3.3 Derivatives of Composite Functions: The Chain Rule

In this section we want to find the derivative of a composite function $f(g(x))$ where $f(x)$ and $g(x)$ are two differentiable functions.

Theorem 3.3.1
If f and g are differentiable then $f(g(x))$ is differentiable with derivative given by the formula

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

This result is known as the **chain rule**. Thus, the derivative of $f(g(x))$ is the derivative of $f(x)$ evaluated at $g(x)$ times the derivative of $g(x)$.

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Since g is differentiable at x, letting

$$v = \frac{g(x+h) - g(x)}{h} - g'(x)$$

we find

$$g(x+h) = g(x) + (v + g'(x))h$$

with $\lim_{h \to 0} v = 0$. Similarly, we can write

$$f(y+k) = f(y) + (w + f'(y))k$$

with $\lim_{k \to 0} w = 0$. In particular, letting $y = g(x)$ and $k = (v + g'(x))h$ we find

$$f(g(x) + (v + g'(x))h) = f(g(x)) + (w + f'(g(x)))(v + g'(x))h.$$

Hence,

$$\begin{aligned} f(g(x+h)) - f(g(x)) &= f(g(x) + (v + g'(x))h) - f(g(x)) \\ &= f(g(x)) + (w + f'(g(x)))(v + g'(x))h - f(g(x)) \\ &= (w + f'(g(x)))(v + g'(x))h \end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\ &= \lim_{h\to 0}(w+f'(g(x)))(v+g'(x))\\ &= f'(g(x))g'(x).\end{aligned}$$

This completes a proof of the theorem ■

Example 3.3.1
Find the derivative of $y = (4x^2+1)^7$.

Solution.
First note that $y = f(g(x))$ where $f(x) = x^7$ and $g(x) = 4x^2+1$. Thus, $f'(x) = 7x^6, f'(g(x)) = 7(4x^2+1)^6$ and $g'(x) = 8x$. So according to the chain rule, $y' = 7(4x^2+1)^6(8x) = 56x(4x^2+1)^6$ ■

Example 3.3.2
Prove the power rule for rational exponents.

Solution.
Suppose that $y = x^{\frac{p}{q}}$, where p and q are integers with $q > 0$. Take the q^{th} power of both sides to obtain $y^q = x^p$. Differentiate both sides with respect to x to obtain $qy^{q-1}y' = px^{p-1}$. Thus,

$$y' = \frac{p}{q}\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

Note that we are assuming that x is chosen in such a way that $x^{\frac{p}{q}}$ is defined ■

Example 3.3.3
Show that $\frac{d}{dx}x^n = nx^{n-1}$ for $x > 0$ and n is any real number.

Solution.
Since $x^n = e^{n\ln x}$ then

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n\ln x}\cdot\frac{n}{x} = nx^{n-1}.■$$

We end this section by finding the derivative of $f(x) = \ln x$ using the chain rule. Write $y = \ln x$. Then $e^y = x$. Differentiate both sides with respect to x to obtain

$$e^y \cdot y' = 1.$$

Solving for y' we find

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$