3.3 DERIVATIVES OF COMPOSITE FUNCTIONS: THE CHAIN RULE

3.3 Derivatives of Composite Functions: The Chain Rule

In this section we want to find the derivative of a composite function $f(g(x))$ where $f(x)$ and $g(x)$ are two differentiable functions.

Theorem 3.3.1
If $f$ and $g$ are differentiable then $f(g(x))$ is differentiable with derivative given by the formula

$$
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).
$$

This result is known as the chain rule. Thus, the derivative of $f(g(x))$ is the derivative of $f(x)$ evaluated at $g(x)$ times the derivative of $g(x)$.

Proof.
By the definition of the derivative we have

$$
\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h}.
$$

Since $g$ is differentiable at $x$, letting

$$
v = \frac{g(x + h) - g(x)}{h} - g'(x)
$$

we find

$$
g(x + h) = g(x) + (v + g'(x))h
$$

with $\lim_{h \to 0} v = 0$. Similarly, we can write

$$
f(y + k) = f(y) + (w + f'(y))k
$$

with $\lim_{k \to 0} w = 0$. In particular, letting $y = g(x)$ and $k = (v + g'(x))h$ we find

$$
f(g(x) + (v + g'(x))h) = f(g(x)) + (w + f'(g(x)))(v + g'(x))h.
$$

Hence,

$$
f(g(x + h)) - f(g(x)) = f(g(x) + (v + g'(x))h) - f(g(x))
$$

$$
= f(g(x)) + (w + f'(g(x)))(v + g'(x))h - f(g(x))
$$

$$
= (w + f'(g(x)))(v + g'(x))h
$$
Thus,
\[
\frac{d}{dx}f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} (w + f'(g(x))(v + g'(x))) = f'(g(x))g'(x).
\]

This completes a proof of the theorem.

**Example 3.3.1**
Find the derivative of \( y = (4x^2 + 1)^7 \).

**Solution.**
First note that \( y = f(g(x)) \) where \( f(x) = x^7 \) and \( g(x) = 4x^2 + 1 \). Thus, \( f'(x) = 7x^6 \), \( f'(g(x)) = 7(4x^2 + 1)^6 \) and \( g'(x) = 8x \). So according to the chain rule, \( y' = 7(4x^2 + 1)^6(8x) = 56x(4x^2 + 1)^6 \).

**Example 3.3.2**
Prove the power rule for rational exponents.

**Solution.**
Suppose that \( y = x^{\frac{p}{q}} \), where \( p \) and \( q \) are integers with \( q > 0 \). Take the \( q \)th power of both sides to obtain \( y^q = x^p \). Differentiate both sides with respect to \( x \) to obtain \( qy^{q-1}y' = px^{p-1} \). Thus,
\[
y' = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{\frac{q(q-1)}{q}}} = \frac{p}{q} \cdot x^{\frac{p}{q}-1}.
\]

Note that we are assuming that \( x \) is chosen in such a way that \( x^{\frac{p}{q}} \) is defined.

**Example 3.3.3**
Show that \( \frac{d}{dx}x^n = nx^{n-1} \) for \( x > 0 \) and \( n \) is any real number.

**Solution.**
Since \( x^n = e^{n \ln x} \) then
\[
\frac{d}{dx}x^n = \frac{d}{dx}e^{n \ln x} = e^{n \ln x} \cdot \frac{n}{x} = nx^{n-1}.
\]
We end this section by finding the derivative of $f(x) = \ln x$ using the chain rule. Write $y = \ln x$. Then $e^y = x$. Differentiate both sides with respect to $x$ to obtain
\[ e^y \cdot y' = 1. \]
Solving for $y'$ we find
\[ y' = \frac{1}{e^y} = \frac{1}{x}. \]