3.1 Derivative Formulas for Power and Polynomials

Finding the derivative function by using the limit of the difference quotient is sometimes difficult for functions with complicated expressions. Fortunately, there is an indirect way for computing derivatives that does not compute limits but instead uses formulas which we will derive in this section and in the coming sections.
We first derive a couple of formulas of differentiation.

Theorem 3.1.1
If f is differentiable and k is a constant then the new function $kf(x)$ is differentiable with derivative given by

$$[kf(x)]' = kf'(x).$$

Proof.
We have,

$$\begin{aligned}[kf'(x) &= \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h} = \lim_{h\to 0}\frac{k(f(x+h)-f(x))}{h}\\ &= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = kf'(x)\end{aligned}$$

where we used the fact that a constant can be taking across the limit sign by the properties of limits ■

Theorem 3.1.2
If $f(x)$ and $g(x)$ are two differentiable functions then the functions $f+g$ and $f-g$ are also differentiable with derivatives

$$[f(x)\pm g(x)]' = f'(x)\pm g'(x).$$

Proof.
Again by using the definition of the derivative and the fact that the limit of a sum/difference is the sum/difference of limits we find

$$\begin{aligned}[f(x)+g(x)]' &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x).\end{aligned}$$

The same proof is valid for the difference formula $\blacksquare$

Next, we state and give a partial proof of a rule for finding the derivative of a power function of the form $f(x) = x^n$.

Theorem 3.1.3 *(Power Rule)*
For any real number n, the derivative of the function $y = x^n$ is given by the formula

$$\frac{dy}{dx} = nx^{n-1}.$$

Proof.
We prove the result when n is a positive integer. We start by writing the definition of the derivative of any function $f(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Letting $h = ax - x$ we can rewrite the previous definition in the form

$$f'(x) = \lim_{a\to 1} \frac{f(ax) - f(x)}{ax - x}.$$

Thus,

$$f'(x) = \lim_{a\to 1} \frac{(ax)^n - x^n}{ax - x} = x^{n-1} \lim_{a\to 1} \frac{a^n - 1}{a - 1}.$$

Dividing $a^n - 1$ by $a - 1$ by the method of synthetic division we find

$$a^n - 1 = (a-1)(1 + a + a^2 + a^3 + \cdots + a^{n-1}).$$

Thus,

$$f'(x) = x^{n-1} \lim_{a\to 1}(1 + a + a^2 + \cdots + a^{n-1}) = nx^{n-1} \ \blacksquare$$

Example 3.1.1
Use the power rule to differentiate the following:
(a) $y = x^{\frac{4}{3}}$ (b) $y = \frac{1}{\sqrt[3]{x}}$ (c) $y = x^{\pi}$.

Solution.
(a) Using the power rule with $n = \frac{4}{3}$ to obtain $y' = \frac{4}{3}x^{\frac{1}{3}}$.
(b) Since $y = x^{-\frac{1}{3}}$, using the power rule with $n = -\frac{1}{3}$ to obtain $y' = -\frac{1}{3}x^{-\frac{4}{3}}$.
(c) Using the power rule with $n = \pi$ to obtain $y' = \pi x^{\pi - 1}$ $\blacksquare$

Remark 3.1.1
The derivative of a function of the form $y = 2^x$ is not $y' = x2^{x-1}$ because $y = 2^x$ is an exponential function and not a power function. A formula for finding the derivative of an exponential function will be discussed in the next section.

Now, combining the results discussed above, we can find the derivative of functions that are combinations of power functions of the form ax^n. In particular, the derivative of a polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is given by the formula

$$f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Example 3.1.2
Find the derivative of the function $y = \sqrt{3}x^7 - \frac{x^5}{5} + \pi$.

Solution.
The derivative is $f'(x) = 7\sqrt{3}x^6 - x^4$ ■

Example 3.1.3
Find the second derivative of $y = 5\sqrt[3]{x} - \frac{10}{x^4} + \frac{1}{2\sqrt{x}}$.

Solution.
Note that the given function can be written in the form $y = 5x^{\frac{1}{3}} - 10x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$. Thus, the first derivative is

$$y' = \frac{5}{3}x^{-\frac{2}{3}} + 40x^{-5} - \frac{1}{4}x^{-\frac{3}{2}}.$$

The second derivative is

$$y'' = -\frac{10}{9}x^{-\frac{5}{3}} - 200x^{-6} + \frac{3}{8}x^{-\frac{5}{2}} \blacksquare$$