43 Perimeter and Area

Perimeters of figures are encountered in real life situations. For example, one might want to know what length of fence will enclose a rectangular field. In this section we will study the perimeters of polygons and circles.
By definition, the **perimeter** of a simple closed plane figure is the length of its boundary. The perimeter is always measured in units of length, such as feet or centimeters.
The perimeter of a polygon is defined to be the sum of the lengths of its sides. Figure 43.1 exhibits the perimeters of some standard plane figures. We will use p to denote the perimeter of a plane figure.

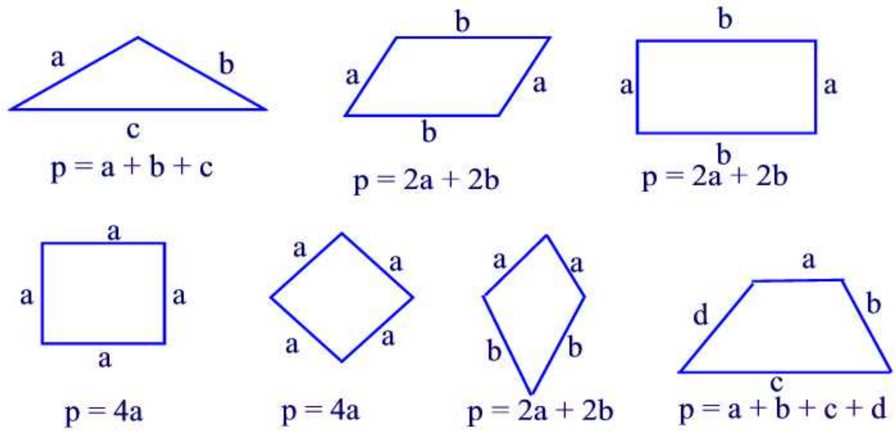


Figure 43.1

The Circumference of a Circle:
The perimeter of a circle is called its **circumference.** Using a tape measure around the circle one can find the circumference. Also, one will notice that the ratio of the circumference of a circle to its diameter is the same for all circles. This common ratio is denoted by π. Thus

$$\frac{C}{d} = \pi \text{ or } C = \pi d.$$

Since the diameter d is twice the radius r, we also have $C = 2\pi r$. It has been shown that π is an irrational number with unending decimal expansion: $\pi = 3.1415926\cdots$.

Example 43.1
A dining room table has 8 sides of equal length. If one side measures 15 inches, what is the perimeter of the table?

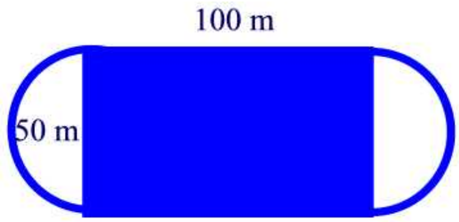
Solution.
If we denote the length of a side by a then the perimeter of the table is $p = 8a$. Since a = 15 in then p=15(8) = 120 in ■

Example 43.2
The circumference of a circle is 6π cm. Find its radius.

Solution.
Since $C = 2\pi r$ and $C = 6\pi$ then $2\pi r = 6\pi$. Solving for r we find $r = \frac{6\pi}{2\pi} = 3\ cm$ ■

Practice Problems

Problem 43.1
An oval track is made up by erecting semicircles on each end of a 50-m by 100-m rectangle as shown in the figure below.



What is the perimeter of the track?

Problem 43.2
Find each of the following:
(a) The circumference of a circle if the radius is 2 m.
(b) The radius of a circle if the circumference is 15π m.

Problem 43.3
Draw a triangle ABC. Measure the length of each side. For each of the following, tell which is greater?
(a) AB + BC or AC
(b) BC + CA or AB
(c) AB + CA or BC

Problem 43.4
Can the following be the lengths of the sides of a triangle? Why or Why not?
(a) 23 cm, 50 cm, 60 cm
(b) 10 cm, 40 cm, 50 cm
(c) 410 mm, 260 mm, 14 cm

Problem 43.5
Find the circumference of a circle with diameter 6π cm.

Problem 43.6
What happens to the circumference of a circle if the radius is doubled?

Problem 43.7
Find the length of the side of a square that has the same perimeter as a rectangle that is 66 cm by 32 cm.

Problem 43.8
A bicycle wheel has a diameter of 26 in. How far a rider travel in one full revolution of the tire? Use 3.14 for π.

Problem 43.9
A car has wheels with radii of 40 cm. How many revolutions per minute must a wheel turn so that the car travels 50 km/h?

Problem 43.10
A lot is 21 ft by 30 ft. To support a fence, an architect wants an upright post at each corner and an upright post every 3 ft in between. How many of these posts are needed?

Area
The number of units required to cover a region in the plane is known as its **area.** Usually squares are used to define a **unit square.** An area of "10 square units" means that 10 unit squares are needed to cover a flat surface. Below, we will find the area of some quadrilaterals and the area of a circle.

Area of a rectangle:
A 3-cm by 5-cm rectangle can be covered by 15 unit squares when the unit square is 1 cm^2, as shown in Figure 43.2(a). Similarly a 2.5-cm by 3.5 cm rectangle can be covered by six whole units, five half-unit squares, and one quarter unit square, giving a total area of 8.75 cm^2 as shown in Figure 43.2(b). This is also the product of the width and the length since $2.5\times 3.5 = 8.75\ cm^2$. It follows that a rectangle of length L and width W has an area A given by the formula

$$A = LW.$$

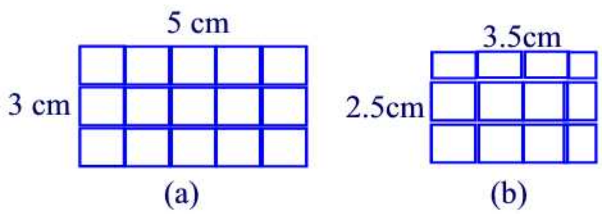


Figure 43.2

Example 43.3
The area of a rectangle of length L and width W is equal to its perimeter. Find the relationship between L and W.

Solution.
Since $A = LW$ and $p = 2(L+W)$ then $LW = 2(L+W)$. That is, $LW = 2L + 2W$. Subtracting 2L from both sides we obtain $L(W-2) = 2W$. Thus, $L = \frac{2W}{W-2}$ ■

Area of a square:
Since a square is a rectangle with length equals width then the area of a square with side length equals to s is given by the formula

$$A = s^2.$$

Area of a triangle:
First, we consider the case of a right triangle as shown in Figure 43.3(a). Construct rectangle ABDC where ΔDCB is a copy of ΔABC as shown in Figure 43.3(b). The area of the rectangle ABDC is bh, and the area of ΔABC is one-half the area of the rectangle. Hence, the area of $\Delta ABC = \frac{1}{2}bh$.
Now, suppose we have an arbitrary triangle as shown in Figure 43.3(c). Then

$$\text{Area of } \Delta ABC = \text{area of } \Delta ADC + \text{area of } \Delta CDB$$

That is,

$$\begin{aligned} A &= \tfrac{1}{2}h \cdot AD + \tfrac{1}{2}h \cdot DB \\ &= \tfrac{h}{2}(AD + DB) \\ &= \tfrac{h \cdot AB}{2} \end{aligned}$$

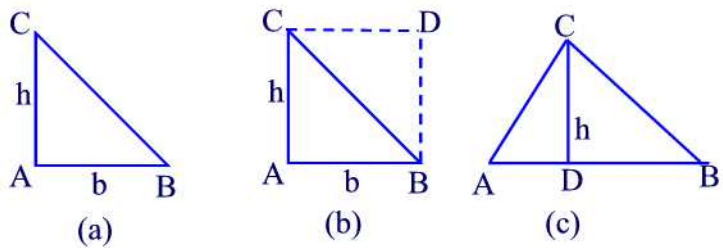


Figure 43.3

Area of a parallelogram:
Consider a parallelogram with a pair of opposite sides b units long, and these sides are h units apart as shown in Figure 43.4(a). The side with length b is called the **base** of the parallelogram and h is the **height** or **altitude.** Removing and replacing a right triangle as shown in Figure 43.4(b) results in a rectangle of length b and width h with the same area as the parallelogram. This means that the area of the parallelogram is given by the formula

$$A = bh.$$

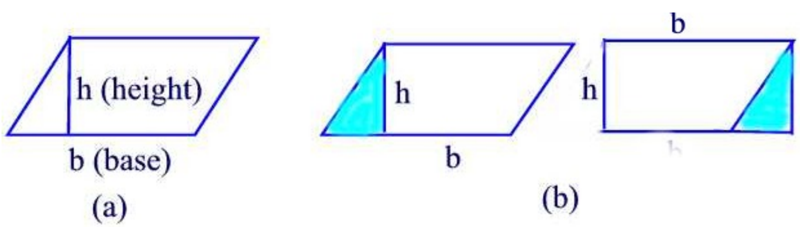
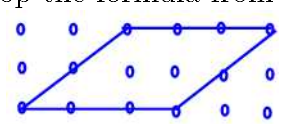


Figure 43.4

Example 43.4
Suppose a fifth grader has not yet learned the area formula for a parallelogram. How could you develop the formula from the geoboard diagram?



Solution.
The geoboard shows that the length of the base is 3 and the height is 2. Using unit parallelogram we see that 6 complete unit parallelograms cover the inside of the large parallelogram. This shows that the area is $2 \cdot 3 = 6$ ∎

Area of a Rhombus:

Consider the rhombus given in Figure 43.5. Since the diagonals bisect each other at right angles then we have

$$TQ = TS = \frac{QS}{2} \text{ and } TP = TR = \frac{PR}{2}$$

But

Area of rhombus = area of triangle PQR + area of triangle PRS

Thus,

$$\begin{aligned} A &= \frac{1}{2}TQ \cdot PR + \frac{1}{2}TS \cdot PR \\ &= \frac{PR}{2}(TQ + TS) \\ &= \frac{PR \cdot QS}{2} \end{aligned}$$

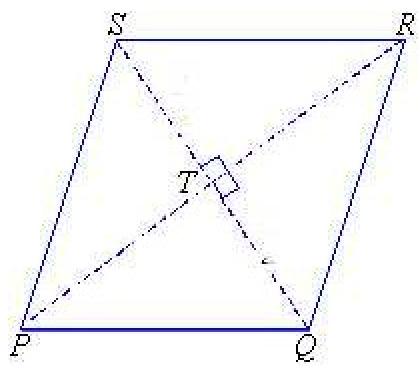


Figure 43.5

Area of a kite:

Consider the kite shown in Figure 43.6. Then

Area of kite = area of triangle DAB + area of triangle DCB

Thus,

$$\begin{aligned} A &= \frac{1}{2}TA \cdot BD + \frac{1}{2}TC \cdot BD \\ &= \frac{BD}{2}(TA + TC) \\ &= \frac{BD \cdot AC}{2} \end{aligned}$$

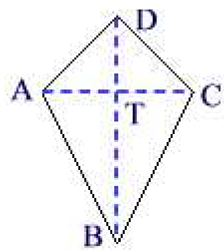


Figure 43.6

Area of a trapezoid:
Consider the trapezoid shown in Figure 43.7(a). Consider two identical trapezoids, and "turn" one around and "paste" it to the other along one side as pictured in Figure 43.7(b).

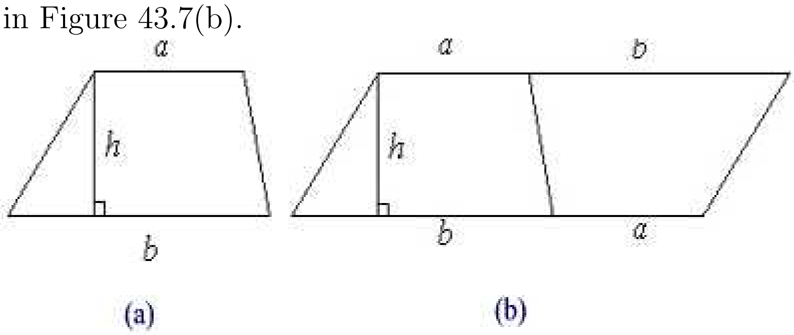


Figure 43.7

The figure formed is a parallelogram having an area of $h(a+b)$, which is twice the area of one of the trapezoids. Thus, the area of a trapezoid with height h and parallel sides a and b is given by the formula

$$A = \frac{h(a+b)}{2}.$$

Area of a circle:
We will try to discover the formula of the area of a circle, say of radius 7 cm, by executing the following steps:
Step 1: Using a compass, draw a circle of radius 7 cm. Then mark the circle's centre and draw its radius. See Figure 43.8(a).
Step 2: Place the centre of the protractor at the centre of the circle and the zero line along the radius. Then mark every 30° around the circle.
Step 3: Using a ruler and a pencil, draw lines joining each 30° mark to the

centre of the circle to form 6 diameters. The diagram thus obtained will have 12 circular sectors as shown in Figure 43.8(c).
Step 4: Colour the parts as shown in Figure 43.8(d).
Step 5: Cut out the circle and then cut along the diameters so that all parts (i.e. sectors) are separated. See Figure 43.8(e).
Step 6: Using a ruler, measure the base and the height of the approximate parallelogram obtained in Step 5. See Figure 43.8(f).

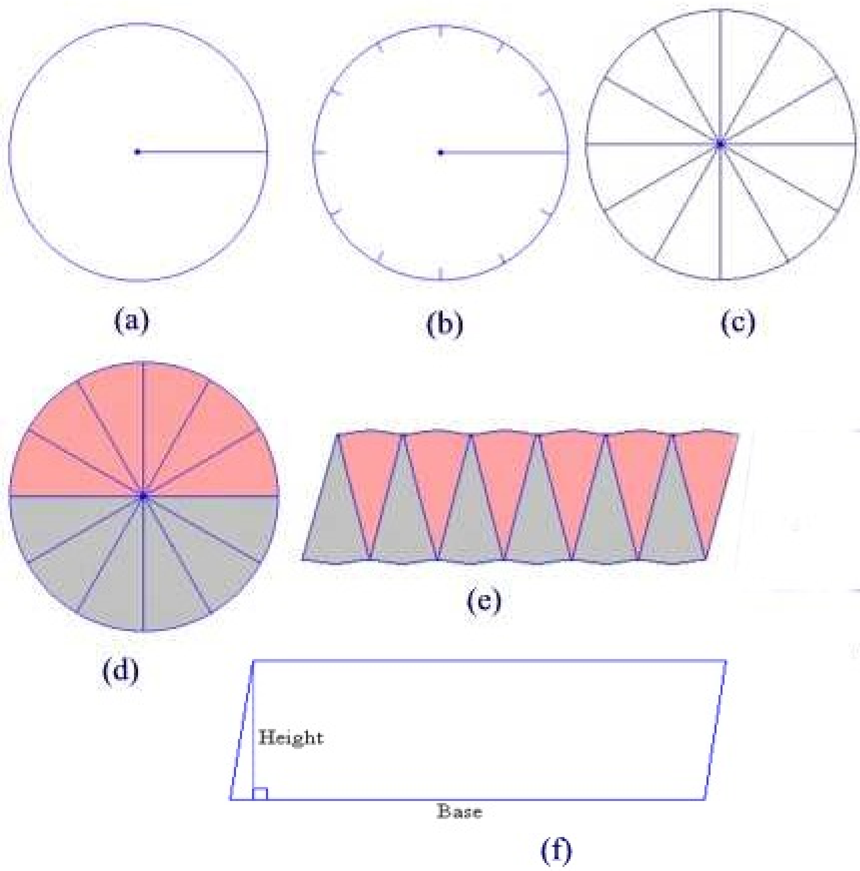


Figure 43.8

Thus, the base is approximately one-half of the circumference of the circle, that is $b = 7\pi$. The height of the parallelogram is just the radius. So $h = 7$. Hence, the area of the circle is about $49\pi = \pi 7^2$.
In general, the area of a circle of radius r is given by the formula

$$A = \pi r^2.$$

The Pythagorean Formula
Given a right triangle with legs of lengths a and b and hypotenuse of length c as shown in Figure 43.9(a). The Pythagorean formula states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse:

$$c^2 = a^2 + b^2.$$

The justification of this is as follows: Start with four copies of the same triangle. Three of these have been rotated $90°, 180°$, and $270°$, respectively as shown in Figure 43.9(b). Each has area ab/2. Let's put them together so that they form a square with side c as shown in Figure 43.9(c).
The square has a square hole with the side $(a - b)$. Summing up its area $(a - b)^2$ and the area of the four triangles $4\left(\frac{ab}{2}\right) = 2ab$ we get

$$c^2 = (a - b)^2 + 2ab = a^2 - 2ab + b^2 + 2ab = a^2 + b^2.$$

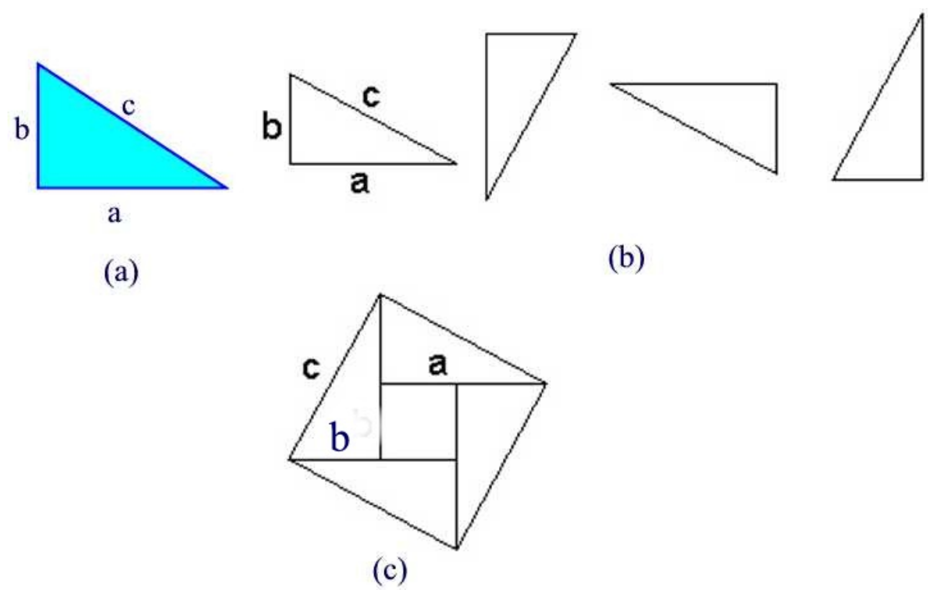


Figure 43.9

Practice Problems

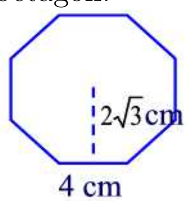
Problem 43.11
Convert each of the following:

(a) 1 cm^2 = ____ mm^2
(b) 124,000,000 m^2 = ____ km^2

Problem 43.12
Find the cost of carpeting a 6.5 m × 4.5 m rectangular room if one meter square of carpet costs \$ 13.85.

Problem 43.13
A rectangular plot of land is to be seeded with grass. If the plot is 22 m × 28 m and 1-kg bag of seed is needed for 85 m^2 of land, how many bags of seed are needed?

Problem 43.14
Find the area of the following octagon.

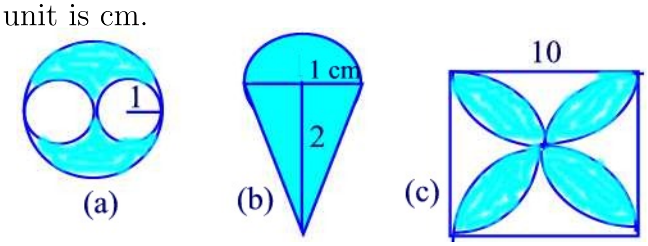


Problem 43.15
(a) If a circle has a circumference of 8π cm, what is its area?
(b) If a circle of radius r and a square with a side of length s have equal areas, express r in terms of s.

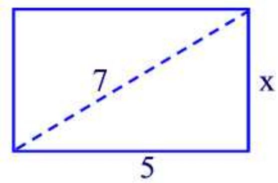
Problem 43.16
A circular flower bed is 6 m in diameter and has a circular sidewalk around it 1 m wide. Find the area of the sidewalk.

Problem 43.17
(a) If the area of a square is 144 cm^2, what is its perimeter?
(b) If the perimeter of a square is 32 cm, what is its area?

Problem 43.18
Find the area of each of the following shaded parts. Assume all arcs are circular. The unit is cm.



Problem 43.19
For the drawing below find the value of x.

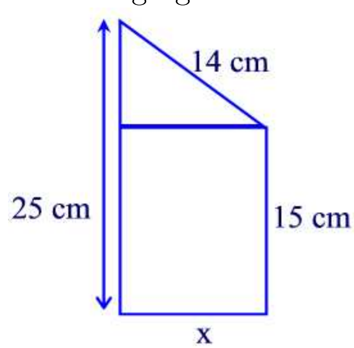


Problem 43.20
The size of a rectangular television screen is given as the length of the diagonal of the screen. If the length of the screen is 24 cm and the width is 18 cm, what is the diagonal length?

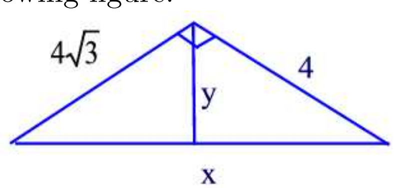
Problem 43.21
If the hypotenuse of a right triangle is 30 cm long and one leg is twice as long as the other, how long are the legs of the triangle?

Problem 43.22
A 15-ft ladder is leaning against a wall. The base of the ladder is 3 ft from the wall. How high above the ground is the top of the ladder?

Problem 43.23
Find the value of x in the following figure.



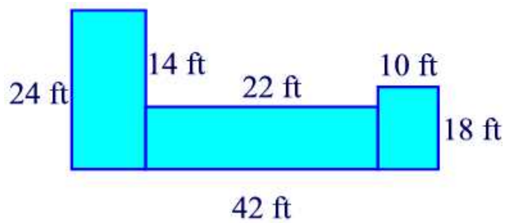
Problem 43.24
Find x and y in the following figure.



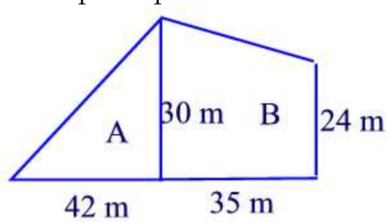
Problem 43.25
Complete the following table which concerns circles:

Radius	Diameter	Circumference	Area	
5 cm				
	24 cm			
			$17\pi\ m^2$	
		20πcm		

Problem 43.26
Suppose the housing authority has valued the house shown here at \$220 per ft^2. Find the assessed value.

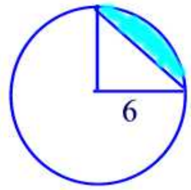


Problem 43.27
Two adjacent lots are for sale. Lot A cost \$ 20,000 and lot B costs \$27,000. Which lot has the lower cost per square meter?



Problem 43.28
If the radius of a circle increases by 30% then by what percent does the area of the circle increase?

Problem 43.29
Find the area of the shaded region.



Problem 43.30
Find the area of the shaded region.