2.2 The Derivative Function

Recall that a function f is differentiable at x if the following limit exists

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.2.1}$$

Thus, we associate with the function f, a new function f' whose domain is the set of points x at which the limit (2.2.1) exists. We call the function f' the **derivative function** of f.

The Derivative Function Graphically

Since the derivative at a point represents the slope of the tangent line, one can obtain the graph of the derivative function from the graph of the original function. It is important to keep in mind the relationship between the graphs of f and f'. If $f'(x) > 0$ then the tangent line must be tilted upward and the graph of f is rising or increasing. Similarly, if $f'(x) < 0$ then the tangent line is tilted downward and the graph of f is falling or decreasing. If $f'(a) = 0$ then the tangent line is horizontal at $x = a$.

Example 2.2.1
Sketch the graph of the derivative of the function shown in Figure 2.2.1.

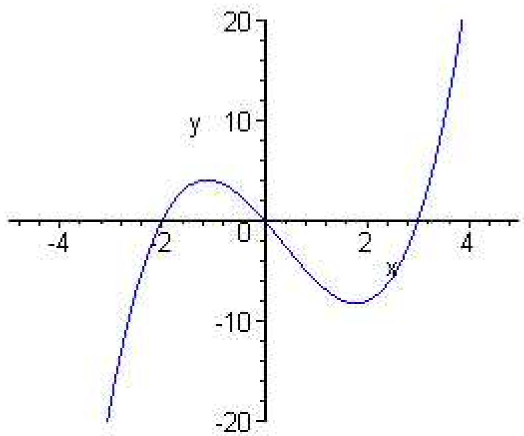

Figure 2.2.1

Solution.
Note that for $x < -1.12$ the derivative is positive and getting less and less positive. At $x \approx -1.12$ we have $f'(-1.12) = 0$. For $-1.12 < x < 0$ the

derivative is negative and getting more and more negative till reaching $x = 0$. For $0 < x < 1.79$ the derivative is less and less negative and at $x = 1.79$ we have $f'(1.79) = 0$. Finally, for $x > 1.79$ the derivative is getting more and more positive. Thus, a possible graph of f' is given in Figure 2.2.2 ■

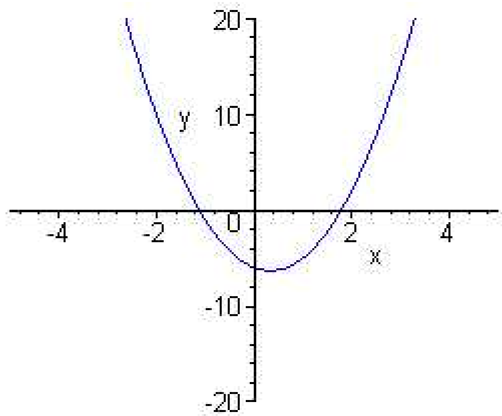

Figure 22.2

The Derivative Function Numerically

Here, we want to estimate the derivative of a function defined by a table. The derivative can be estimated by using the average rate of change or the difference quotient

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

If a is a left-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$

where $b > a$. If a is a right-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(a) - f(b)}{a - b}$$

where $b < a$. If a is an interior point then $f'(a)$ is estimated by

$$f'(a) \approx \frac{1}{2}\left(\frac{f(a) - f(b)}{a - b} + \frac{f(c) - f(a)}{c - a}\right)$$

where $b < a < c$.

Example 2.2.2
Find approximate values for $f'(x)$ at each of the $x-$values given in the following table

x	0	5	10	15	20
$f(x)$	100	70	55	46	40

Solution.
We have

$$\begin{aligned}
f'(0) &\approx \frac{f(5)-f(0)}{5} = -6\\
f'(5) &\approx \frac{1}{2}\left(\frac{f(10)-f(5)}{5} + \frac{f(5)-f(0)}{5}\right) = -4.5\\
f'(10) &\approx \frac{1}{2}\left(\frac{f(15)-f(10)}{5} + \frac{f(10)-f(5)}{5}\right) = -2.4\\
f'(15) &\approx \frac{1}{2}\left(\frac{f(20)-f(15)}{5} + \frac{f(15)-f(10)}{5}\right) = -1.5\\
f'(20) &\approx \frac{f(20)-f(15)}{5} = -1.2 \blacksquare
\end{aligned}$$

The Derivative Function From a Formula

Now, if a formula for f is given then by applying the definition of $f'(x)$ as the limit of the difference quotient we can find a formula of f' as shown in the following two problems.

Example 2.2.3 *(Derivative of a Constant Function)*
Suppose that $f(x) = k$ for all x. Find a formula for $f'(x)$.

Solution.
We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0} \frac{k-k}{h} = 0.
\end{aligned}$$

Thus, $f'(x) = 0 \blacksquare$

Example 2.2.4 *(Derivative of a Linear Function)*
Find the derivative of the linear function $f(x) = mx + b$.

Solution.
We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{m(x+h) + b - (mx+b)}{h} \\ &= \lim_{h \to 0} \frac{mh}{h} = m. \end{aligned}$$

Thus, $f'(x) = m$ ■

Example 2.2.5
Find $f'(1)$ if $f(t) = 100(1.1)^t$.

Solution.
The average rate of change of f in $[1, 1.1]$ is

$$\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{100(1.1)^{1.1} - 100(1.1)}{0.1} \approx 10.534.$$

The average rate of change of f in $[1, 1.01]$ is

$$\frac{f(1.01) - f(1)}{1.01 - 1} = \frac{100(1.1)^{1.01} - 100(1.1)}{0.01} \approx 10.489.$$

The average rate of change of f in $[1, 1.001]$ is

$$\frac{f(1.001) - f(1)}{1.001 - 1} = \frac{100(1.1)^{1.001} - 100(1.1)}{0.001} \approx 10.485.$$

The average rate of change of f in $[1, 1.0001]$ is

$$\frac{f(1.0001) - f(1)}{1.0001 - 1} = \frac{100(1.1)^{1.0001} - 100(1.1)}{0.0001} \approx 10.484.$$

Thus, $f'(1) \approx 10.484$ ■