34 Probability and Counting Techniques

If you recall that the classical probability of an event $E \subseteq S$ is given by

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ and $n(S)$ denote the number of elements of E and S respectively. Thus, finding $P(E)$ requires counting the elements of the sample space S. Sometimes the sample space is so large that shortcuts are needed to count all the possibilities.

All the examples discussed thus far have been experiments consisting of one action such as tossing three coins or rolling two dice. We now want to consider experiments that consist of doing two or more actions in succession. For example, consider the experiment of drawing two balls in succession and with replacement from a box containing one red ball (R), one white ball (W), and one green ball (G). The outcomes of this experiment, i.e. the elements of the sample space can be found in two different ways by using

Organized Table or an Orderly List

An organized table of our experiment looks like

	R	W	G
R	RR	RW	RG
W	WR	WW	WG
G	GR	GW	GG

Thus, there are nine equally likely outcomes so that

$$S = \{RR, RW, RG, WR, WW, WG, GR, GW, GG\}$$

Tree Diagrams

An alternative way to generate the sample space is to use a tree diagram as shown in Figure 34.1.

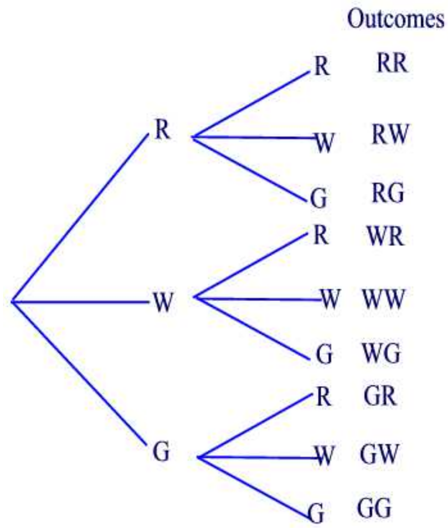

Figure 34.1

Example 34.1
Show the sample space for tossing one penny and rolling one die. (H = heads, T = tails)

Solution.
According to Figure 34.2, the sample space is

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

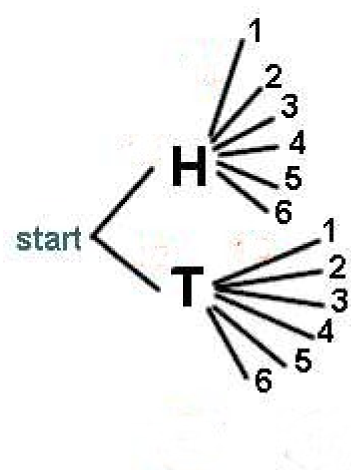

Figure 34.2

Fundamental Principle of Counting
If there are many stages to an experiment and several possibilities at each stage, the tree diagram associated with the experiment would become too large to be manageable. For such problems the counting of the outcomes is simplified by means of algebraic formulas. The commonly used formula is the **multiplication rule of counting** which states:
" If a choice consists of k steps, of which the first can be made in n_1 ways, for each of these the second can be made in n_2 ways,... and for each of these the kth can be made in n_k ways, then the whole choice can be made in $n_1 \cdot n_2 \cdot ... n_k$ ways."

Example 34.2
How many license-plates with 3 letters followed by 3 digits exist?

Solution.
A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third letter, (4) choose the first digit, (5) choose the second digit, and (6) choose the third digit. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ ways.$\blacksquare$

Example 34.3
How many numbers in the range 1000 - 9999 have no repeated digits?

Solution.
A 4-step process: (1) Choose first digit, (2) choose second digit, (3) choose third digit, (4) choose fourth digit. Every step can be done in a number of ways that does not depend on previous choices, and each number can be specified in this manner. So there are $9 \cdot 9 \cdot 8 \cdot 7 = 4,536$ ways.$\blacksquare$

Example 34.4
How many license-plates with 3 letters followed by 3 digits exist if exactly one of the digits is 1?

Solution.
In this case, we must pick a place for the 1 digit, and then the remaining digit places must be populated from the digits $\{0, 2, \cdots 9\}$. A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third

letter, (4) choose which of three positions the 1 goes, (5) choose the first of the other digits, and (6) choose the second of the other digits. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 3 \cdot 9 \cdot 9 = 4,270,968$ ways.■

Practice Problems

Problem 34.1
If each of the 10 digits is chosen at random, how many ways can you choose the following numbers?

(a) A two-digit code number, repeated digits permitted.
(b) A three-digit identification card number, for which the first digit cannot be a 0.
(c) A four-digit bicycle lock number, where no digit can be used twice.
(d) A five-digit zip code number, with the first digit not zero.

Problem 34.2
(a) If eight horses are entered in a race and three finishing places are considered, how many finishing orders can they finish?
(b) If the top three horses are Lucky one, Lucky Two, and Lucky Three, in how many possible orders can they finish?

Problem 34.3
You are taking 3 shirts(red, blue, yellow) and 2 pairs of pants (tan, gray) on a trip. How many different choices of outfits do you have?

Problem 34.4
The state of Maryland has automobile license plates consisting of 3 letters followed by three digits. How many possible license plates are there?

Problem 34.5
A club has 10 members. In how many ways can the club choose a president and vice-president if everyone is eligible?

Problem 34.6
A lottery allows you to select a two-digit number. Each digit may be either 1,2 or 3. Use a tree diagram to show the sample space and tell how many different numbers can be selected.

Problem 34.7
In a medical study, patients are classified according to whether they have blood type A, B, AB, or O, and also according to whether their blood pressure is low, normal, or high. Use a tree diagram to represent the various outcomes that can occur.

Problem 34.8
If a travel agency offers special weekend trips to 12 different cities, by air, rail, or bus, in how many different ways can such a trip be arranged?

Problem 34.9
If twenty paintings are entered in art show, in how many different ways can the judges award a first prize and a second prize?

Problem 34.10
In how many ways can the 52 members of a labor union choose a president, a vice-president, a secretary, and a treasurer?

Problem 34.11
Find the number of ways in which four of ten new movies can be ranked first, second, third, and fourth according to their attendance figures for the first six months.

Problem 34.12
To fill a number of vacancies, the personnel manager of a company has to choose three secretaries from among ten applicants and two bookkeepers from among five applicants. In how many different ways can the personnel manager fill the five vacancies?

Problem 34.13
A box contains three red balls and two blue balls. Two balls are to be drawn without replacement. Use a tree diagram to represent the various outcomes that can occur. What is the probability of each outcome?

Problem 34.14
Repeat the previous exercise but this time replace the first ball before drawing the second.

Problem 34.15
If a new-car buyer has the choice of four body styles, three engines, and ten colors, in how many different ways can s/he order one of these cars?

Finding Probabilities Using the Fundamental Principle of Counting
The Fundamental Principle of Counting can be used to compute probabilities as shown in the following example.

Example 34.5
A quizz has 5 multiple-choice questions. Each question has 4 answer choices, of which 1 is correct answer and the other 3 are incorrect. Suppose that you guess all the answers.

(a) How many ways are there to answer the 5 questions?
(b) What is the probability of getting all 5 questions right?
(c) What is the probability of getting exactly 4 questions right and 1 wrong?
(d) What is the probability of doing well (getting at leat 4 right)?

Solution.
(a) We can look at this question as a decision consisting of five steps. There are 4 ways to do each step so that by the Fundamental Principle of Counting there are

$$(4)(4)(4)(4)(4) = 1024 \text{ ways}$$

(b) There is only one way to answer each question correctly. Using the Fundamental Principle of Counting there is $(1)(1)(1)(1)(1) = 1$ way to answer all 5 questions correctly out of 1024 possibilities. Hence,

$$\text{P(all 5 right)} = \tfrac{1}{1024}$$

(c) The following table lists all possible responses that involve at least 4 right answers, R stands for right and W stands for a wrong answer

Five Responses	Number of ways to fill out the test
WRRRR	$(3)(1)(1)(1)(1) = 3$
RWRRR	$(1)(3)(1)(1)(1) = 3$
RRWRR	$(1)(1)(3)(1)(1) = 3$
RRRWR	$(1)(1)(1)(3)(1) = 3$
RRRRW	$(1)(1)(1)(1)(3) = 3$

So there are 15 ways out of the 1024 possible ways that result in 4 right answers and 1 wrong answer so that

$$P(4 \text{ right,1 wrong}) = \frac{15}{1024} \approx 1.5\%$$

(d) "At least 4" means you can get either 4 right and 1 wrong or all 5 right. Thus,

$$P(\text{at least 4 right}) = P(4 \text{ right, 1 wrong}) + P(5 \text{ right}) = \frac{15}{1024} + \frac{1}{1024} = \frac{16}{1024} \approx 0.016 \blacksquare$$

Probability Trees

Probability trees can be used to compute the probabilities of combined outcomes in a sequence of experiments.

Example 34.6

Construct the probability tree of the experiment of flipping a fair coin twice.

Solution.

The probability tree is shown in Figure 34.3 ■

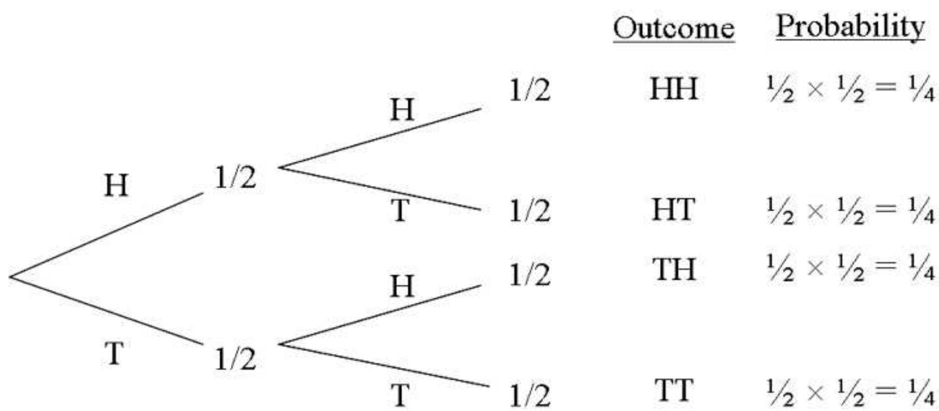

Figure 34.3

The probabilities shown in Figure 34.3 are obtained by following the paths leading to each of the four outcomes and multiplying the probabilities along the paths. This procedure is an instance of the following general property

Multiplication Rule for Probabilities for Tree Diagrams

For all multistage experiments, the probability of the outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Example 34.7
Suppose that out of 500 computer chips there are 9 defective. Construct the probability tree of the experiment of sampling two of them without replacement.

Solution.
The probability tree is shown in Figure 34.4■

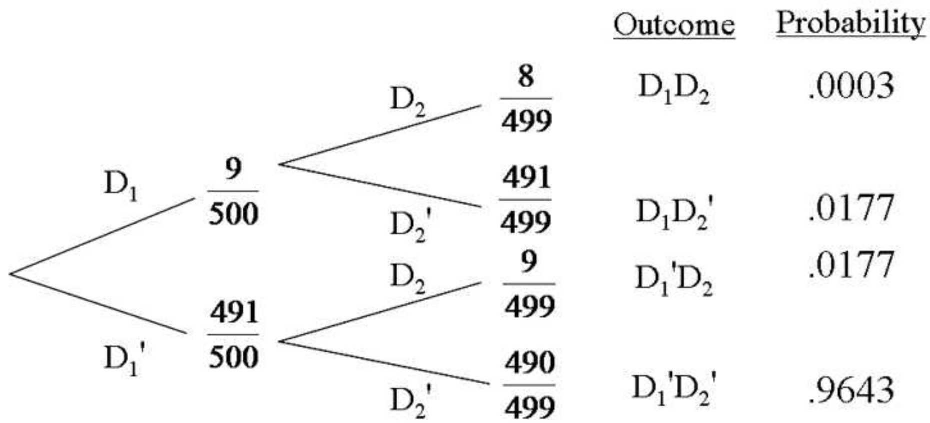

Figure 34.4

Practice Problems

Problem 34.16
A jar contains three red gumballs and two green gumballs. An experiment consists of drawing gumballs one at a time from the jar, without replacement, until a red one is obtained. Find the probability of the following events.

A: Only one draw is needed.
B: Exactly two draws are needed.
C: Exactly three draws are needed.

Problem 34.17
Consider a jar with three black marbles and one red marble. For the experiment of drawing two marbles with replacement, what is the probability of drawing a black marble and then a red marble in that order?

Problem 34.18
A jar contains three marbles, two black and one red. Two marbles are drawn with replacement. What is the probability that both marbles are black? Assume that the marbles are equally likely to be drawn.

Problem 34.19
A jar contains four marbles-one red, one green, one yellow, and one white. If two marbles are drawn without replacement from the jar, what is the probability of getting a red marble and a white marble?

Problem 34.20
A jar contains 3 white balls and 2 red balls. A ball is drawn at random from the box and not replaced. Then a second ball is drawn from the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Problem 34.21
Suppose that a ball is drawn from the box in the previous problem, its color recorded, and then it is put back in the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Problem 34.22
Suppose there are 19 balls in an urn. They are identical except in color. 16 of the balls are black and 3 are purple. You are instructed to draw out one ball, note its color, and set it aside. Then you are to draw out another ball and note its color. What is the probability of drawing out a black on the first draw and a purple on the second?

Binary Experiments
Binary experiments are experiments with exactly two outcomes such as coin-tossing. Our first question is to find the total number of outcomes when a coin is tossed n times which is equivalent to saying that n coins are tossed. So what we have here is a decision consisting of n steps each step has two outcomes (head/tail) so by the Fundamental Principle of Counting there are

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n\ factors} = 2^n \text{ outcomes}$$

Next, let's list the outcomes. For simplicity, assume $n = 3$. In this case, the tree diagram of Figure 34.5 lists all the outcomes of tossing one, two, and three coins.

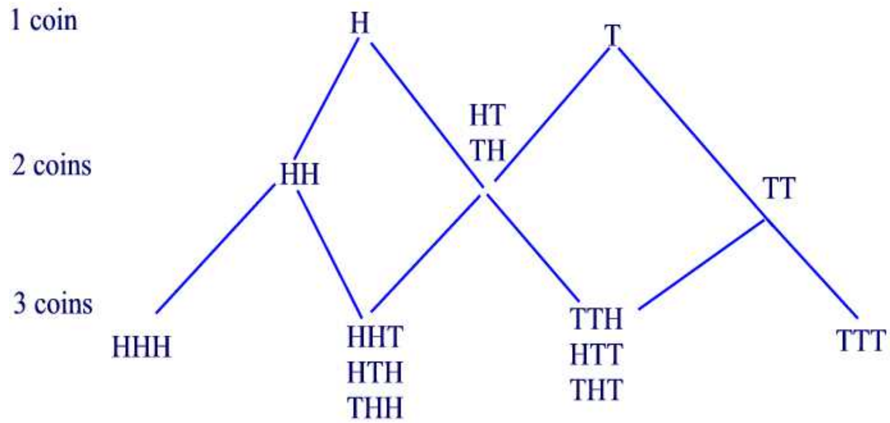

Figure 34.5

From Figure 34.5 we can create a tree diagram that counts the coin outcomes with a given number of heads as shown in Figure 34.6.

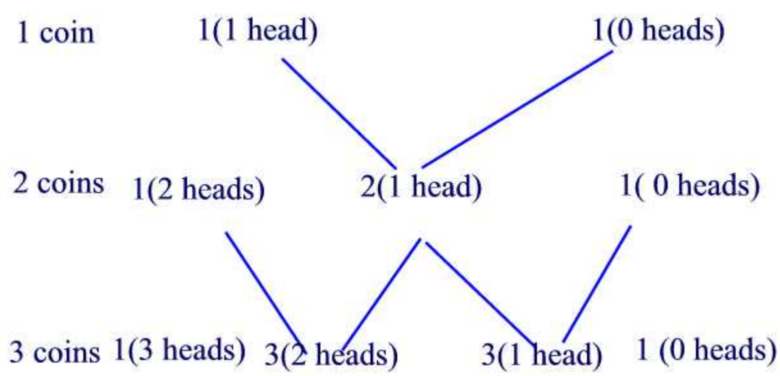

Figure 34.6

The pattern of Figure 34.6 holds for any number of coins and thus leads to the following diagram known as **Pascal's triangle**.

	1
1 coin	1 1
2coins	1 2 1
3coins	1 3 3 1
4 coins	1 4 6 4 1
5 coins	1 5 10 10 5 1
6 coins	1 6 15 20 15 6 1

Example 34.8
Six fair coins are tossed.

(a) Find the probability of getting exactly 3 heads.
(b) Find the probability of getting at least four heads.

Solution.
(a) The 6-coins row of Pascal's triangle may be interpreted as follows

$$1(6H)\ 6(5H)\ 15(4H)\ 20(3H)\ 15(2H)\ 6(1H)\ 1(0H)$$

Thus, there are 20 ways of getting exactly three heads, and the probability of 3 heads is $\frac{20}{64} = \frac{5}{16}$
(b) The first numbers $-1, 6,$ and $15-$ represent the number of outcomes for which there are at least 4 heads. Thus, the probability of getting at least four heads is

$$\frac{1+6+15}{64} = \frac{22}{64} = \frac{11}{32} \blacksquare$$

Practice Problems

Problem 34.23
The row of Pascal's triangle that starts 1,4,... would be useful in finding probabilities for an experiment of tossing four coins.
(a) Interpret the meaning of each number.
(b) Find the probability of exactly one head and three tails.
(c) Find the probability of at least one tail turning up.
(d) Should you bet in favor of getting exactly two heads or should you bet against it?

Problem 34.24
Four coins are tossed.

(a) Draw a tree diagram to represent the arrangements of heads (H) and tails (T).
(b) How many outcomes involve all heads? three heads, one tail? two heads, two tails? one head, three tails? no heads?
(c) How do these results relate to Pascal's triangle?

Problem 34.25
A true-false problem has 6 questions.

(a) How many ways are there to answer the 6-question test?
(b) What is the probability of getting at least 5 right by guessing the answers at random?

Problem 34.26
(a) Write the 7th row of Pascal's triangle.
(b) What is the probability of getting at least four heads when tossing seven coins?

Problem 34.27
Assume the probability is $\frac{1}{2}$ that a child born is a boy. What is the probability that if a family is going to have four children, they will all be boys?