47 Similar Triangles

An overhead projector forms an image on the screen which has the same shape as the image on the transparency but with the size altered. Two figures that have the same shape but not necessarily the same size are called **similar**. In the case of triangles, this means that the two triangles will have the same angles and their sides will be in the same proportion (for example, the sides of one triangle might all be 3 times the length of the sides of the other). More formally, we have the following definition:
Two triangles ΔABC and ΔDEF are similar, written $\Delta ABC \sim \Delta DEF$ if and only if
(i) $m(\angle A) = m(\angle D), m(\angle B) = m(\angle E)$, and $m(\angle C) = m(\angle F)$

(ii) $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

The common ratio in (ii) is called the **scaled-factor.** An example of two similar triangles is shown in Figure 47.1

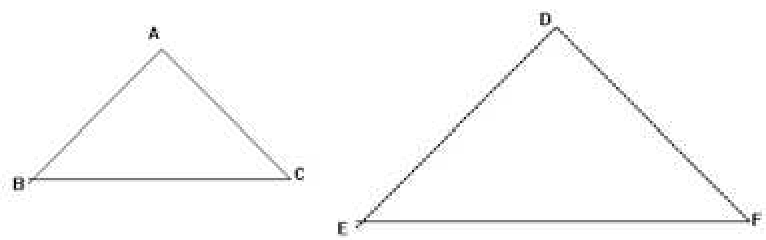


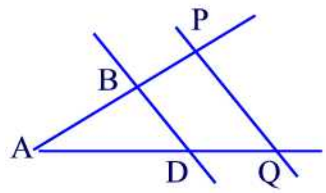
Figure 47.1

Example 47.1
Suppose in Figure 47.1 that EF = 6 cm, BC = 2cm and AB = 3cm. What is the value of DE?

Solution.
Since DE = 3AB and AB = 3 then DE = 9 cm ■

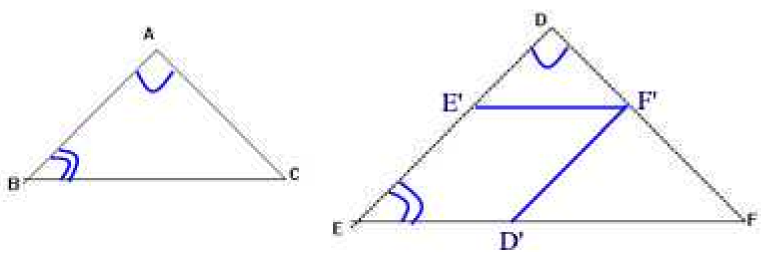
The proofs of various properties of similar triangles depend upon certain properties of parallel lines. Mainly, we need the following theorem which we state without proof.

Theorem 47.1
Let PQ be a line intersecting an angle $\angle BAD$. Then $\frac{AP}{AB} = \frac{AQ}{AD}$ if and only if the lines PQ and BD are parallel.



When we attempted to prove two triangles to be congruent we had a few tests SSS, SAS, ASA. In a similar way we have a few tests to help us determine whether two triangles are similar.
If the measures of two angles of a triangle are given, then the measure of the third angle is known automatically. Thus, the shape of the triangle is completely determined. Since similar triangles have the same shape, we have the following similarity condition.

Theorem 47.2 *(AA Test)*
If two angles of one triangle are congruent with the corresponding two angles of another triangle, then the two triangles are similar.



Proof.
Let $ABC \leftrightarrow DEF$ denote a correspondence of ΔABC and ΔDEF in which $m(\angle A) = m(\angle D)$ and $m(\angle B) = m(\angle E)$. Then $m(\angle A) + m(\angle B) = m(\angle D) + m(\angle E)$. But $180° - m(\angle C) = m(\angle A) + m(\angle B)$ and $180° - m(\angle F) = m(\angle D) + m(\angle E)$. Hence, $m(\angle C) = m(\angle F)$.
Now, let F' be a point on $\overline{DF}$ and E' be a point on $\overline{DE}$ such that $DF' = AC$ and $DE' = AB$. Then by SAS test for congruent triangles we have $\Delta ABC \cong DE'F'$. Hence, $m(\angle C) = m(\angle DF'E')$ and $m(\angle DF'E') = m(\angle DFE)$. This last equality implies that $FE || F'E'$ by the Corresponding Angles Theorem.

Hence, Theorem 47.1 implies

$$\frac{DF'}{DF} = \frac{DE'}{DE}.$$

Since $DF' = AC$ and $DE' = AB$ then we have

$$\frac{AC}{DF} = \frac{AB}{DE}.$$

Now, let D' be a point on $\overline{FE}$ such that $FD' = CB$. Then by repeating the above argument we find

$$\frac{AC}{DF} = \frac{BC}{FE}.$$

It follows that

$$\frac{AC}{DF} = \frac{BC}{FE} = \frac{AB}{DE}.$$

This shows $\Delta ABC \sim \Delta DEF$ ■

Example 47.2
Consider Figure 47.2.

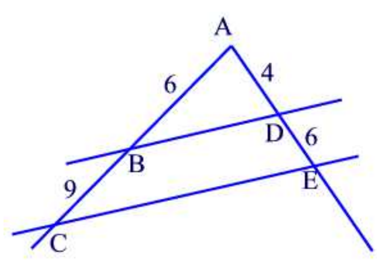


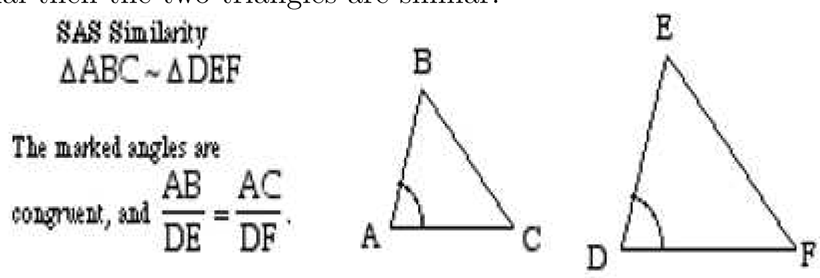
Figure 47.2

Show that $\frac{BD}{CE} = \frac{AD}{AE} = \frac{AB}{AC} = \frac{2}{5}$

Solution.
Since $m(\angle A) = m(\angle A)$ and $m(\angle ADB) = m(\angle AEC)$ then by the AA test we have $\Delta ABD \sim \Delta ACE$. Hence,

$$\frac{BD}{CE} = \frac{AD}{AE} = \frac{AB}{AC} = \frac{6}{15} = \frac{2}{5} \blacksquare$$

Theorem 47.3 *(SAS Test)*
If two sides of one triangle are proportional to the two corresponding sides of the second triangle and the angles between the two sides of each triangle are equal then the two triangles are similar.



Proof.
Let E' be a point on the line segement $\overline{DE}$ such that $DE' = AB$. Also, let F' be a point on $\overline{DF}$ such that $DF' = AC$. See Figure 47.3.

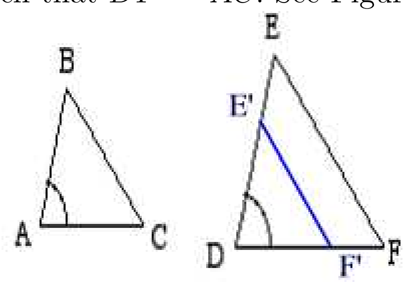


Figure 47.3

Since

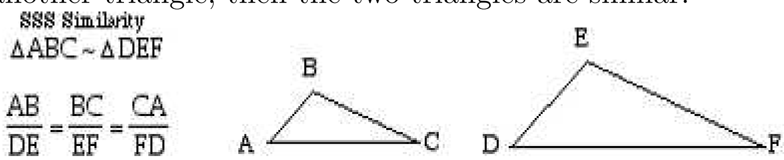
$$\frac{AB}{DE} = \frac{AC}{DF}$$

then

$$\frac{DE'}{DE} = \frac{DF'}{DF}.$$

By Theorem 47.1, the lines EF and $E'F'$ are parallel. Thus, $m(\angle DEF) = m(\angle DE'F')$ and $m(\angle DFE) = m(\angle DF'E')$. By the AA similarity theorem we have $\Delta DE'F' \sim \Delta DEF$. But by the SAS congruence test, we have $\Delta ABC \cong \Delta DE'F'$ and in particular $\Delta ABC \sim \Delta DE'F'$ since two congruent triangles are also similar. Hence, $\Delta ABC \sim \Delta DEF$ ■

Theorem 47.4 *(SSS Test)*
If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.



Proof.
Let E' be a point on the line segement $\overline{DE}$ such that $DE' = AB$. Also, let F' be a point on $\overline{DF}$ such that $DF' = AC$. See Figure 47.4.

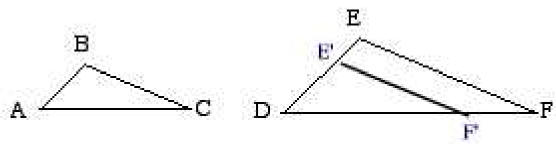
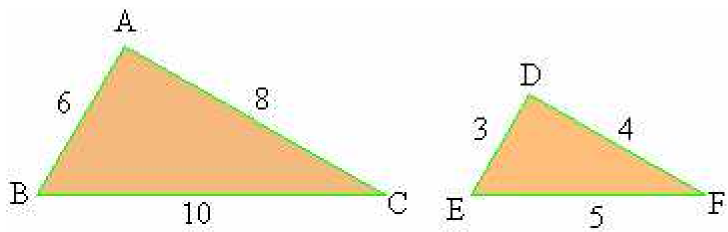


Figure 47.4

Since
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$
then
$$\frac{DE'}{DE} = \frac{DF'}{DF} = \frac{BC}{EF}.$$
By Theorem 47.1, the lines EF and $E'F'$ are parallel. By SAS similarity theorem , we have $\Delta DE'F' \sim \Delta DEF$. Thus,
$$\frac{DE'}{DE} = \frac{DF'}{DF} = \frac{E'F'}{EF}.$$
But $\frac{DE'}{DE} = \frac{BC}{EF}$. Hence, $E'F' = BC$. By the SSS congruence theorem we have $\Delta ABC \cong \Delta DE'F'$. Hence, $\Delta ABC \sim \Delta DE'F'$. Since $\Delta DE'F' \sim \Delta DEF$ then $\Delta ABC \sim \Delta DEF$ $\blacksquare$

Example 47.3
Show that the triangles in the figure below are similar.



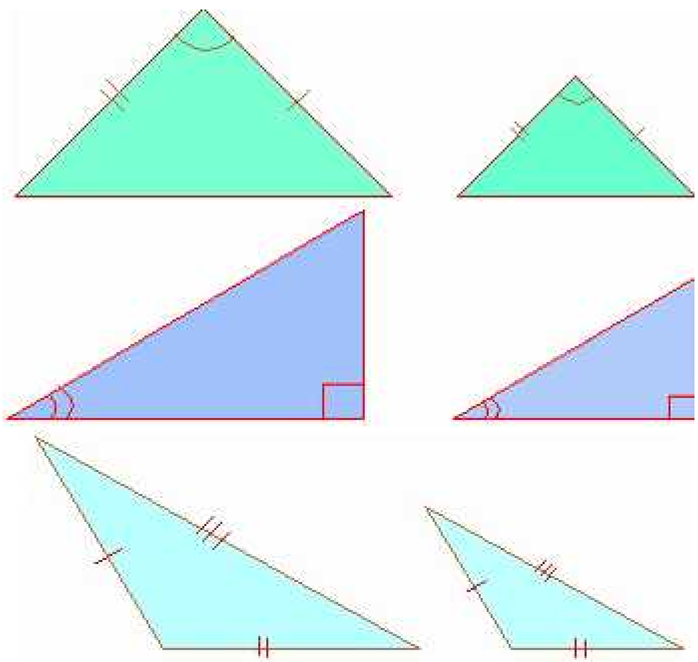
Solution.
Since $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = 2$ then by Theorem 47.4 $\Delta ABC \sim \Delta DEF$ ■

Practice Problems

Problem 47.1
Which of the following triangles are always similar?
(a) right triangles
(b) isosceles triangles
(c) equilateral triangles

Problem 47.2
Show that if $\Delta ABC \sim \Delta A'B'C'$ and $\Delta A'B'C' \sim \Delta A''B''C''$ then $\Delta ABC \sim \Delta A''B''C''$.

Problem 47.3
Each pair of triangles is similar. By which test can they be proved to be similar ?



Problem 47.4
Suppose $\Delta ABC \sim \Delta DEF$ with scaled factor k.
(a) Compare the perimeters of the two triangles.
(b) Compare the areas of the two triangles.

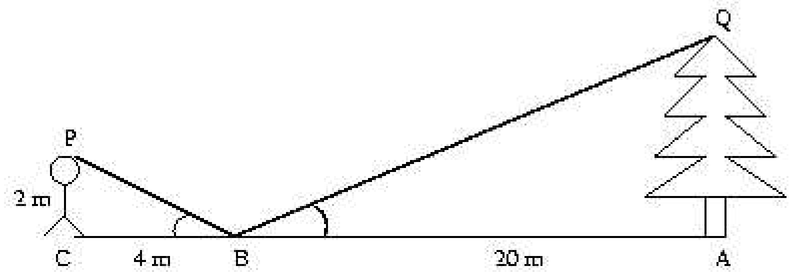
Problem 47.5
Areas of two similar triangles are 144 sq.cm. and 81 sq.cm. If one side of the first triangle is 6 cm then find the corresponding side of the second triangle.

Problem 47.6
The side of an equilateral triangle ΔABC is 5 cm. Find the length of the side of another equilateral triangle ΔPQR whose area is four times area of ΔABC.

Problem 47.7
The corresponding sides of two similar triangles are 4 cm and 6 cm. Find the ratio of the areas of the triangles.

Problem 47.8
A clever outdoorsman whose eye-level is 2 meters above the ground, wishes

to find the height of a tree. He places a mirror horizontally on the ground 20 meters from the tree, and finds that if he stands at a point C which is 4 meters from the mirror B, he can see the reflection of the top of the tree. How high is the tree?

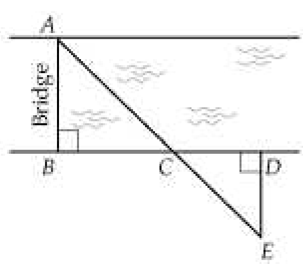


Problem 47.9
A child 1.2 meters tall is standing 11 meters away from a tall building. A spotlight on the ground is located 20 meters away from the building and shines on the wall. How tall is the child's shadow on the building?

Problem 47.10
On a sunny day, Michelle and Nancy noticed that their shadows were different lengths. Nancy measured Michelle's shadow and found that it was 96 inches long. Michelle then measured Nancy's shadow and found that it was 102 inches long.

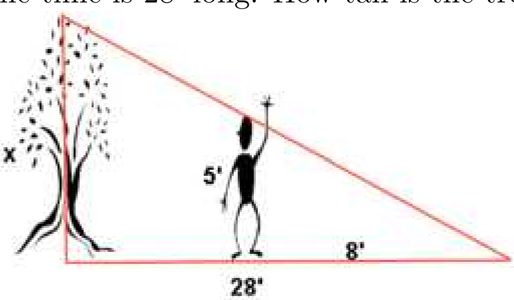
(a) Who do you think is taller, Nancy or Michelle? Why?
(b) If Michelle is 5 feet 4 inches tall, how tall is Nancy?
(c) If Nancy is 5 feet 4 inches tall, how tall is Michelle?

Problem 47.11
An engineering firm wants to build a bridge across the river shown below. An engineer measures the following distances: BC = 1,200 feet, CD = 40 feet, and DE = 20 feet.

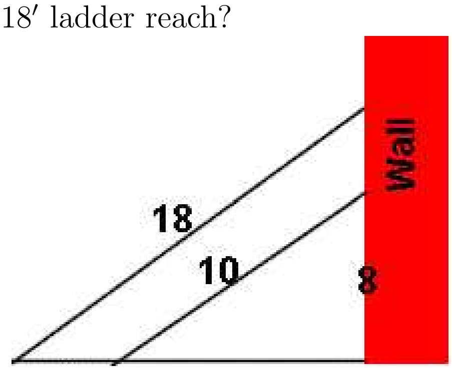


(a) Prove ΔABC is similar to ΔEDC.
(b) Railings cost \$4 per foot. How much will it cost to put railings on both sides of the bridge?

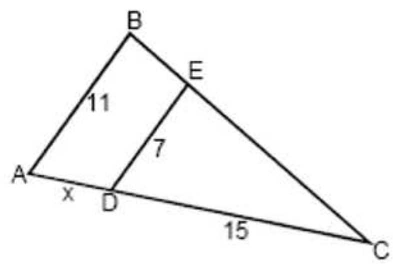
Problem 47.12
At a certain time of the day, the shadow of a $5'$ boy is $8'$ long. The shadow of a tree at this same time is $28'$ long. How tall is the tree?



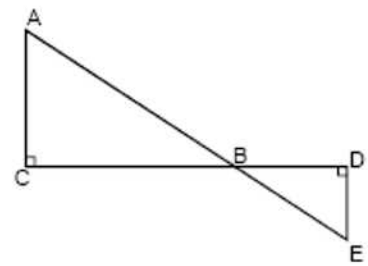
Problem 47.13
Two ladders are leaned against a wall such that they make the same angle with the ground. The $10'$ ladder reaches $8'$ up the wall. How much further up the wall does the $18'$ ladder reach?



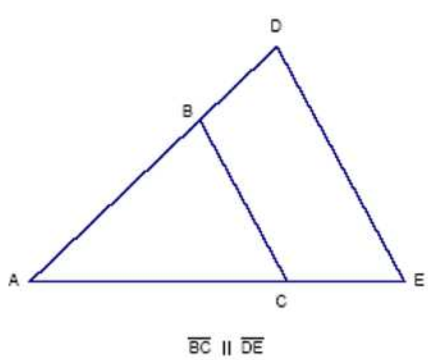
Problem 47.14
Given that lines DE and AB are parallel in the figure below, determine the value of x, i.e. the distance between points A and D.

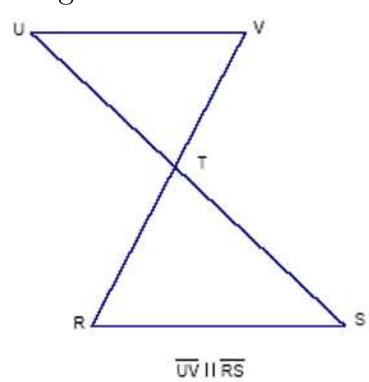


Problem 47.15
In the figure below, lines AC and DE are vertical, and line CD is horizontal. Show that $\Delta ABC \sim \Delta EBD$.

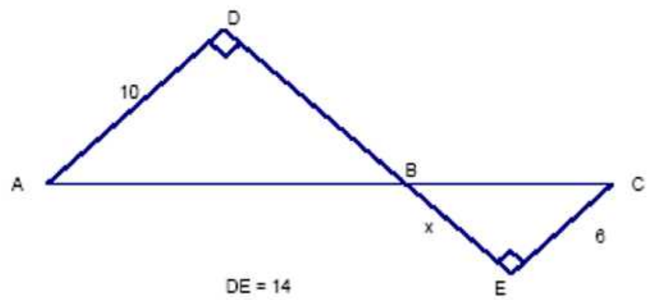


Problem 47.16
Find a pair of similar triangles in each of these figures:

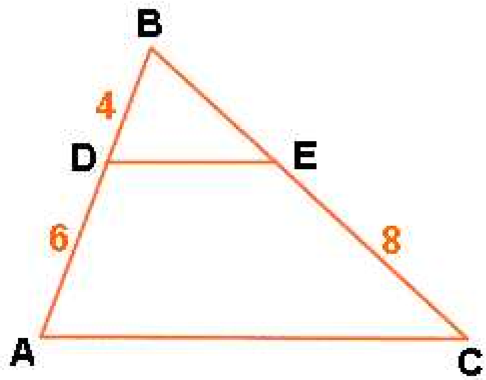




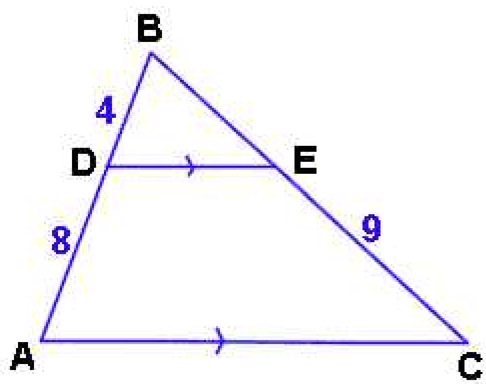
Problem 47.17
Find x :



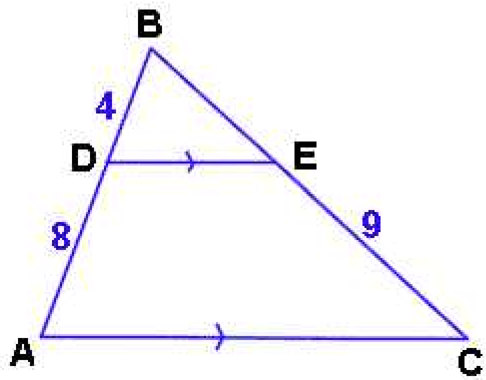
Problem 47.18
In the diagram, DE is parallel to AC. Also, BD = 4, DA = 6 and EC = 8. Find BC to the nearest tenth.



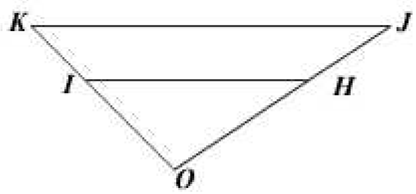
Problem 47.19
Find BC.



Problem 47.20
Find BE



Problem 47.21
Copy and complete the given table. It is given that $\frac{OH}{HJ} = \frac{OI}{IK}$.



OH	HJ	OJ	OI	IK	OK
15	5	?	45	?	?