

41 Three Dimensional Shapes

Space figures are the first shapes children perceive in their environment. In elementary school, children learn about the most basic classes of space figures such as prisms, pyramids, cylinders, cones and spheres. This section concerns the definitions of these space figures and their properties.

Planes and Lines in Space

As a basis for studying space figures, first consider relationships among lines and planes in space. These relationships are helpful in analyzing and defining space figures.

Two planes in space are either **parallel** as in Figure 41.1(a) or intersect as in Figure 41.1(b).

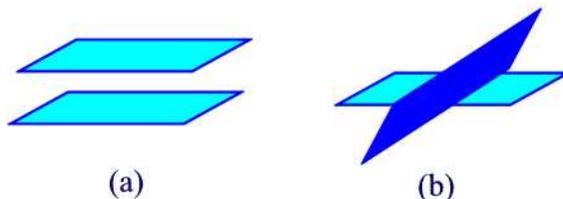


Figure 41.1

The angle formed by two intersecting planes is called the **dihedral angle**. It is measured by measuring an angle whose sides line in the planes and are perpendicular to the line of intersection of the planes as shown in Figure 41.2.

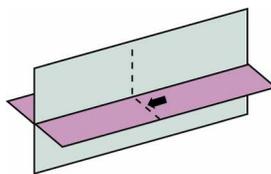


Figure 41.2

Some examples of dihedral angles and their measures are shown in Figure 41.3.

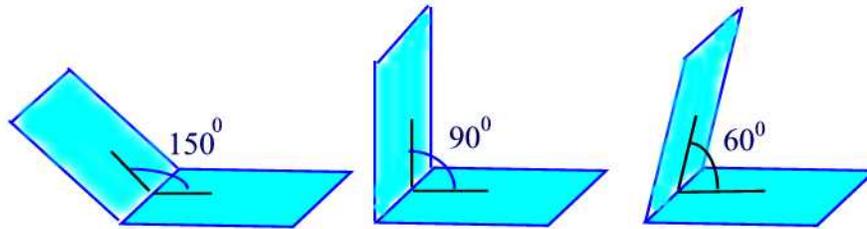


Figure 41.3

Two nonintersecting lines in space are **parallel** if they belong to a common plane. Two nonintersecting lines that do not belong to the same plane are called **skew lines**. If a line does not intersect a plane then it is said to be **parallel to the plane**. A line is said to be **perpendicular to a plane** at a point A if every line in the plane through A intersects the line at a right angle. Figures illustrating these terms are shown in Figure 41.4.

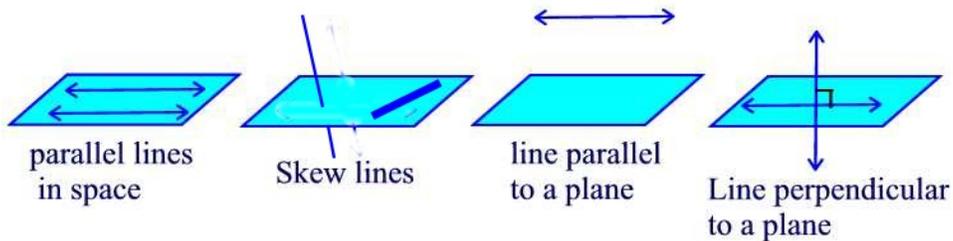


Figure 41.4

Polyhedra

To define a polyhedron, we need the terms "simple closed surface" and "polygonal region." By a **simple closed surface** we mean any surface without holes and that encloses a hollow region-its interior. An example is a **sphere**, which is the set of all points at a constant distance from a single point called the **center**. The set of all points on a simple closed surface together with all interior points is called a **solid**. For example, the shell of a hardboiled egg can be viewed as a simple closed surface whereas the shell together with the white and the yolk of the egg form a solid. A **polygonal region** is a polygon together with its interior as shown in Figure 41.5.

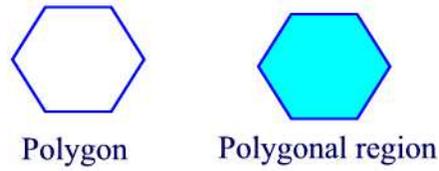


Figure 41.5

A **polyhedron** (polyhedra is the plural) is a simple closed surface bounded by polygonal regions. All of the surfaces are flat, not curved. The polygonal regions are called the **faces**. The sides of each face are called the **edges**. The vertices of the polygons are also called **vertices** of the polyhedron. Figure 41.6(a) and (b) are examples of polyhedra, but (c) and (d) are not.

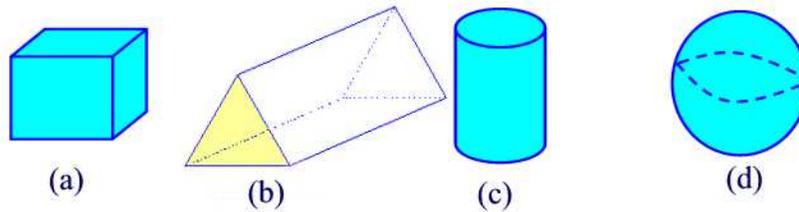


Figure 41.6

Polyhedra are named according to the number of faces. For example, a **tetrahedron** has four faces, a **pentahedron** has five faces, a **hexahedron** has six faces, and so on.

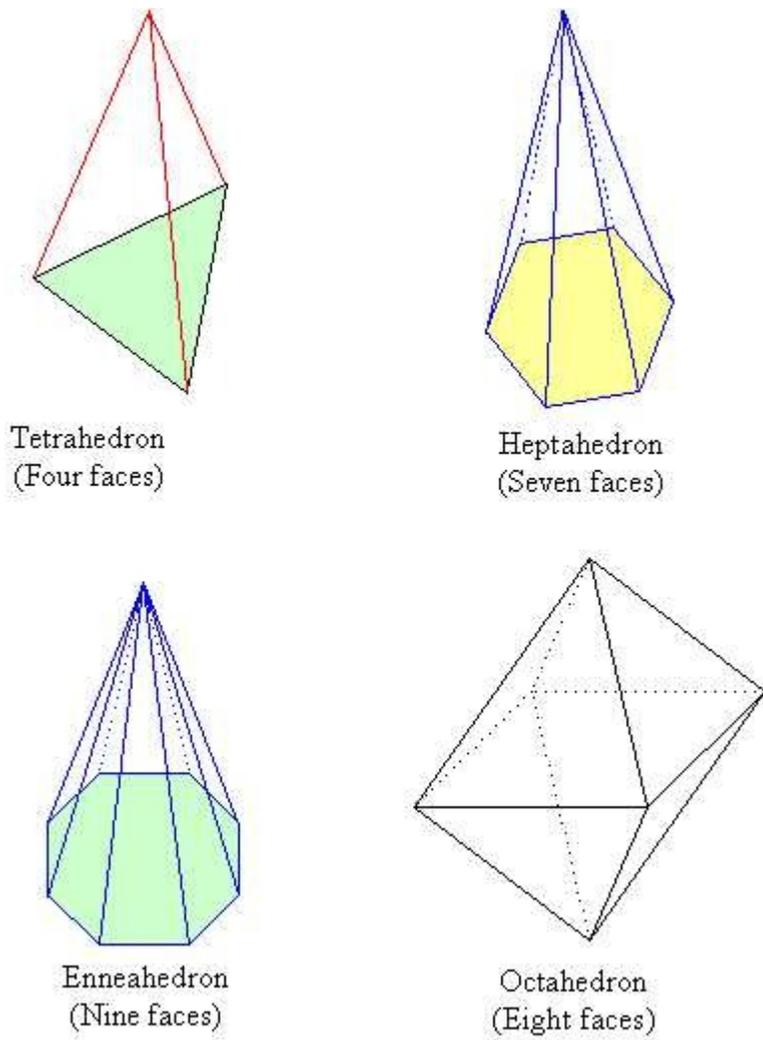


Figure 41.7

Polyhedra can be classified into several types:

- **Prisms:** A prism is a polyhedron, with two parallel faces called **bases**. The other faces are called **lateral faces** and are always parallelogram. If the lateral faces are rectangles then the prism is called a **right prism**. Otherwise, the prism is called an **oblique prism**. Figure 41.8 shows some types of prisms.

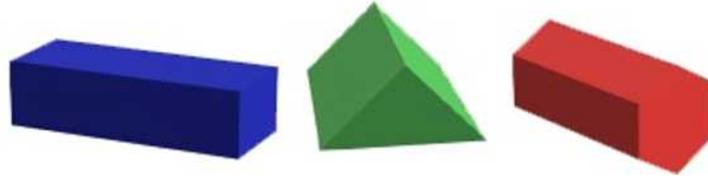


Figure 41.8

• **Pyramids:** A polyhedron is a **pyramid** if it has 3 or more triangular faces sharing a common vertex, called the **apex**. The base of a pyramid may be any polygon. Pyramids are named according to the type of polygon forming the base. Pyramids whose bases are regular polygons are called **regular pyramids**. If in addition, the lateral faces are isosceles triangles then the pyramids are called **right regular pyramids**. Otherwise, they are called **oblique regular pyramids**. See Figure 41.9.

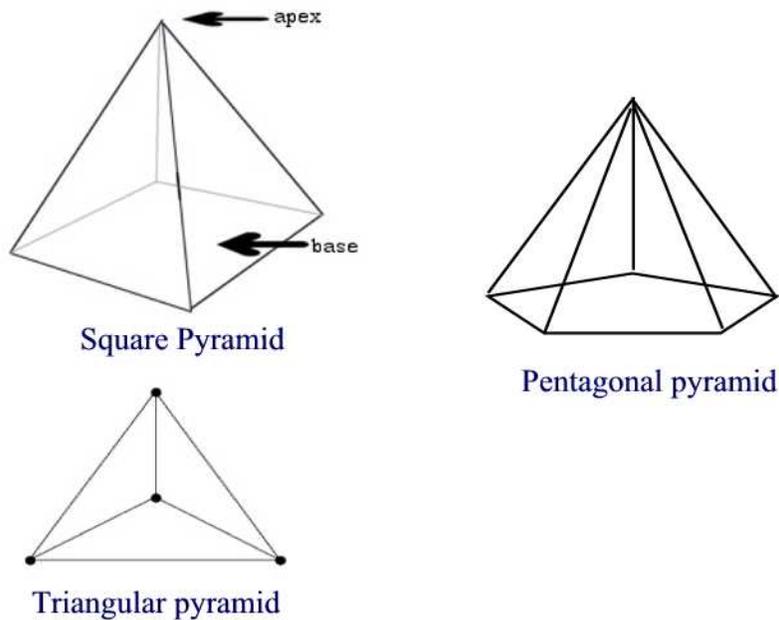
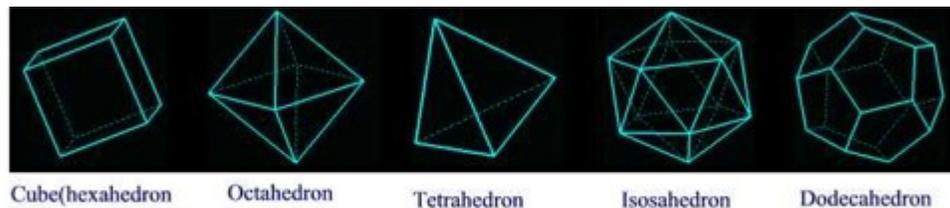


Figure 41.9

Regular Polyhedra

A polyhedron is said to be **regular** if all its faces are identical regular polygonal regions and all dihedral angles have the same measures. Regular polyhedra are also known as **platonic solids**. There are five regular polyhedra analyzed in the table below and shown in Figure 41.10.

Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Hexahedron	6	12	8
Icosahedron	20	30	12
Dodecahedron	12	30	20



Cube(hexahedron) Octahedron Tetrahedron Icosahedron Dodecahedron

Figure 41.10

Curved Shapes: Cylinders, Cones, and Spheres

Some everyday objects suggest space figures that are not polyhedra: cylinders (can of soup), cones (ice cream cone), and spheres (a basketball).

- **Cylinders:** A **cylinder** is the space figure obtained by translating the points of a simple closed region in one plane to a parallel plane as shown in Figure 41.11(a). The simple closed regions are called the **base** of the cylinder. The remaining points constitute the **lateral surface** of the cylinder.

The most common types of cylinders in our environment are right circular cylinders. The line joining the centers of the two circles is called the **axis**. If the axis is perpendicular to the circles then the cylinder is called a **right circular cylinder** (See Figure 41.11(b)). Otherwise it is called an **oblique circular cylinder**. (See Figure 41.11(c))

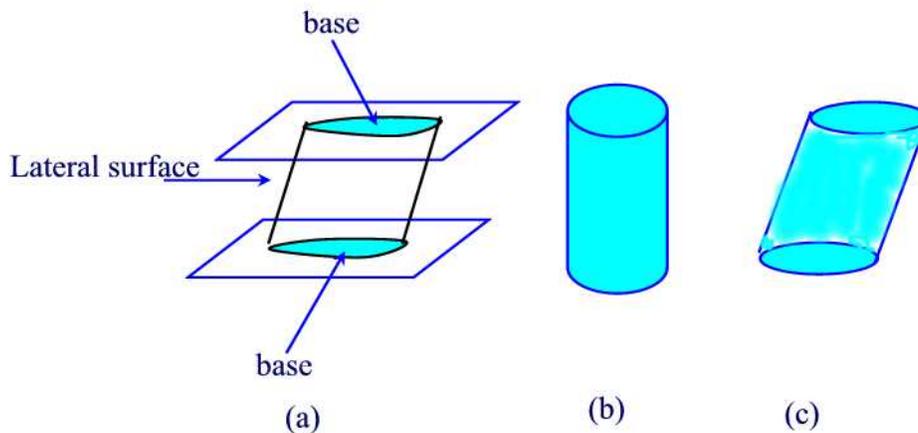


Figure 41.11

• **Cones:** Suppose we have a simple closed region in a plane and a point P not in the plane of the region. The union of line segments connecting P to each point in the simple closed region is called a **cone**. See Figure 41.12(a). P is called the **vertex**. The simple closed region is called the **base**. The points in the cone not in the base constitute the **lateral surface**. If the base is a circle then the cone is called a **circular cone**. A circular cone where the line connecting the vertex to the center of the base is perpendicular to the base is called a **right circular cone**. See Figure 41.12(b). Otherwise, the cone is called **oblique circular cone**. See Figure 41.12(c).

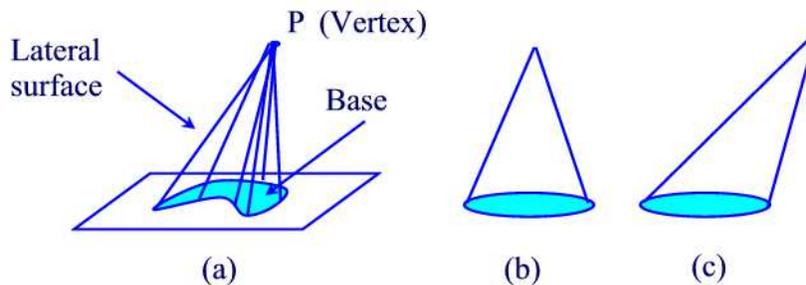


Figure 41.12

• **Spheres:** We have already encountered this concept. Recall that a sphere is the collection of all points in the space that are the same distance from a fixed point called **center**. Any line segment joining the center to a point on the surface of the sphere is called a **radius**. Any line segment going through

the center and connecting two points on the sphere is called a **diameter**. See Figure 41.13.

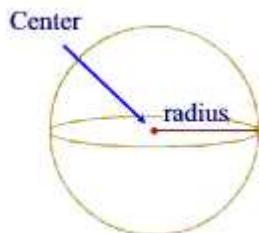


Figure 41.13

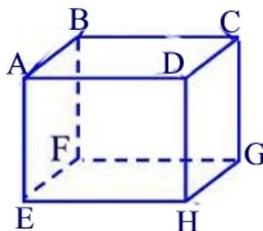
Practice Problems

Problem 41.1

Determine the measures of all dihedral angles of a right prism whose bases are regular octagons.

Problem 41.2

Consider the cube given in the figure below.



- How many planes are determined by the faces of the cube?
- Which edges of the cube are parallel to edge \overline{AB} ?
- Which edges of the cube are contained in lines that are skew to the line going through A and B?

Problem 41.3

If a pyramid has an octagon for a base, how many lateral faces does it have?

Problem 41.4

Determine the number of faces, vertices, and edges for a hexagonal pyramid.

Problem 41.5

Sketch drawings to illustrate different possible intersections of a square pyramid and a plane.

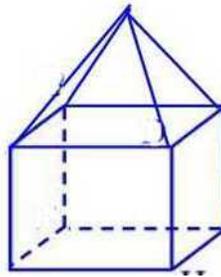
Problem 41.6

Sketch each of the following.

- (a) A plane and a cone that intersect in a circle.
- (b) A plane and a cylinder that intersect in a segment.
- (c) Two pyramids that intersect in a triangle.

Problem 41.7

For the following figure, find the number of faces, edges, and vertices.



Problem 41.8

For each of the following figures, draw all possible intersections of a plane with

- (a) Cube
- (b) Cylinder

Problem 41.9

A diagonal of a prism is any segment determined by two vertices that do not lie in the same face. How many diagonals does a pentagonal prism have?

Problem 41.10

For each of the following, draw a prism and a pyramid that have the given region as a base:

- (a) Triangle
- (b) Pentagon
- (c) Regular hexagon

Problem 41.11

If possible, sketch each of the following:

- (a) An oblique square prism

- (b) An oblique square pyramid
- (c) A noncircular right cone
- (d) A noncircular cone that is not right

Problem 41.12

A simple relationship among the number of faces(F), the number of edges(E), and the number of vertices(V) of any convex polyhedron exists and is known as **Euler's formula**. Use the following table to find this relationship.

Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Hexahedron	6	12	8
Icosahedron	20	30	12
Dodecahedron	12	30	20

Problem 41.13

What is the sum of the angles at the each vertex of a

- (a) Tetrahedron
- (b) Octahedron
- (c) Cube
- (d) Icosahedron
- (e) Dodecahedron

Problem 41.14

Determine for each of the following the minimum number of faces possible:

- (a) Prism
- (b) Pyramid
- (c) Hexahedron

Problem 41.15

True or false?

- (a) Through a given point not on plane P, there is exactly one line parallel to P.
- (b) Every set of four points is contained in one plane.
- (c) If a line is perpendicular to one of two parallel planes then it is perpendicular to the other.

Problem 41.16

Tell whether each of the following suggests a polygon or a polygonal region.

- (a) A picture frame.
- (b) A page in this book.
- (c) A stop sign.

Problem 41.17

A certain prism has 20 vertices. How many faces and edges does it have?

Problem 41.18

Why are there no such things as skew planes?

Problem 41.19

A prism has a base with n sides.

- (a) How many faces does it have?
- (b) How many vertices does it have?
- (c) How many edges does it have?

Problem 41.20

A pyramid has a base with n sides.

- (a) How many faces does it have?
- (b) How many vertices does it have?
- (c) How many edges does it have?