

4.1 Local Maxima and Minima

We start this section by reviewing what the first and second derivatives of a function tell us about its graph:

- If $f'(x) > 0$ on an open interval I then $f(x)$ is increasing on I .
- If $f'(x) < 0$ on an open interval I then $f(x)$ is decreasing on I .
- If $f''(x) > 0$ on an open interval I then $f(x)$ is concave up on I .
- If $f''(x) < 0$ on an open interval I then $f(x)$ is concave down on I .

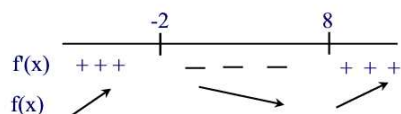
Example 4.1.1

Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

- Find the intervals where the function is increasing/decreasing.
- Find the intervals where the function is concave up/down.

Solution.

(a) Finding the first derivative we obtain $f'(x) = 3x^2 - 18x - 48 = 3(x - 8)(x + 2)$. Constructing the chart of signs below



we see that $f(x)$ is increasing on $(-\infty, -2) \cup (8, \infty)$ and decreasing on $(-2, 8)$.
 (b) Finding the second derivative, we obtain $f''(x) = 6x - 18 = 6(x - 3)$. So, $f(x)$ is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$ ■

Points of interest on the graph of a function are those points that are the highest on the curve, or the lowest, in a specific interval. Such points are called **local extrema**. The highest point, say $f(a)$, is called a **local maximum** and satisfies $f(x) \leq f(a)$ for all x in an interval I . A **local minimum** is a point $f(a)$ such that $f(a) \leq f(x)$ for all x in an interval I containing a .

Example 4.1.2

Find the local maxima and the local minima of the function given in Figure 4.1.1.

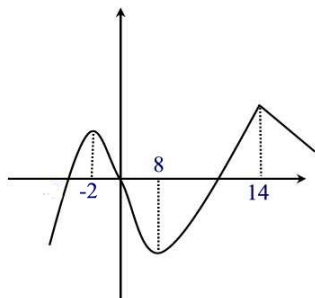


Figure 4.1.1

Solution.

The local maxima occur at $x = -2$ and $x = 14$ whereas the local minimum occurs at $x = 8$ ■

Next, we will discuss two procedures for finding local extrema. We notice from the previous example that local extrema occur at points p where the derivative is either zero or undefined. We call p a **critical number**, $f(p)$ a **critical value**, and $(p, f(p))$ a **critical point**. Most of the critical numbers that we will encounter in this book are of the form $f'(p) = 0$ type. The following theorem asserts that local extrema occur at the critical points.

Theorem 4.1.1

Suppose that f is defined on an interval I and has a local maximum or minimum at an interior point a . If f is differentiable at a then $f'(a) = 0$.

Remark 4.1.1

By Theorem 4.1.1, local extrema are always critical points. The converse of this statement is not true in general. That is, there are critical points that are not local extrema of a function. An example of this situation is given next.

Example 4.1.3

Show that $f(x) = x^3$ has a critical point at $x = 0$ but 0 is neither a local maximum nor a local minimum.

Solution.

Finding the derivative to obtain $f'(x) = 3x^2$. Setting this to 0 we find the

critical point $x = 0$. Since $f'(x)$ does not change sign at 0 we conclude that 0 is neither a local maximum nor a local minimum ■

The graph in Example 4.1.2 suggests two tests for finding local extrema. The first is known as the **first derivative test** and the second as the **second derivative test**.

First-Derivative Test

Suppose that a continuous function f has a critical point at p of the type $f'(p) = 0$.

- If f' changes sign from negative to positive at p , then f has a local minimum at p .
- If f' changes sign from positive to negative at p , then f has a local maximum at p . See Figure 4.1.2.

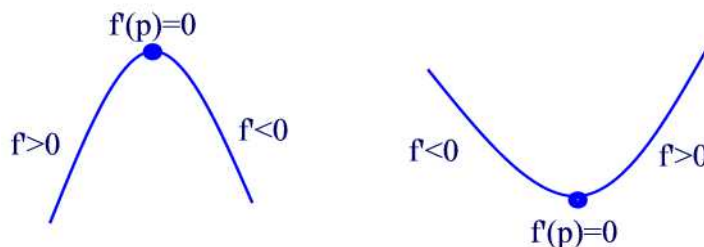


Figure 4.1.2

Example 4.1.4

Find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution.

Using the chart of signs of f' discussed in Example 4.1.1, we find that $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 8$ ■

Second-Derivative Test

Let f be a continuous function such that $f'(p) = 0$.

- if $f''(p) > 0$ then f has a local minimum at p . See Figure 4.1.3.
- if $f''(p) < 0$ then f has a local maximum at p . See Figure 4.1.3.
- if $f''(p) = 0$ then the test fails. In this case, it is recommended that you

use the first derivative test.

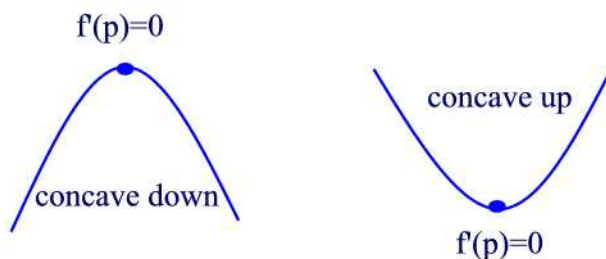


Figure 4.1.3

Example 4.1.5

Use the second derivative test to find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution.

The second derivative of $f(x)$ is given by $f''(x) = 6(x - 3)$. The critical numbers are -2 and 8 . Since $f''(-2) = -30 < 0$, $x = -2$ is a local maximum. Since $f''(8) = 30 > 0$, $x = 8$ is a local minimum ■

Example 4.1.6

Find the local extrema of the function $f(x) = x^4$.

Solution.

Let's try and find the local extrema by using the second derivative test. Since $f'(x) = 4x^3$, $x = 0$ is the only critical number. Since $f''(x) = 12x^2$, we find $f''(0) = 0$. So the second derivative test is inconclusive. Now, using the first derivative test, we see that $f'(x)$ changes sign from negative to positive at $x = 0$. Thus, $x = 0$ is a local minimum ■