Arkansas Tech University
MATH 2243: Business Calculus
Dr. Marcel B. Finan

19 The Product and Quotient Rules

At this point we don't have the tools to find the derivative of either the function $f(x) = x^3e^{x^2}$ or the function $g(x) = \frac{x^2}{e^x}$. Looking closely at the function $f(x)$ we notice that this function is the product of two functions, namely, x^3 and e^{x^2}. On the other hand, the function $g(x)$ is the ratio of two functions. Thus, we hope to have a rule for differentiating a product of two functions and one for differentiating the ratio of two functions.
We start by finding the derivative of the product $u(x)v(x)$, where u and v are differentiable functions:

$$\begin{array}{rcl}(u(x)v(x))' & = & \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ & = & \lim_{h\to 0}\frac{u(x+h)(v(x+h)-v(x))+v(x)(u(x+h)-u(x))}{h}\\ & = & \lim_{h\to 0}u(x+h)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\\ & = & u(x)v'(x)+u'(x)v(x).\end{array}$$

Note that since u is differentiable so it is continuous and therefore

$$\lim_{h\to 0}u(x+h) = u(x).$$

The formula

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{d}{dx}(v(x)) + \frac{d}{dx}(u(x))v(x). \qquad (1)$$

is called the **product rule**.

Example 19.1
Find the derivative of $f(x) = x^3e^{x^2}$.

Solution.
Let $u(x) = x^3$ and $v(x) = e^{x^2}$. Then $u'(x) = 3x^2$ and $v'(x) = 2xe^{x^2}$. Thus, by the product rule we have

$$f'(x) = x^3(2x)e^{x^2} + 3x^2e^{x^2} = 2x^4e^{x^2} + 3x^2e^{x^2} = x^2e^{x^2}(2x^2+3)\blacksquare$$

The **quotient rule** is obtained from the product rule as follows: Let $f(x) = \frac{u(x)}{v(x)}$. Then $u(x) = f(x)v(x)$. Using the product rule, we find $u'(x) = f(x)v'(x) + f'(x)v(x)$. Solving for $f'(x)$ to obtain

$$f'(x) = \frac{u'(x) - f(x)v'(x)}{v(x)}.$$

Now replace $f(x)$ by $\frac{u(x)}{v(x)}$ to obtain

$$\begin{aligned} \left(\frac{u(x)}{v(x)}\right)' &= \frac{u'(x) - \frac{u(x)}{v(x)}v'(x)}{v(x)} \\ &= \frac{\frac{u'(x)v(x) - u(x)v'(x)}{v(x)}}{v(x)} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}. \end{aligned}$$

Example 19.2
Find the derivative of $g(x) = \frac{x^2}{e^x}$.

Solution.
Let $u(x) = x^2$ and $v(x) = e^x$. Then by the quotient rule we have

$$\begin{aligned} f'(x) &= \frac{(x^2)'e^x - x^2(e^x)'}{(e^x)^2} \\ &= \frac{2xe^x - x^2e^x}{e^{2x}} \\ &= \frac{e^x x(2-x)}{e^x e^x} = \frac{x(2-x)}{e^x} \quad \blacksquare \end{aligned}$$

Example 19.3
Prove the power rule for integer exponents: $x^n = nx^{n-1}$ for integers n.

Solution.
In Section 15, we proved the result for positive integers. The result is trivially true when the exponent is zero. So suppose that $y = x^n$ with n a negative integer. Then $y = \frac{1}{x^{-n}}$ where $-n$ is a positive integer. Applying both the quotient rule and the power rule we find

$$y' = \frac{(1)'x^{-n} - (1)(x^{-n})'}{(x^{-n})^2} = \frac{(0)(x^{-n}) - (-nx^{-n-1})}{x^{-2n}} = nx^{n-1}. \blacksquare$$