

2.1 Instantaneous Rate of Change

In this section, we discuss the concept of the instantaneous rate of change of a given function. As an application, we use the velocity of a moving object. The motion of an object along a line at a particular instant is very difficult to define precisely. The modern approach consists of computing the average velocity over smaller and smaller time intervals. To be more precise, let $s(t)$ be the **position** function or displacement of a moving object at time t . We would like to compute the velocity of the object at the instant $t = t_0$.

Average Velocity

We start by finding the **average velocity** of the object over the time interval $t_0 \leq t \leq t_0 + \Delta t$ given by the expression

$$\bar{v} = \frac{\text{Distance Traveled}}{\text{Elapsed Time}} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}. \quad (2.1.1)$$

Geometrically, the average velocity over the time interval $[t_0, t_0 + \Delta t]$ is just the slope of the line joining the points $(t_0, s(t_0))$ and $(t_0 + \Delta t, s(t_0 + \Delta t))$ on the graph of $s(t)$. (See Figure 2.1.1)

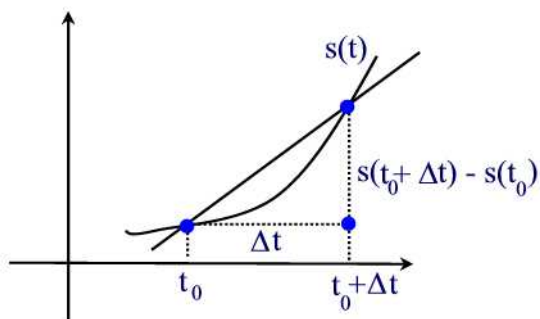


Figure 2.1.1

Example 2.1.1

A freely falling body experiencing no air resistance falls $s(t) = 16t^2$ feet in t seconds. Complete the following table

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity							

Solution.

From time $t = 1.8$ to time $t = 2$, formula (2.1.1) gives

$$\frac{s(2) - s(1.8)}{2 - 1.8} = \frac{-16(2)^2 - [-16(1.8)^2]}{0.2} = -60.8 \text{ ft/sec.}$$

Using formula (2.1.1) on each of the remaining intervals, we find

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity	60.8	62.4	63.84	63.98	64.0016	64.016	64.16

Instantaneous Velocity and Speed

The next step is to calculate the average velocity on smaller and smaller time intervals (that is, make Δt close to zero). The average velocity in this case approaches what we would intuitively call the **instantaneous velocity** at time $t = t_0$ which is defined using the **limit** notation by

$$v(t_0) = \lim_{\Delta t \rightarrow 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}. \quad (2.1.2)$$

Geometrically, the instantaneous velocity at t_0 is the slope of the tangent line to the graph of $s(t)$ at the point $(t_0, s(t_0))$.(See Figure 2.1.2)

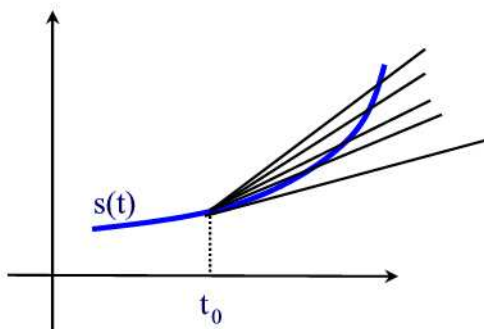


Figure 2.1.2

Example 2.1.2

For the distance function in Example 2.1.1, find the instantaneous velocity at $t = 2$.

Solution.

Examining the bottom row of the table in Example 2.1.1, we see that the average velocity seems to be approaching the value 64 as we shrink the time intervals. Thus, it is reasonable to expect the velocity to be $v(2) = 64$ ft/sec ■

We define the **speed** of a moving object to be the absolute value of the velocity function. Sometimes there is confusion between the words “speed” and “velocity”. Speed is a nonnegative number that indicates how fast an object is moving, whereas velocity indicates both speed and direction (relative to a coordinate system). For example, if the object is moving along a vertical line we define a positive velocity when the object is going upward and a negative velocity when the object is going downward.

If the position function $s(t)$ is replaced by an arbitrary function $f(x)$ then we define the **instantaneous rate of change** of a function $y = f(x)$ at $x = a$ to be

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2.1.3)$$

where

$$\frac{f(x) - f(a)}{x - a} \quad (2.1.4)$$

is the **average rate of change** of $f(x)$ from $x = a$ to $x = b$, also known as the **difference quotient** of $f(x)$ from $x = a$ to $x = b$.

The instantaneous rate of change of a function $f(x)$ from $x = a$ to $x = a + h$ is called the **derivative of f at a** and we denote it by $f'(a)$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}. \quad (2.1.5)$$

If the derivative exists, i.e., can be found, then we say that the function is **differentiable** at $x = a$. The process of finding the derivative of a function is called **differentiation**. If a function has no derivative at a point then we say that it is non-differentiable there.

Since the instantaneous rate of change represents the slope of a tangent line, $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. The equation of the tangent line is given by

$$y - f(a) = f'(a)(x - a). \quad (2.1.6)$$

Example 2.1.3

- (a) Find $f'(1)$ for $f(x) = x^2$.
 (b) Find the equation of the tangent line to the graph of $f(x)$ at the point $(1, f(1))$.

Solution.

Completing the following chart

x	[0.9,1]	[0.99,1]	[0.999,1]	[1,1.0001]	[1,1.001]	[1,1.01]	[1,1.1]
$\frac{f(b)-f(a)}{b-a}$	1.9	1.99	1.999	2.0001	2.001	2.01	2.1

we see that $f'(1) = 2$.

- (b) The equation of the tangent line is

$$y - f(1) = f'(1)(x - 1)$$

or

$$y - 1 = 2(x - 1).$$

In point-intercept form, we have $y = 2x - 1$ ■

Example 2.1.4 (*Numerical Estimation of the Derivative*)

Find approximate values for $f'(x)$ at each of the x -values given in the following table

x	0	5	10	15	20
$f(x)$	100	70	55	46	40

Solution.

The derivative can be estimated by using the average rate of change or the difference quotient

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

If a is a left-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$

where $b > a$. If a is a right-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(a) - f(b)}{a - b}$$

where $b < a$. If a is an interior point then $f'(a)$ is estimated by

$$f'(a) \approx \frac{1}{2} \left(\frac{f(a) - f(b)}{a - b} + \frac{f(c) - f(a)}{c - a} \right)$$

where $b < a < c$. For example,

$$\begin{aligned} f'(0) &\approx \frac{f(5) - f(0)}{5} = -6 \\ f'(5) &\approx \frac{1}{2} \left(\frac{f(10) - f(5)}{5} + \frac{f(5) - f(0)}{5} \right) = -4.5 \\ f'(10) &\approx \frac{1}{2} \left(\frac{f(15) - f(10)}{5} + \frac{f(10) - f(5)}{5} \right) = -2.4 \\ f'(15) &\approx \frac{1}{2} \left(\frac{f(20) - f(15)}{5} + \frac{f(15) - f(10)}{5} \right) = -1.5 \\ f'(20) &\approx \frac{f(20) - f(15)}{5} = -1.2. \end{aligned}$$

The quantity $\frac{f(10) - f(5)}{10 - 5}$ is known as the **right slope estimation** of $f'(5)$. Similarly, we can estimate $f'(5)$ by using a **left slope estimation**, i.e.,

$$f'(5) \approx \frac{f(5) - f(0)}{5 - 0} = -6.$$

An improved estimation consists of taking the average of the left slope and the right slope, that is,

$$f'(5) \approx \frac{-3 - 6}{2} = -4.5 \blacksquare$$