

36 Odds, Expected Value, and Conditional Probability

What's the difference between probabilities and odds? To answer this question, let's consider a game that involves rolling a die. If one gets the face 1 then he wins the game, otherwise he loses. The probability of winning is $\frac{1}{6}$ whereas the probability of losing is $\frac{5}{6}$. The odds of winning is 1:5 (read 1 to 5). This expression means that the probability of losing is five times the probability of winning. Thus, probabilities describe the frequency of a favorable result in relation to all possible outcomes whereas the "odds in favor" compare the favorable outcomes to the unfavorable outcomes. More formally,

$$\text{odds in favor} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}}$$

If E is the event of all favorable outcomes then its complementary, \overline{E} , is the event of unfavorable outcomes. Hence,

$$\text{odds in favor} = \frac{n(E)}{n(\overline{E})}$$

Also, we define the **odds against** an event as

$$\text{odds against} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{n(\overline{E})}{n(E)}$$

Any probability can be converted to odds, and any odds can be converted to a probability.

Converting Odds to Probability

Suppose that the odds for an event E is a:b. Thus, $n(E) = ak$ and $n(\overline{E}) = bk$ where k is a positive integer. Since E and \overline{E} are complementary then

$n(S) = n(E) + n(\bar{E})$. Therefore,

$$\begin{aligned}P(E) &= \frac{n(E)}{n(S)} \\&= \frac{n(E)}{n(E)+n(\bar{E})} \\&= \frac{ak}{ak+bk} = \frac{a}{a+b} \\P(\bar{E}) &= \frac{n(\bar{E})}{n(S)} \\&= \frac{n(\bar{E})}{n(E)+n(\bar{E})} \\&= \frac{bk}{ak+bk} = \frac{b}{a+b}\end{aligned}$$

Example 36.1

If the odds in favor of an event E is 5 to 4, compute $P(E)$ and $P(\bar{E})$.

Solution.

We have

$$P(E) = \frac{5}{5+4} = \frac{5}{9}$$

and

$$P(\bar{E}) = \frac{4}{5+4} = \frac{4}{9} \blacksquare$$

Converting Probability to Odds

Given $P(E)$, we want to find the odds in favor of E and the odds against E .

The odds in favor of E are

$$\begin{aligned}\frac{n(E)}{n(\bar{E})} &= \frac{n(E)}{n(S)} \cdot \frac{n(S)}{n(\bar{E})} \\&= \frac{P(E)}{P(\bar{E})} \\&= \frac{P(E)}{1-P(E)}\end{aligned}$$

and the odds against E are

$$\frac{n(\bar{E})}{n(E)} = \frac{1-P(E)}{P(E)}$$

Example 36.2

For each of the following, find the odds in favor of the event's occurring:

- (a) Rolling a number less than 5 on a die.
- (b) Tossing heads on a fair coin.
- (c) Drawing an ace from an ordinary 52-card deck.

Solution.

(a) The probability of rolling a number less than 5 is $\frac{4}{6}$ and that of rolling 5 or 6 is $\frac{2}{6}$. Thus, the odds in favor of rolling a number less than 5 is $\frac{4}{6} \div \frac{2}{6} = \frac{2}{1}$ or 2:1

(b) Since $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$ then the odds in favor of getting heads is $(\frac{1}{2}) \div (\frac{1}{2})$ or 1:1

(c) We have $P(\text{ace}) = \frac{4}{52}$ and $P(\text{not an ace}) = \frac{48}{52}$ so that the odds in favor of drawing an ace is $(\frac{4}{52}) \div (\frac{48}{52}) = \frac{1}{12}$ or 1:12 ■

Remark 36.1

A probability such as $P(E) = \frac{5}{6}$ is just a ratio. The exact number of favorable outcomes and the exact total of all outcomes are not necessarily known.

Practice Problems**Problem 36.1**

If the probability of a boy's being born is $\frac{1}{2}$, and a family plans to have four children, what are the odds against having all boys?

Problem 36.2

If the odds against Deborah's winning first prize in a chess tournament are 3 to 5, what is the probability that she will win first prize?

Problem 36.3

What are the odds in favor of getting at least two heads if a fair coin is tossed three times?

Problem 36.4

If the probability of rain for the day is 60%, what are the odds against its raining?

Problem 36.5

On a tote board at a race track, the odds for Gameylegs are listed as 26:1. Tote boards list the odds that the horse will lose the race. If this is the case, what is the probability of Gameylegs's winning the race?

Problem 36.6

If a die is tossed, what are the odds in favor of the following events?

- (a) Getting a 4
- (b) Getting a prime
- (c) Getting a number greater than 0
- (d) Getting a number greater than 6.

Problem 36.7

Find the odds against E if $P(E) = \frac{3}{4}$.

Problem 36.8

Find $P(E)$ in each case.

- (a) The odds in favor of E are 3:4
- (b) The odds against E are 7:3

Expected Value

A cube has three red faces, two green faces, and one blue face. A game consists of rolling the cube twice. You pay \$ 2 to play. If both faces are the same color, you are paid \$ 5(that is you win \$3). If not, you lose the \$2 it costs to play. Will you win money in the long run? Let W denote the event that you win. Then $W = \{RR, GG, BB\}$ and

$$P(W) = P(RR) + P(GG) + P(BB) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6} = \frac{7}{18} \approx 39\%.$$

Thus, $P(L) = \frac{11}{18} = 61\%$. Hence, if you play the game 18 times you expect to win 7 times and lose 11 times on average. So your winnings in dollars will be $3 \times 7 - 2 \times 11 = -1$. That is, you can expect to lose \$1 if you play the game 18 times. On the average, you will lose $\frac{1}{18}$ per game (about 6 cents). This can be found also using the equation

$$3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}$$

We call this number the expected value. More formally, let the outcomes of an experiment be a sequence of real numbers n_1, n_2, \dots, n_k , and suppose that the outcomes occur with respective probabilities p_1, p_2, \dots, p_k . Then the **expected value** of the experiment is

$$E = n_1p_1 + n_2p_2 + \dots + n_kp_k.$$

Example 36.3

Suppose that an insurance company has broken down yearly automobile claims for drivers from age 16 through 21 as shown in the following table.

Amount of claim	Probability
0	0.80
2,000	0.10
4,000	0.05
6,000	0.03
8,000	0.01
10,000	0.01

How much should the company charge as its average premium in order to break even on costs for claims?

Solution.

Finding the expected value

$$E = 0(0.80) + 2,000(0.10) + 4,000(0.05) + 6,000(0.03) + 8,000(0.01) + 10,000(0.01) = 760$$

Since average claim value is \$760, the average automobile insurance premium should be set at \$760 per year for the insurance company to break even■

Example 36.4

An American roulette wheel has 38 compartments around its rim. Two of these are colored green and are numbered 0 and 00. The remaining compartments are numbered 1 through 36 and are alternately colored black and red. When the wheel is spun in one direction, a small ivory ball is rolled in the opposite direction around the rim. When the wheel and the ball slow down, the ball eventually falls in any one compartments with equal likelihood if the wheel is fair. One way to play is to bet on whether the ball will fall in a red slot or a black slot. If you bet on red for example, you win the amount of the bet if the ball lands in a red slot; otherwise you lose. What is the expected win if you consistently bet \$5 on red?

Solution.

The probability of winning is $\frac{18}{38}$ and that of losing is $\frac{20}{38}$. Your expected win is

$$\frac{18}{38} \times 5 - \frac{20}{38} \times 5 \approx -0.26$$

On average you should expect to lose 26 cents per play ■

Practice Problems

Problem 36.9

Compute the expected value of the score when rolling two dice.

Problem 36.10

A game consists of rolling two dice. You win the amounts shown for rolling the score shown.

Score	2	3	4	5	6	7	8	9	10	11	12
\$ won	4	6	8	10	20	40	20	10	8	6	4

Compute the expected value of the game.

Problem 36.11

Consider the spinner in Figure 36.1, with the payoff in each sector of the circle. Should the owner of this spinner expect to make money over an extended period of time if the charge is \$2.00 per spin?



Figure 36.1

Problem 36.12

You play a game in which two dice are rolled. If a sum of 7 appears, you win \$10; otherwise, you lose \$2.00. If you intend to play this game for a long time, should you expect to make money, lose money, or come out about even? Explain.

Problem 36.13

Suppose it costs \$8 to roll a pair of dice. You get paid the sum of the numbers in dollars that appear on the dice. What is the expected value of this game?

Problem 36.14

An insurance company will insure your dorm room against theft for a semester. Suppose the value of your possessions is \$800. The probability of your being robbed of \$400 worth of goods during a semester is $\frac{1}{100}$, and the probability of your being robbed of \$800 worth of goods is $\frac{1}{400}$. Assume that these are the only possible kinds of robberies. How much should the insurance company charge people like you to cover the money they pay out and to make an additional \$20 profit per person on the average?

Problem 36.15

Consider a lottery game in which 7 out of 10 people lose, 1 out of 10 wins \$50, and 2 out of 10 wins \$35. If you played 10 times, about how much would you expect to win?

Problem 36.16

Suppose a lottery game allows you to select a 2-digit number. Each digit may be either 1, 2, 3, 4, or 5. If you pick the winning number, you win \$10. Otherwise, you win nothing. What is the expected payoff?

Conditional Probability and Independent Events

When the sample space of an experiment is affected by additional information, the new sample space is reduced in size. For example, suppose we toss a fair coin three times and consider the following events:

- A : getting a tail on the first toss
- B : getting a tail on all three tosses

Since

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

then $P(A) = \frac{4}{8} = \frac{1}{2}$ and $P(B) = \frac{1}{8}$. What if we were told that event A has occurred (that is, a tail occurred on the first toss), and we are now asked to find $P(B)$. The sample space is now reduced to $\{THH, THT, TTH, TTT\}$. The probability that all three are tails given that the first toss is a tail is $\frac{1}{4}$.

The notation we use for this situation is $P(B|A)$, read "the probability of B given A," and we write $P(B|A) = \frac{1}{4}$. Notice that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

This is true in general, and we have the following:

Given two events A and B belonging to the same sample S . The **conditional probability** $P(B|A)$ denotes the probability that event B will occur given that event A has occurred. It is given by the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Example 36.5

Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{aligned} A &= \{\text{Observing an even number of dots on the upper face of the die}\}, \\ B &= \{\text{Observing a number of dots less than or equal to 3 on the upper face of the die}\}. \end{aligned}$$

Find the probability of the event A , given the event B .

Solution.

Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$ then $A \cap B = \{2\}$ and therefore $P(A|B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$. ■

If $P(B|A) = P(B)$, i.e., the occurrence of the event A does not affect the probability of the event B , then we say that the two events A and B are **independent**. In this case the above formula gives

$$P(A \cap B) = P(A) \cdot P(B).$$

This formula is known as the "multiplication rule of probabilities". If two events are not independent, we say that they are **dependent**. In this case, $P(B|A) \neq P(B)$.

Example 36.6

Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{aligned} A &= \{\text{Observing an even number of dots on the upper face of the die}\}, \\ B &= \{\text{Observing a number of dots less than or equal to 4 on the upper face of the die}\}. \end{aligned}$$

Are A and B independent?

Solution.

Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ then $A \cap B = \{2, 4\}$ and therefore $P(A|B) = \frac{\frac{2}{4}}{\frac{4}{6}} = \frac{1}{2} = P(A)$. Thus, A and B are independent. ■

Practice Problems**Problem 36.17**

Suppose that A is the event of rolling a sum of 7 with two fair dice. Make up an event B so that

- (a) A and B are independent.
- (b) A and B are dependent.

Problem 36.18

When tossing three fair coins, what is the probability of getting two tails given that the first coin came up heads?

Problem 36.19

Suppose a 20-sided die has the following numerals on its face: 1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. The die is rolled once and the number on the top face is recorded. Let A be the event the number is prime, and B be the event the number is odd. Find $P(A|B)$ and $P(B|A)$.

Problem 36.20

What is the probability of rolling a 6 on a fair die if you know that the roll is an even number?

Problem 36.21

A red die and a green die are rolled. What is the probability of obtaining an even number on the red die and a multiple of 3 on the green die?

Problem 36.22

Two coins are tossed. What is the probability of obtaining a head on the first coin and a tail on the second coin?

Problem 36.23

Consider two boxes: Box 1 contains 2 white and 2 black balls, and box 2 contains 2 white balls and three black balls. What is the probability of drawing a black ball from each box?

Problem 36.24

A container holds three red balls and five blue balls. One ball is drawn and discarded. Then a second ball is drawn.

- (a) What is the probability that the second ball drawn is red if you drew a red ball the first time?
- (b) What is the probability of drawing a blue ball second if the first ball was red?
- (c) What is the probability of drawing a blue ball second if the first ball was blue?

Problem 36.25

Consider the following events.

- A: rain tomorrow
- B: You carry an umbrella
- C: coin flipped tomorrow lands on heads

Which of two events are dependent and which are independent?

Problem 36.26

You roll a regular red die and a regular green die. Consider the following events.

- A: a 4 on the red die
- B: a 3 on the green die
- C: a sum of 9 on the two dice

Tell whether each pair of events is independent or dependent.

- (a) A and B
- (b) B and C