{-x}$. Using the product rule for the second time we find $f''(x) = e^{-x}(x-2)$. Thus, a candidate for a point of inflection is $x = 2$. Since $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$, $x = 2$ is a point of inflection ∎

Remark 4.2.1

We have seen that not every value of x where the derivative is zero or undefined is a local maximum or minimum. The same thing applies for points of inflection. That is, it is not always true that if the second derivative is 0 or undefined then automatically you have a point of inflection. It is critical that f'' changes sign at such a point in order to have a point of inflection.

Example 4.2.2

Consider the function $f(x) = x^4$. Show that $f''(0) = 0$ but 0 is not a point of inflection.

Solution.

The second derivative is given by the formula $f''(x) = 12x^2$. Clearly, $f''(0) = 0$. Since $f''(x) \geq 0$, that is, $f''(x)$ does not change sign, 0 is not a point of inflection ∎

Example 4.2.3

Graph a function with the following properties:

- f has a critical point at $x = 4$ and an inflection point at $x = 8$.
- $f' < 0$ for $x < 4$ and $f' > 0$ for $x > 4$.
- $f'' > 0$ for $x < 8$ and $f'' < 0$ for $x > 8$.
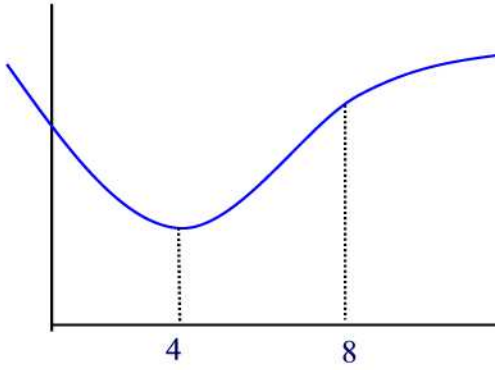
Solution.
The graph is shown in Figure 4.2.1.

Figure 4.2.1