Arkansas Tech University
MATH 2243: Business Calculus
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7 Exponential Growth and Decay

In this section, we consider some applications of exponential functions.

Doubling Time
In some exponential models one is interested in finding the time for an exponential growing quantity to double. We call this time the **doubling time**. To find it, we start with the equation $a \cdot b^t = 2a$ or $b^t = 2$. Taking the ln of both sides we find $i \ln b = \ln 2$. Solving for t we find $t = \frac{\ln 2}{\ln b}$.

Example 7.1
Find the doubling time of a population growing according to $P = P_0 e^{0.2t}$.

Solution.
Setting the equation $P_0 e^{0.2t} = 2P_0$ and dividing both sides by P_0 to obtain $e^{0.2t} = 2$. Take ln of both sides to obtain $0.2t = \ln 2$. Thus, $t = \frac{\ln 2}{0.2} \approx 3.47$.■

Doubling-Time and the Rule of 70
Recall that $b = 1 + r$ where r is the percent growth or decay rate. Thus, the doubling time is given by the formula

$$t = \frac{\ln 2}{\ln(1+r)}.$$

It is shown in Calculus that for small values of r we can estimate $\ln(1+r)$ with r so that $\frac{r}{\ln(1+r)} \approx 1$. Writing $r = i\%$ and using the fact that $\ln 2 \approx 0.7$, we can use the following formula for finding the doubling time when r is small

$$t = \frac{\ln 2}{\ln(1+r)} = \frac{\ln 2}{r} \cdot \frac{r}{\ln(1+r)} \approx \frac{0.7}{r} = \frac{70}{i}.$$

This formula is known as the **Rule of 70.**

Example 7.2
Use the Rule of 70 to estimate the doubling time in Example 7.1.

Solution.
Since $r = e^{0.2} - 1 = 22.1\%$, we have $i = 22.1$ so that $t = \frac{70}{22.1} = 3.12$ ■

Half-Life
On the other hand, if a quantity is decaying exponentially then the time required for the quantity to reduce into half is called the **half-life**. To find it, we start with the equation $ab^t = \frac{a}{2}$ and we divide both sides by a to obtain $b^t = 0.5$. Take the ln of both sides to obtain $t \ln b = \ln(0.5)$. Solving for t we find $t = \frac{\ln(0.5)}{\ln b}$.

Example 7.3
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Solution.
Since the problem involves exponential decay, if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50b^t$ with $0 < b < 1$. But $Q(13) = 25$. That is, $50b^{13} = 25$ or $b^{13} = 0.5$. Thus $b = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t$.■

Compound Interest
The term **compound interest** refers to a procedure for computing interest whereby the interest for a specified interest period is added to the original principal. The resulting sum becomes a new principal for the next interest period. The interest earned in the earlier interest periods earn interest in the future interest periods.
Suppose that you deposit P dollars into a savings account that pays annual interest r and the bank agrees to pay the interest at the end of each time period (usually expressed as a fraction of a year). If the number of periods in a year is n then we say that the interest is **compounded** n times per year (e.g.,'yearly'=1, 'quarterly'=4, 'monthly'=12, etc.). Thus, at the end of the first period the balance will be

$$B = P + \frac{r}{n}P = P\left(1 + \frac{r}{n}\right).$$

At the end of the second period the balance is given by

$$B = P\left(1 + \frac{r}{n}\right) + \frac{r}{n}P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2.$$

Continuing in this fashion, we find that the balance at the end of the first year, i.e. after n periods, is

$$B = P\left(1 + \frac{r}{n}\right)^n.$$

If the investment extends to another year than the balance would be given by

$$P\left(1 + \frac{r}{n}\right)^{2n}.$$

For an investment of t years the balance is given by

$$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Since $\left(1 + \frac{r}{n}\right)^{nt} = \left[\left(1 + \frac{r}{n}\right)^n\right]^t$, the function B can be written in the form $B(t) = Pb^t$ where $b = \left(1 + \frac{r}{n}\right)^n$. That is, B is an exponential function.

Remark 7.1
Interest given by banks are known as **nominal rate** (e.g. "in name only").

When interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between nominal rate and **effective rate.** The effective annual rate tells how much interest the investment actually earns in one year period. The quantity $(1+\frac{r}{n})^n - 1$ is known as the **effective interest rate**.

Example 7.4
Translating a value to the future is referred to as **compounding**. What will be the maturity value of an investment of $\$15,000$ invested for four years at 9.5% compounded semi-annually?

Solution.
Using the formula for compound interest with $P = \$15,000, t = 4, n = 2$, and $r = .095$ we obtain

$$B = 15,000\left(1 + \frac{0.095}{2}\right)^8 \approx \$21,743.20\blacksquare$$

Example 7.5
Translating a value to the present is referred to as **discounting.** What principal invested today will amount to $\$8,000$ in 4 years if it is invested at 8% compounded quarterly?

Solution.
The principal is found using the formula

$$P = B\left(1 + \frac{r}{n}\right)^{-nt} = 8,000\left(1 + \frac{0.08}{4}\right)^{-16} \approx \$5,827.57\blacksquare$$

Example 7.6
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

Solution.

$$\text{effective rate} = \left(1 + \frac{0.05}{4}\right)^4 - 1 \approx 0.051 = 5.1\%\blacksquare$$

Continuous Compound Interest
When the compound formula is used over smaller time periods the interest becomes slightly larger and larger. That is, frequent compounding earns a higher effective rate, though the increase is small.
This suggests compounding more and more, or equivalently, finding the value of B in the long run. In Calculus, it can be shown that the expression $\left(1+\frac{r}{n}\right)^n$ approaches e^r as $n \to \infty$, where e (named after Euler) is a number whose value is $e = 2.71828\cdots$. The balance formula reduces to $B = Pe^{rt}$. This formula is known as the **continuous compound formula**. In this case, the annual effective interest rate is found using the formula $e^r - 1$.

Example 7.7
Find the effective rate if \$1000 is deposited at 5% annual interest rate compounded continuously.

Solution.
The effective interest rate is $e^{0.05} - 1 \approx 0.05127 = 5.127\%$ ■

Example 7.8
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

Solution.
The effective rate corresponding to the first option is

$$\left(1 + \frac{0.08}{4}\right)^4 - 1 \approx 8.24\%$$

That of the second option

$$e^{0.0795} - 1 \approx 8.27\%$$

Thus, we see that 7.95% compounded continuously is better than 8% compounded quarterly. ■

Present and Future Value
Many business deals involve payments in the future. For example, when a car or a home is bought on credits, payments are made over a period of time. The **future value**, FV, of a payment P is the amount to which P would have grown if deposited today in an interest bearing bank account. The **present value**, PV, of a future payment FV, is the amount that would have to be deposited in a bank account today to produce exactly FV in the account at the relevant time future.
If interest is compounded n times a year at a rate r for t years, then the relationship between FV and PV is given by the formula

$$FV = PV(1 + \frac{r}{n})^{nt}.$$

In the case of continuous compound interest, the forumla is given by

$$FV = PVe^{rt}.$$

Example 7.9
You need \$10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

Solution.
We have $FV = \$10,000, r = 0.08, t = 3$ and we want to find PV. Solving the formula $FV = PVe^{rt}$ for PV we find $PV = FVe^{-rt}$. Substituting to obtain, $PV = 10,000e^{-0.24} \approx \$7,866.28$.■