46 Congruence of Triangles

Two triangles are congruent if one can be moved on top of the other, so that edges and vertices coincide. The corresponding sides have the same lengths, and corresponding angles are congruent. That is, two triangles ΔABC and $\Delta A'B'C'$ are **congruent** if $m(\angle A) = m(\angle A'), m(\angle B) = m(\angle B'), m(\angle C) = m(\angle C'), AB = A'B', AC = A'C'$, and $BC = B'C'$. We write $\Delta ABC \cong \Delta A'B'C'$.

Example 46.1
Suppose that $\Delta ABC \cong \Delta A'B'C', AB = A'B', AB = 2x + 10$, and $A'B' = 4x - 20$. Find x.

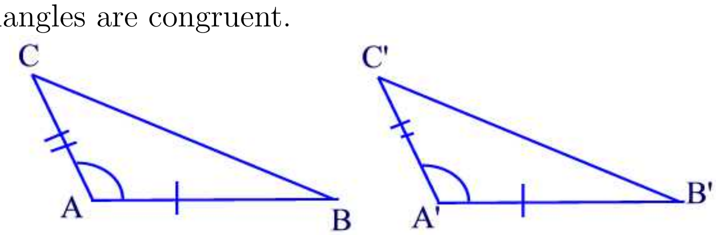
Solution.
Since $AB = A'B'$ then

$$\begin{array}{rcr} 4x - 20 & = & 2x + 10 \\ 4x - 2x & = & 10 + 20 \\ 2x & = & 30 \\ x & = & 15 \blacksquare \end{array}$$

Remark 46.1
It is very important to maintain the vertices in the proper order. Not doing so is a common mistake.

Simpler conditions can be applied to verify that two triangles are congruent. The first one involves two sides and the included angle. We adopt this result as an axiom, that is we accept this result as true by assumption, not by a proof as in the case of a theorem.

Axiom(Side-Angle-Side)
If two triangles have two sides and the included angles equal, respectively, then the triangles are congruent.

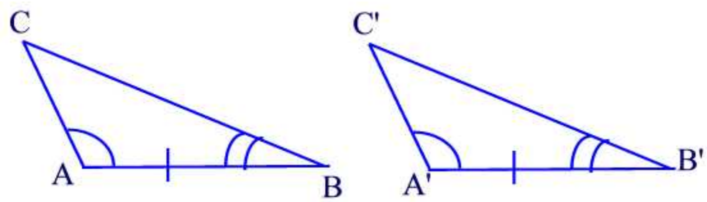


Example 46.2
In ΔABC, if $AB = AC$ then $m(\angle B) = m(\angle C)$.

Solution.
Consider the correspondence of vertices $A \leftrightarrow A, B \leftrightarrow C$, and $C \leftrightarrow B$. Under this correspondence, two sides and the included angle of ΔABC are congruent respectively to the corresponding sides and included angle of ΔACB. Hence, by SAS the triangles are congruent. Therefore, the corresponding angles are congruent and $m(\angle B) = m(\angle C)$ ■

The second congruence property that we consider involves two angles and the side included.

Theorem 46.1 *(Angle-Side-Angle)*
If two angles and the included side of a triangle are congruent, respectively, to two angles and the included side of another triangle, then the two triangles are congruent.



Proof.
We start with two triangles ΔABC and $\Delta A'B'C'$, where $m(\angle A) = m(\angle A'), AB = A'B'$, and $m(\angle B) = m(\angle B')$. We will show $BC = B'C'$ and then apply SAS theorem above. (The same sort of argument shows $AC = A'C'$.) In relating BC and B'C', there are three possibilities: BC = B'C', $BC < B'C'$, or $BC > B'C'$. If the first case holds, we are done. Now suppose $BC < B'C'$, then we can find a point G between B' and C' so that $BC = B'G$, and then $\Delta ABC \cong \Delta A'B'G$ by SAS. In particular, this implies $m(\angle C'A'B') = m(\angle CAB) = m(\angle GA'B')$. However, the segement $\overline{A'G}$ lies inside the angle $\angle C'A'B'$, so $m(\angle GA'B') < m(\angle C'A'B')$, which contradicts $m(\angle GA'B') = m(\angle C'A'B')$. So we cannot have $BC < B'C'$. A similar argument arrives at a contradiction if $BC > B'C'$, and so the only possibility is that $BC = B'C'$ ■

Example 46.3
In Figure 46.1, we have $AC = CD$. Show that $\Delta ABC \cong \Delta DEC$.

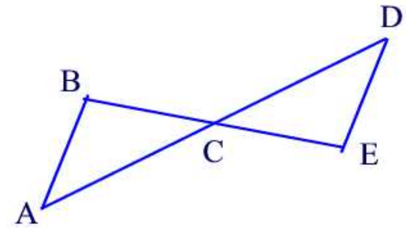
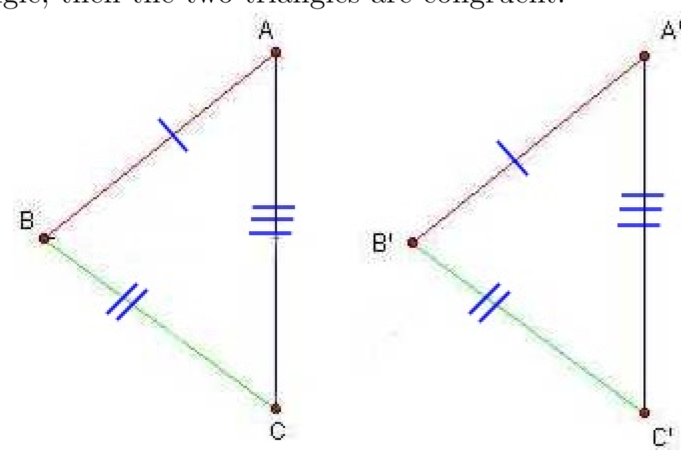


Figure 46.1

Solution.
Note that AD is a transversal line crossing two parallel lines. Then $m(\angle ACB) = m(\angle DCE)$ (vertical angles) and $m(\angle CAB) = m(\angle CDE)$ (alternate interior angles). Moreover, AC = CD so by ASA the two triangles are congruent ■

The third property of congruence involves the three sides of the triangles.

Theorem 46.2 *(Side-Side-Side)*
If a triangle has all three sides congruent to the corresponding sides of a second triangle, then the two triangles are congruent.



Proof.
Let B'' be the point such that $m(\angle ACB'') = m(\angle A'C'B')$ and $m(\angle CAB'') = m(\angle C'A'B')$ and $B'' \neq B$. See Figure 46.2

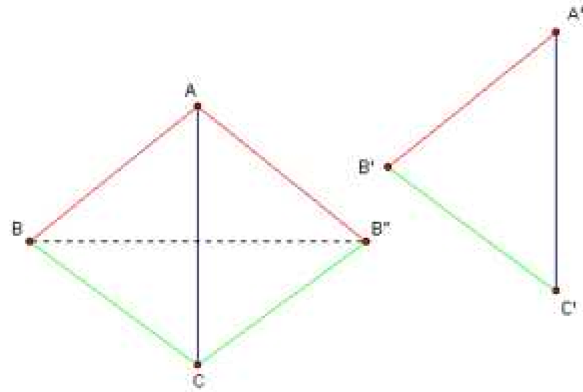


Figure 46.2

This implies by angle-side-angle congruence that $\Delta AB''C \cong \Delta A'B'C'$. Also triangles $\Delta ABB''$ and $\Delta CBB''$ are two isosceles triangles. Thus, the base angles are congruent. This implies $m(\angle ABB'') = m(\angle AB''B)$ and $m(\angle CBB'') = m(\angle CB''B)$. Thus, $m(\angle ABC) = m(\angle AB''C)$. Hence, by SAS, $\Delta ABC \cong \Delta AB''C$. Therefore triangles ABC and $A'B'C'$ are congruent $\blacksquare$

Example 46.4
In Figure 46.3, we are given that AC = CD, AB = BD. Show that $\Delta ABC \cong \Delta DBC$.

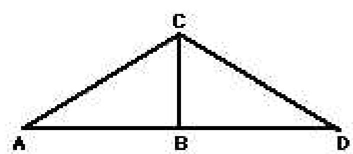
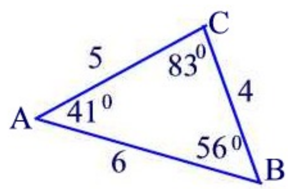


Figure 46.3

Solution.
Since AC=CD, AB=BD, and BC = CB then by SSS we have $\Delta ABC \cong \Delta DBC$ $\blacksquare$

Practice Problems

Problem 46.1
Suppose $\Delta JKL \cong \Delta ABC$, where ΔABC is shown below.

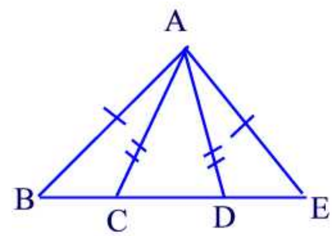


Find the following
(a) KL (b) LJ (c) $m(\angle L)$ (d) $m(\angle J)$

Problem 46.2
Using congruence of triangles show that equilateral triangles are equiangular.

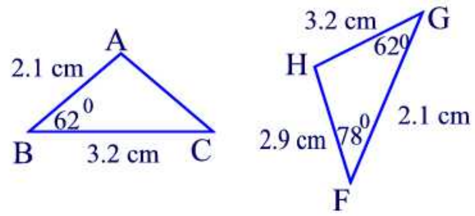
Problem 46.3
Let the diagonals of a parallelogram ABCD intersect at a point M.
(a) Show that $\Delta ABM \cong \Delta CDM$.
(b) Use part (a) to explain why M is the midpoint of both diagonals of the parallelogram.

Problem 46.4
In the figure below, AB = AE and AC=AD.



(a) Show that $m(\angle B) = m(\angle E)$
(b) Show that $m(\angle ACD) = m(\angle ADC)$
(c) Show that $\Delta ABC \cong \Delta AED$
(d) Show that $BC = DE$.

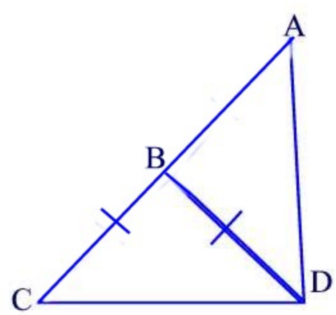
Problem 46.5
Consider the following figure.



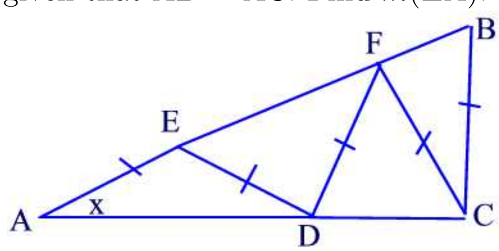
(a) Find AC
(b) Find $m(\angle H), m(\angle A)$, and $m(\angle C)$.

Problem 46.6
Show that if $\Delta ABC \cong \Delta A'B'C'$ and $\Delta A'B'C' \cong \Delta A''B''C''$ then $\Delta ABC \cong \Delta A''B''C''$.

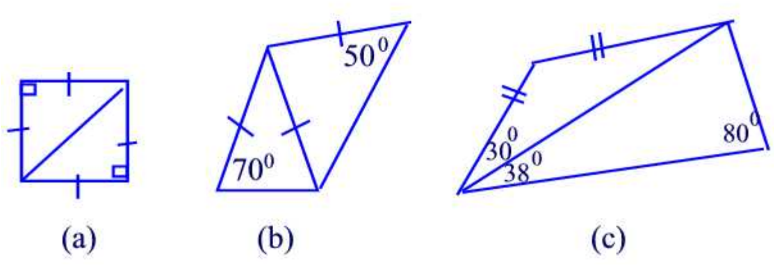
Problem 46.7
In the figure below, given that AB=BC=BD. Find $m(\angle ADC)$.



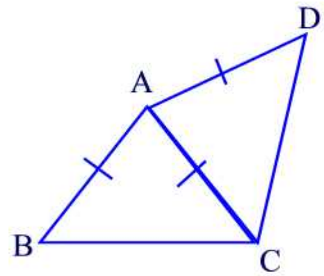
Problem 46.8
In the figure below given that AB = AC. Find $m(\angle A)$.



Problem 46.9
Find all missing angle measures in each figure.



Problem 46.10
An eighth grader says that AB=AC=AD , as shown in the figure below, then $m(\angle B) = m(\angle C) = m(\angle D)$. Is this right? If not, what would you tell the child?

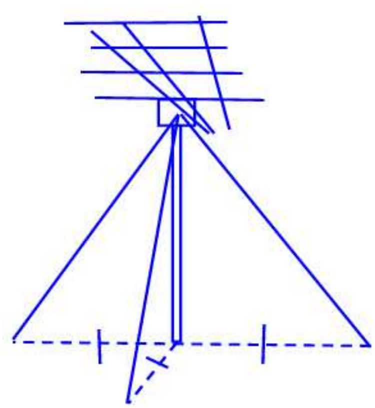


Problem 46.11
What type of figure is formed by joining the midpoints of a rectangle?

Problem 46.12
If two triangles are congruent what can be said about their perimeters? areas?

Problem 46.13
In a pair of right triangles, suppose two legs of one are congruent to respectively to two legs of the other. Explain whether the triangles are congruent and why.

Problem 46.14
A rural homeowner had his television antenna held in place by three guy wires, as shown in the following figure. If the distance to each of the stakes from the base of the antenna are the same, what is true about the lengths of the wires? Why?



Problem 46.15
For each of the following, determine whether the given conditions are sufficient to prove that $\Delta PQR \cong \Delta MNO$. Justify your answer.
(a) PQ = MN, PR = MO, $m(\angle P) = m(\angle M)$
(b) PQ = MN, PR = MO, QR = NO
(c) PQ = MN, PR = MO, $m(\angle Q) = m(\angle N)$

Problem 46.16
Given that $\Delta RST \cong \Delta JLK$, complete the following statements.
(a) $\Delta TRS \cong \Delta$____
(b) $\Delta SRT \cong \Delta$____
(c) $\Delta TSR \cong \Delta$____
(d) $\Delta JKL \cong \Delta$____

Problem 46.17
You are given ΔRST and ΔXYZ with $m(\angle S) = m(\angle Y)$.
(a) To show $\Delta RST \cong \Delta XYZ$ by the SAS congruence property, what more would you need to know?
(b) To show that $\Delta RST \cong \Delta XYZ$ by the ASA congruence property, what more would you need to know?

Problem 46.18
You are given ΔABC and ΔGHI with AB = GH. To show that $\Delta ABC \cong \Delta GHI$ by the SSS congruence property, what more would you need to know?

Problem 46.19
Suppose that ABCD is a kite with AB = AD and BC = DC. Show that the diagonal $\overline{AC}$ divides the kite into two congruent triangles.

Problem 46.20
Show that the diagonals of a kite are perpendicular to each other.

Problem 46.21
(a) Show that the diagonal of a parallelogram divides it into two congruent triangles.
(b) Use part (a) to show that the opposite sides of a parallelogram are congruent.
(c) Use part (a) to show that the opposite angles of a parallelogram are congruent.