2.4 The Second Derivative

Let $f(x)$ be a differentiable function. If the limit

$$\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$

exists then we say that the function $f'(x)$ is **differentiable** and we denote its derivative by $f''(x)$ or using Leibniz notation

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We call $f''(x)$ the **second derivative** of $f(x)$.

Now, recall that if $f'(x) > 0$ (resp. $f'(x) < 0$) over an interval I then the function $f(x)$ is increasing (resp. decreasing on I). So if $f''(x) > 0$ on I then $f'(x)$ is increasing on I. So either $f'(x)$ gets more and more positive or less and less negative. This occurs only when the graph of f is concave up. Similarly, if $f''(x) < 0$ on I then $f'(x)$ is decreasing. So either $f(x)$ is getting less and less positive or more and more negative. This means that the graph of f is concave down.

Remark 2.4.1
Note that when a curve is concave up then the tangent lines lie below the curve whereas when it is concave down then the tangent lines lie above the curve.

Example 2.4.1
Give the signs of f' and f'' for the function whose graph is given in Figure 2.4.1.

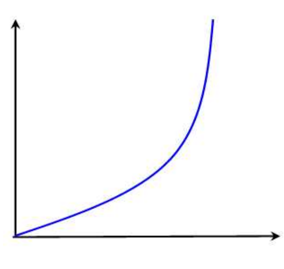

Figure 2.4.1

Solution.
Since f is always increasing, f' is always positive. Since the graph is concave up, f'' is always positive ■

Example 2.4.2
Find where the graph of $f(x) = x^3 + 3x + 1$ is concave up and where it is concave down.

Solution.
Finding the first and second derivatives of f we obtain $f'(x) = 3x^2 + 3$ and $f''(x) = 6x$. Thus, the graph of f is concave up for $x > 0$, i.e., in the interval $(0, \infty)$ and concave down for $x < 0$, i.e., in the interval $(-\infty, 0)$ ■

As an application to the second derivative, we consider the motion of an object determined by the position function $s(t)$. Recall that the velocity of the object is defined to be the first derivative of $s(t)$, i.e.

$$v(t) = s'(t) = \frac{ds}{dt}$$

and the absolute value of $v(t)$ is the speed. When the object speeds up we say that he/she accelerates and when the object slows down we say that he/she decelerates. We define the **acceleration** of an object as the derivative of the velocity function and consequently as the second derivative of the position function

$$a(t) = \frac{d^2s}{dt^2} = \frac{ds}{dt}.$$

Example 2.4.3
A particle is moving along a straight line. If its distance, s, to the right of a fixed point is given by Figure 2.4.2, estimate:
(a) When the particle is moving to the right and when it is moving to the left.
(b) When the particle has positive acceleration and when it has negative acceleration.

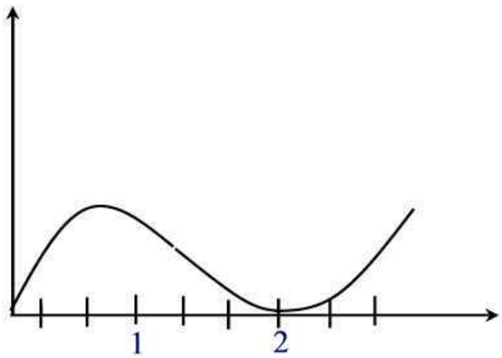

Figure 2.4.2

Solution.
(a) When s is increasing then the particle moves to the right. This occurs when $0 < t < \frac{2}{3}$ and for $t > 2$. On the other hand, the particle moves to the left when s is decreasing. This happens when $\frac{2}{3} < t < 2$.
(b) Positive acceleration occurs when the graph is concave up. This occurs when $t > \frac{4}{3}$. The particle has negative acceleration when the curve is concave down, i.e., for $t < \frac{4}{3}$ ■