

45 Volume

Surface area measures the area of the two-dimensional boundary of a three-dimensional figure; it is the area of the outside surface of a solid. Volume, on the other hand, is a measure of the space a figure occupies, or the space "inside" the three-dimensional figure. "How much cardboard does it take to make the box?" is a question about surface area; "How much cereal can be put into the box?" is a question about volume. Surface area is measured in squares of length units, such as square feet (ft^2), square centimeters (cm^2) or square inches (in^2). However, volume is measured in cubes of length units, such as cubic feet (ft^3), cubic centimeters (cm^3) or cubic inches (in^3).

Rectangular Prisms

Consider a rectangular box of length 4 cm, width 2 cm, and height 3 cm. Then this box occupy the same space as 24 boxes each of length 1 cm, width 1 cm, and height 1 cm. Note that $2 \times 4 \times 3 = 24$. Thus, the volume of the box is $V = 24 cm^3$. This suggests that the volume of a rectangular prism with length l , width w , and height h is given by the formula

$$V = lwh.$$

Since lw is the area B of the base of the prism then we can say that

$$\text{Volume of rectangular prism} = \text{area of base} \times \text{height}$$

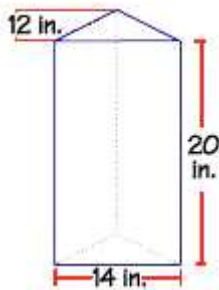
This formula is valid for any prism whose base area is B and height is h

Remark 45.1

Note that a cube is a rectangular prism with $l = w = h = s$ so that its volume is $V = s^3$

Example 45.1

Find the volume of the prism given below.



Solution.

First we have to find the area of the triangle that forms the base of this prism. The area of a triangle is one-half the base of the triangle times the height of the triangle. The triangle has a base of 14 inches and a height of 12 inches. If we substitute these values into the formula, we get 84 square inches for the area of the triangle.

$$B = \frac{1}{2} \times 14 \times 12 = 84 \text{ in}^2$$

Since $h = 20 \text{ in}$ then the total volume is

$$V = 84 \times 20 = 1680 \text{ in}^3 \blacksquare$$

Right Pyramids

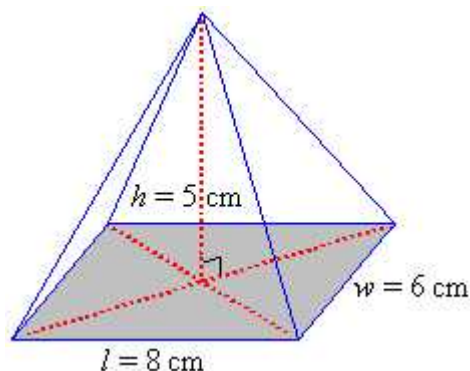
Consider a pyramid with rectangular base of area B and height h . Also, consider a rectangular box of area of base B and height h . Fill the pyramid with rice and pour the content into the box. Repeat this process three times. You will notice that the cube is completely full with rice. This shows that the volume of the pyramid is one-third that of the box. That is,

$$V = \frac{1}{3}Bh.$$

This formula is true for any right pyramid.

Example 45.2

Find the volume of a rectangular-based pyramid whose base is 8 cm by 6 cm and height is 5 cm.



Solution.

The volume is

$$V = \frac{1}{3} \times 8 \times 6 \times 5 = 80 \text{ cm}^3 \blacksquare$$

Right Cylinders

Since we know how to calculate the volume of a prism, we use a regular prism to approach a cylinder with base radius r and height h : a regular prism whose base is a quadrilateral, a pentagon, a hexagon, a heptagon, an octagon, or a many-sided regular polygon. As the number of the sides increases, the perimeter of the regular polygon approaches the circumference of the circle, and the area of the base polygon approaches the base area of the cylinder. As a result, the volume of the regular prism approaches the volume of the cylinder as shown in Figure 45.2

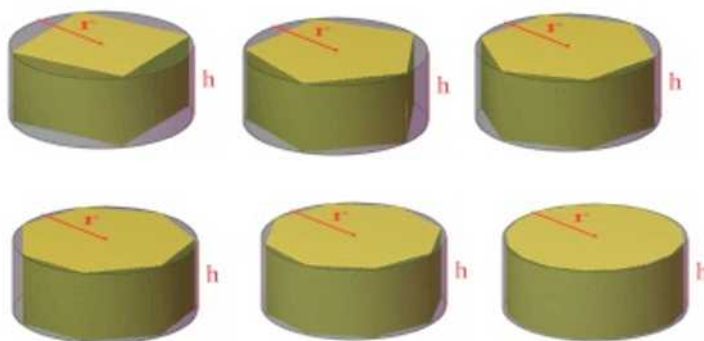


Figure 45.2

Eventually, the regular prism and the cylinder fit together perfectly when the number of the sides is large enough. Since the volume of the regular prism is the product of its base area and its height, and since its base area and the cylinder base fit together perfectly, the volume of the cylinder is also the product of its base area and its height. Since the base area is πr^2 then the volume of the cylinder is:

$$V = \pi r^2 h.$$

Example 45.3

A steel pipe is 35.0 cm long and has an inside radius of 8.0 cm and an outside radius of 10.0 cm. How much steel is needed to build the pipe? (The density

of steel is about 7.8 g/cm^3)



Solution.

Given:

Inner radius $r_1 = 8.0 \text{ cm}$
Outer radius $r_2 = 10.0 \text{ cm}$
Steel pipe length $h = 35.0 \text{ cm}$

The volume of the pipe is the difference of two cylinders: one with the radius of 10.0 cm and the other one with the radius of 8.0 cm. That is,

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= 3.14 \times 10.0^2 \times 35.0 - 3.14 \times 8.0^2 \times 35.0 \\ &= 3956.4 \text{ cm}^3 \end{aligned}$$

The weight of the pipe:

$$w = Volume \times density = 30859.9 \text{ g} \approx 30.86 \text{ kg} \blacksquare$$

Right Circular Cones

Consider a cone with radius r and height h . Take a cylinder of radius r and height h . Fill the cone with rice til its top and pour the content into the cylinder. Repeat this process three times. After the third time you will notice that the cylinder is completely full. This shows that the volume of the cone is one third the volume of the cylinder. But the volume of the cylinder is $\pi r^2 h$. Hence, the volume of the cone is

$$V = \frac{1}{3} \pi r^2 h$$

Spheres

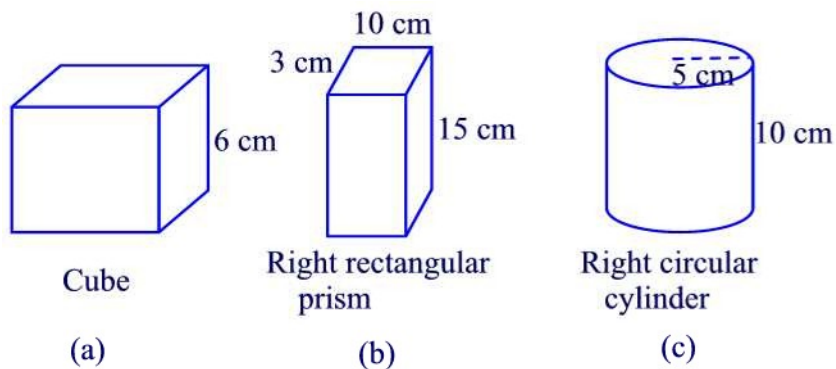
We find the volume of a sphere as follows: Take half a sphere of radius r and fill it completely with rice. Empty the rice into a right cylinder of radius r and height r . You will notice that the rice fill up $\frac{2}{3}$ of the cylinder. This shows that the volume of the hemisphere is equal to $\frac{2}{3}$ the volume of the cylinder. But the volume of the cylinder is πr^3 . Hence, the volume of a hemisphere is $\frac{2}{3}\pi r^3$. The volume of the sphere is twice the volume of a hemisphere and is given by the formula

$$V = \frac{4}{3}\pi r^3.$$

Practice Problems

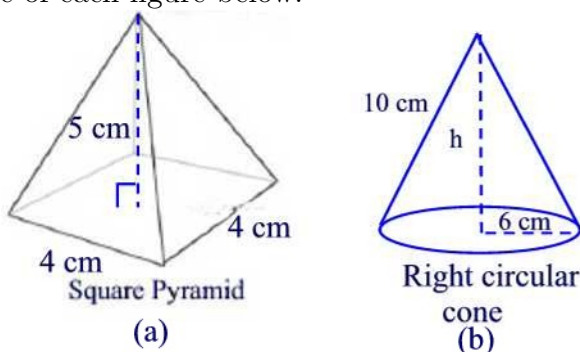
Problem 45.1

Find the volume of each figure below.



Problem 45.2

Find the volume of each figure below.



Problem 45.3

Maggie is planning to build a new one-story house with floor area of 2000 ft^2 . She is thinking of putting in a 9-ft ceiling. If she does this, how many cubic feet of space will she have to heat or cool?

Problem 45.4

Two cubes have sides lengths 4 cm and 6 cm, respectively. What is the ratio of their volumes?

Problem 45.5

What happens to the volume of a sphere if its radius is doubled?

Problem 45.6

An olympic-sized pool in the shape of a right rectangular prism is $50 \text{ m} \times 25 \text{ m}$. If it is 2 m deep throughout, how many liters of water does it hold? Recall that $1 \text{ m}^3 = 1000 \text{ L}$.

Problem 45.7

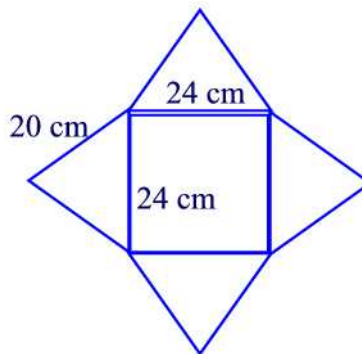
A standard straw is 25 cm long and 4 mm in diameter. How much liquid can be held in the straw at one time?

Problem 45.8

The pyramid of Khufu is 147 m high and its square base is 231 m on each side. What is the volume of the pyramid?

Problem 45.9

A square right regular pyramid is formed by cutting, folding, and gluing the following pattern.



- (a) What is the slant height of the pyramid?
- (b) What is the lateral surface area of the pyramid?
- (c) Use the Pythagorean formula to find the height of the pyramid.
- (d) What is the volume of the pyramid?

Problem 45.10

A cube 10 cm on a side holds 1 liter. How many liters does a cube 20 cm on a side hold?

Problem 45.11

A right circular cone has height r and a circular base of radius $2r$. Compare the volume of the cone to that of a sphere of radius r .

Problem 45.12

A store sell two types of freezers. Freezer A costs \$350 and measures 2 ft by 2 ft by 4.5 ft. Freezer B cots \$480 and measures 3 ft by 3 ft by 3.5 ft. Which freezer is the better buy?

Problem 45.13

Write a sentence that tells the difference between the surface area and volume of a prism.

Problem 45.14

A cylindrical water tank has a radius of 6.0 m. About how high must be filled to hold $400.0 m^3$?

Problem 45.15

Roll an 8.5 by 11 in sheet of paper into a cylindrical tube. What is the diameter?

Problem 45.16

A cylindrical pipe has an inner radius r , an outer radius R , and length l . Find its volume.

Problem 45.17

A basketball has a diameter of 10 in. What is its volume?

Problem 45.18

A standard tennis can is a cylinder that holds three tennis ball.

- (a) Which is greater the circumference of the can or its height?
- (b) Find the volume of the can?

Problem 45.19

A cylindrical aquarium has a circular base with diameter 2 ft and height 3 ft. How much water does it hold, in cubic feet?

Problem 45.20

The circumference of a beach ball is 73 inches. How many cubic inches of air does the ball hold? Round your answer to the nearest cubic inch.