8 Building New Functions from Old Ones

In this section we discuss various ways for building new functions from old ones. New functions can be obtained by composing functions, using arithmetic combinations, and finally by making changes to either the input or the output of a function.

Composition of Functions

The first procedure for building new functions from old ones known as the composition of functions.

We start with an example of a real-life situation where composite functions are applied.

Example 8.1

You have two money machines, both of which increase any money inserted into them. The first machine doubles your money. The second adds five dollars. The money that comes out is described by \( f(x) = 2x \) in the first case, and \( g(x) = x + 5 \) in the second case, where \( x \) is the number of dollars inserted. The machines can be hooked up so that the money coming out of one machine goes into the other. Find formulas for each of the two possible composition machines.

Solution.

Suppose first that \( x \) dollars enters the first machine. Then the amount of money that comes out is \( f(x) = 2x \). This amount enters the second machine. The final amount coming out is given by \( g(f(x)) = f(x) + 5 = 2x + 5 \).

Now, if \( x \) dollars enters the second machine first, then the amount that comes out is \( g(x) = x + 5 \). If this amount enters the second machine then the final amount coming out is \( f(g(x)) = 2(x + 5) = 2x + 10 \).

The function \( f(g(x)) \) is called the composition of the functions \( f \) and \( g \); the function \( g(f(x)) \) is called the composition of the functions \( g \) and \( f \).

In general, suppose we are given two functions \( f \) and \( g \) such that the range of \( g \) is contained in the domain of \( f \) so that the output of \( g \) can be used as input for \( f \). We define a new function, called the composition of \( f \) and \( g \), by the formula \( f(g(x)) \) where \( g \) is the inside function and \( f \) is the outside function.
function. In a similar way, we can define the composition of \( g \) and \( f \) to be the function

\[ g(f(x)) \]

so that the output of \( f \) is the input of \( g \).

Using a Venn diagram (See Figure 16) we have

![Venn diagram](image)

**Figure 16**

Composition of Functions Defined by Tables

**Example 8.2**

Complete the following table

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>( f(g(x)) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution.**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>( f(g(x)) )</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Composition of Functions Defined by Formulas

**Example 8.3**

Suppose that \( f(x) = 2x + 1 \) and \( g(x) = x^2 - 3 \).

(a) Find \( f(g(x)) \) and \( g(f(x)) \).
(b) Calculate \( f(g(5)) \) and \( g(f(-3)) \).
(c) Are \( f(g(x)) \) and \( g(f(x)) \) equal?
Solution.
(a) \( f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5. \) Similarly, \( g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2. \)
(b) \( f(g(5)) = 2(5)^2 - 5 = 45 \) and \( g(f(-3)) = 4(-3)^2 + 4(-3) - 2 = 22. \)
(c) \( f(g(x)) \neq g(f(x)). \)

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

Example 8.4
Suppose that \( f(x) = 2x + 1 \) and \( g(x) = x^2 - 3. \)

(a) Find \( f(f(x)). \)
(b) Find \( f(f(g(x))). \)

Solution.
(a) \( f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3. \)
(b) \( f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9. \)

Decomposition of Functions
If a formula for \( f(g(x)) \) is given then the process of finding the formulas for \( f \) and \( g \) is called decomposition.

Example 8.5
Decompose \( f(g(x)) = \sqrt{1 - 4x^2}. \)

Solution.
One possible answer is \( f(x) = \sqrt{x} \) and \( g(x) = 1 - 4x^2. \) Another possible answer is \( f(x) = \sqrt{1 - x^2} \) and \( g(x) = 2x. \)

Arithmetic Combinations of Functions
A second way to constructing new functions from old ones is to use the operations of addition, subtraction, multiplication, and division.
Let \( f(x) \) and \( g(x) \) be two given functions. Then for all \( x \) in the common domain of these two functions we define new functions as follows:

- **Sum**: \( (f + g)(x) = f(x) + g(x). \)
- **Difference**: \( (f - g)(x) = f(x) - g(x). \)
- **Product**: \( (f \cdot g)(x) = f(x) \cdot g(x). \)
• Division: \[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \text{ provided that } g(x) \neq 0.
\]

In the following example we see how to construct the four functions discussed above when the individual functions are defined by formulas.

**Example 8.6**
Let \( f(x) = x + 1 \) and \( g(x) = \sqrt{x+3} \). Find the common domain and then find a formula for each of the functions \( f + g, f - g, f \cdot g, \frac{f}{g} \).

**Solution.**
The domain of \( f(x) \) consists of all real numbers whereas the domain of \( g(x) \) consists of all numbers \( x \geq 3 \). Thus, the common domain is the interval \([-3, \infty)\). For any \( x \) in this domain we have
\[
\begin{align*}
(f + g)(x) &= x + 1 + \sqrt{x+3} \\
(f - g)(x) &= x + 1 - \sqrt{x+3} \\
(f \cdot g)(x) &= x\sqrt{x+3} + \sqrt{x+3} \\
\left( \frac{f}{g} \right)(x) &= \frac{x+1}{\sqrt{x+3}} \text{ provided } x > -3.
\end{align*}
\]

In the next example, we see how to evaluate the four functions when the individual functions are given in numerical forms.

**Example 8.7**
Suppose the functions \( f \) and \( g \) are given in numerical forms. Complete the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>2</td>
<td>7</td>
<td>-1</td>
<td>-5</td>
<td>-3</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-1</td>
<td>-5</td>
<td>-11</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>( (f + g)(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (f - g)(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (f \cdot g)(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \left( \frac{f}{g} \right)(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution.**

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>2</td>
<td>7</td>
<td>-1</td>
<td>-5</td>
<td>-3</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-1</td>
<td>-5</td>
<td>-11</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>( (f + g)(x) )</td>
<td>7</td>
<td>-3</td>
<td>-4</td>
<td>6</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>( (f - g)(x) )</td>
<td>9</td>
<td>7</td>
<td>18</td>
<td>-8</td>
<td>-13</td>
<td>-12</td>
</tr>
<tr>
<td>( (f \cdot g)(x) )</td>
<td>-8</td>
<td>-10</td>
<td>-77</td>
<td>-7</td>
<td>-40</td>
<td>-27</td>
</tr>
<tr>
<td>( \left( \frac{f}{g} \right)(x) )</td>
<td>-8</td>
<td>-\frac{2}{5}</td>
<td>\frac{x}{11}</td>
<td>-\frac{1}{7}</td>
<td>\frac{5}{8}</td>
<td>-\frac{1}{5}</td>
</tr>
</tbody>
</table>
Transformations of Functions
We close this section by giving a summary of the various transformations obtained when either the input or the output of a function is altered.

**Vertical Shifts:** The graph of \( f(x) + k \) with \( k > 0 \) is a vertical translation of the graph of \( f(x) \), a distance \( k \) upward, whereas for \( k < 0 \) it is a shift by a distance \( k \) downward.

**Horizontal Shifts:** The graph of \( f(x + k) \) with \( k > 0 \) is a horizontal translation of the graph of \( f(x) \), a distance \( k \) to the left, whereas for \( k < 0 \) it is a shift by a distance \( k \) to the right.

**Reflections about the \( x \)-axis:** For a given function \( f(x) \), the graph of \(-f(x)\) is a reflection of the graph of \( f(x) \) about the \( x \)-axis.

**Reflections about the \( y \)-axis:** For a given function \( f(x) \), the graph of \( f(-x) \) is a reflection of the graph of \( f(x) \) about the \( y \)-axis.

**Vertical Stretches and Compressions:** If a function \( f(x) \) is given, then the graph of \( kf(x) \) is a vertical stretch of the graph of \( f(x) \) by a factor of \( k \) for \( k > 1 \), and a vertical compression for \( 0 < k < 1 \).

What about \( k < 0 \)? If \(|k| > 1\) then the graph of \( kf(x) \) is a vertical stretch of the graph of \( f(x) \) followed by a reflection about the \( x \)-axis. If \( 0 < |k| < 1 \) then the graph of \( kf(x) \) is a vertical compression of the graph of \( f(x) \) by a factor of \( k \) followed by a reflection about the \( x \)-axis.

**Horizontal Stretches and Compressions:** If a function \( f(x) \) is given, then the graph of \( f(kx) \) is a horizontal stretch of the graph of \( f(x) \) by a factor of \( \frac{1}{k} \) for \( 0 < k < 1 \), and a horizontal compression for \( k > 1 \).

What about \( k < 0 \)? If \(|k| > 1\) then the graph of \( f(kx) \) is a horizontal compression of the graph of \( f(x) \) followed by a reflection about the \( y \)-axis. If \( 0 < |k| < 1 \) then the graph of \( f(kx) \) is a horizontal stretch of the graph of \( f(x) \) by a factor of \( \frac{1}{k} \) followed by a reflection about the \( y \)-axis.

**Example 8.8**
Write an equation for a graph obtained by vertically stretching the graph of \( f(x) = x^2 \) by a factor of 2, followed by a vertical upward shift of 1 unit.

**Solution.**
The function is given by the formula \( y = 2f(x) + 1 = 2x^2 + 1 \).