3.2 Derivative Formulas for Exponential and Logarithmic Functions

We start this section by looking at the limit

$$\lim_{h\to 0}\frac{e^h-1}{h}.$$

The chart below suggests that the limit is 1.

h	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$\frac{e^h-1}{h}$	0.995	0.9995	0.99995	undefined	1.0000	1.0005	1.005

Now, let's try and find the derivative of the function $f(x) = e^x$ at any number x. By the definition of the derivative and the limit above we see that

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} \\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x. \end{aligned}$$

This means that e^x is its own derivative:

$$\frac{d}{dx}(e^x) = e^x.$$

Now, suppose that the x in e^x is replaced by a differentiable function of x, say $u(x)$. We would like to find the derivative of e^u with respect to x, i.e., what is $\frac{d}{dx}(e^u)$?

Theorem 3.2.1

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}.$$

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}(e^u) = \lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h}.$$

Since u is differentiable at x, by letting

$$v = \frac{u(x+h) - u(x)}{h} - u'(x)$$

we find

$$u(x+h) = u(x) + (v + u'(x))h$$

with $\lim_{h\to 0} v = 0$. Similarly, we can write

$$e^{y+k} = e^y + (w + e^y)k$$

with $\lim_{k\to 0} w = 0$. In particular, letting $y = u(x)$ and $k = (v + u'(x))h$ we find

$$e^{u(x)+(v+u'(x))h} = e^{u(x)} + (w + e^{u(x)})(v + u'(x))h.$$

Hence,

$$\begin{aligned} e^{u(x+h)} - e^{u(x)} &= e^{u(x)+(v+u'(x))h} - e^{u(x)} \\ &= e^{u(x)} + (w + e^{u(x)})(v + u'(x))h - e^{u(x)} \\ &= (w + e^u)(v + u'(x))h \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dx}(e^u) &= \lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h} \\ &= \lim_{h\to 0} (w + e^u)(v + u'(x)) \\ &= e^u u' \blacksquare \end{aligned}$$

Next, we want to find the derivative of the function $f(x) = a^x$, where $a > 0$ and $a \neq 1$. First, note that $f(x) = a^x = (e^{\ln a})^x = e^{x \ln a}$. Thus, by Theorem 3.2.1 we see that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x \ln a}\frac{d}{dx}(x \ln a) = a^x \ln a.$$

Example 3.2.1
Find the derivative of each of the following functions:

(a) $f(x) = 3^x$ (b) $y = 2 \cdot 3^x + 5 \cdot e^{3x-4}$.

Solution.

(a) $f'(x) = 3^x \ln 3$.
(b) $y' = 2(3^x)' + 5(e^{3x-4})' = 2 \cdot 3^x \ln 3 + 5(3)e^{3x-4} = 2 \cdot 3^x \ln 3 + 15 \cdot e^{3x-4}$ ■

We end this section, by finding the derivative of the function $f(x) = \ln x$. In the next section, we will prove the formula

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$