

2.3 Leibniz Notation for The Derivative

When dealing with mathematical models that involve derivatives it is convenient to denote the prime notation of the derivative of a function $y = f(x)$ by $\frac{dy}{dx}$. That is,

$$\frac{dy}{dx} = f'(x).$$

This notation is called **Leibniz notation** (due to W.G. Leibniz). For example, we can write $\frac{dy}{dx} = 2x$ for $y' = 2x$.

When using Leibniz notation to denote the value of the derivative at a point a we will write

$$\left. \frac{dy}{dx} \right|_{x=a}$$

Thus, to evaluate $\frac{dy}{dx} = 2x$ at $x = 2$ we would write

$$\left. \frac{dy}{dx} \right|_{x=2} = 2x|_{x=2} = 2(2) = 4.$$

Remark 2.3.1

Even though $\frac{dy}{dx}$ appears as a fraction but it is not. It is just an alternative notation for the derivative. A concept called **differential** will provide meaning to symbols like dy and dx .

One of the advantages of Leibniz notation is the recognition of the units of the derivative. For example, if the position function $s(t)$ is expressed in meters and the time t in seconds then the units of the velocity function $\frac{ds}{dt}$ are meters/sec.

In general, the units of the derivative are the units of the dependent variable divided by the units of the independent variable. That is, the units of the derivative are the units of the numerator divided by the units of the denominator.

Example 2.3.1

The cost, C (in dollars) to produce x gallons of ice cream can be expressed as $C = f(x)$. What are the units of measurements and the meaning of the statement $\left. \frac{dC}{dx} \right|_{x=200} = 1.4$?

Solution.

$\frac{dC}{dx}$ is measured in dollars per gallon. The notation

$$\left. \frac{dC}{dx} \right|_{x=200} = 1.4$$

means that if 200 gallons of ice cream have already been produced then the cost of producing the next gallon will be roughly 1.4 dollars ■

Example 2.3.2

The derivative of the velocity function v is called **acceleration** and is denoted by a . Suppose that v is measured in meters/seconds, what are the units of a ?

Solution.

The units of a are meters/seconds/seconds=meters/seconds² ■

Local Linear Approximation

Finally, one can use the derivative at a point to approximate values of the function at nearby points. For example, if we know the values of $f(a)$ and $f'(a)$ then for a nearby point b the value of $f(b)$ is found by the formula

$$f(b) \approx f'(a)(b - a) + f(a). \quad (2.3.1)$$

Example 2.3.3

Climbing health care costs have been a source of concern for some time. Use the data in the table below to estimate the average per capita expenditure in 1991 and 2010 assuming that the costs climb at the same rate since 1990.

Year	1970	1975	1980	1985	1990
Per capita expenditure (\$)	349	591	1055	1596	2714

Solution.

Between 1985 and 1990 the rate of increase in the costs is $\frac{2714-1596}{5} = \$223.60$ per year. Since we are assuming that the costs continue to increase at the same rate, we find

$$C(1991) \approx C(1990) + C'(1990)(1991 - 1990) = 2714 + 223.60 = \$2937.60$$

and

$$C(2010) = 2714 + 223.60(10) = \$7,186 \blacksquare$$

Relative Rate of Change

When a function changes values from $t = a$ to $t = b$, we call the difference $f(b) - f(a)$, the **absolute change** of f . That is, $f(b) - f(a)$ tells us how much $f(a)$ has changed when the variable t changes from $t = a$ to $t = b$.

If, instead, one is interested in the percent change of $f(a)$ when t changes from $t = a$ to $t = b$, then we use the **relative change formula**

$$\frac{f(b) - f(a)}{f(a)}. \quad (2.3.2)$$

Now, if we use Formula (2.3.1) with $b = a + 1$ then we find

$$f(a + 1) \approx f(a) + f'(a)$$

or

$$f'(a) \approx f(a + 1) - f(a). \quad (2.3.3)$$

That is, $f'(a)$ is an estimate of how much the function will change by increasing the input by 1. For example, if $C(q)$ is the cost of producing q units of a product then the expression $C'(2) = 3$ says that the cost of producing the third item is about \$3. Hence, $f'(a)$ provides an estimate of the absolute change of the function.

To find the percent change of f from $t = a$ to $t = a + 1$, we use (2.3.2) with $b = a + 1$ and (2.3.3) we find

$$\frac{f'(a)}{f(a)}. \quad (2.3.4)$$

We refer to (2.3.4) as the **relative rate of change** of f at $t = a$.

Example 2.3.4

Let $f(t)$ denote the world's population in billions of people since 2009.

- Interpret the statements $f(3) = 7$ and $f'(3) = 0.085$ in terms of the world's population.
- Calculate the relative rate of change at f at $t = 3$ and interpret its meaning.

Solution.

- The statement $f(3) = 7$ tells us that the world's population is expected to be about 7 billions in the year 2012. The statement $f'(3) = 0.085$ tells

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us that the world's population is expected to increase by 0.085 billion in the year 2013.

(b) The relative rate of change is

$$\frac{f'(3)}{f(3)} \approx 0.012 = 1.2\%.$$

In 2013, the population of the world increased by 1.2% from the year 2012 ■