

## 33 Probability: Some Basic Terms

In this and the coming sections we discuss the fundamental concepts of probability at a level at which no previous exposure to the topic is assumed.

Probability has been used in many applications ranging from medicine to business and so the study of probability is considered an essential component of any mathematics curriculum.

So what is probability? Before answering this question we start with some basic definitions.

An **experiment** is any situation whose outcome cannot be predicted with certainty. Examples of an experiment include rolling a die, flipping a coin, and choosing a card from a deck of playing cards.

By an **outcome** or **simple event** we mean any result of the experiment. For example, the experiment of rolling a die yields six outcomes, namely, the outcomes 1,2,3,4,5, and 6.

The **sample space**  $S$  of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be  $S = \{1, 2, 3, 4, 5, 6\}$  where each digit represents a face of the die.

An **event** is a subset of the sample space. For example, the event of rolling an odd number with a die consists of three simple events  $\{1, 3, 5\}$ .

### Example 33.1

Consider the random experiment of tossing a coin three times.

- (a) Find the sample space of this experiment.
- (b) Find the outcomes of the event of obtaining more than one head.

### Solution.

- (a) The sample space is composed of eight simple events:

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

- (b) The event of obtaining more than one head is the set  $\{THH, HTH, HHT, HHH\}$ . ■

### Example 33.2

Consider the random experiment of rolling a die.

- (a) Find the sample space of this experiment.
- (b) Find the event of rolling the die an even number.

**Solution.**

- (a) The sample space is composed of six simple events:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

- (b) The event of rolling the die an even number is the set  $\{2, 4, 6\}$ .■

**Example 33.3**

An experiment consists of the following two stages: (1) first a fair die is rolled and the number of dots recorded, (2) if the number of dots appearing is even, then a fair coin is tossed and its face recorded, and if the number of dots appearing is odd, then the die is tossed again, and the number of dots recorded. Find the sample space of this experiment.

**Solution.**

The sample space for this experiment is the set of 24 ordered pairs

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, H), (2, T), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, H), (4, T), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, H), (6, T)\} \blacksquare.$$

**Probability** is the measure of occurrence of an event. If the event is impossible to occur then its probability is 0. If the occurrence is certain then the probability is 1. The closer to 1 the probability is, the more likely the event is. The probability of occurrence of an event  $E$  (called its **success**) will be denoted by  $P(E)$ . If an event has no outcomes, that is as a subset of  $S$  if  $E = \emptyset$  then  $P(\emptyset) = 0$ . On the other hand, if  $E = S$  then  $P(S) = 1$ .

**Example 33.4**

A hand of 5 cards is dealt from a deck. Let  $E$  be the event that the hand contains 5 aces. List the elements of  $E$ .

**Solution.**

Recall that a standard deck of 52 playing cards can be described as follows:

hearts (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
clubs (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
diamonds (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
spades (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King

Cards labeled Jack, Queen, or King are called **face cards**.

Since there are only 4 aces in the deck, event  $E$  cannot occur. Hence  $E$  is an impossible event and  $E = \emptyset$  so that  $P(E) = 0$  ■

When the outcome of an experiment is just as likely as another, as in the example of tossing a coin, the outcomes are said to be **equally likely**. Various probability concepts exist nowadays. The **classical** probability concept applies *only* when all possible outcomes are equally likely, in which case we use the formula

$$P(E) = \frac{\text{number of outcomes favorable to event}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)},$$

where  $n(E)$  denotes the number of elements in  $E$ .

Since for any event  $E$  we have  $\emptyset \subseteq E \subseteq S$  then  $0 \leq n(E) \leq n(S)$  so that  $0 \leq \frac{n(E)}{n(S)} \leq 1$ . It follows that  $0 \leq P(E) \leq 1$ .

**Example 33.5**

Which of the following numbers cannot be the probability of some event?

- (a) 0.71 (b)  $-0.5$  (c) 150% (d)  $\frac{4}{3}$ .

**Solution.**

Only (a) can represent the probability of an event. The reason that  $-0.5$  is not because it is a negative number. As for 150% this is a number greater than 1 and so cannot be a probability of an event. The same is true for  $\frac{4}{3}$ . ■

**Example 33.6**

What is the probability of drawing an ace from a well-shuffled deck of 52 playing cards?

**Solution.**

Since there are four aces in a deck of 52 playing cards then the probability of getting an ace is  $\frac{4}{52} = \frac{1}{13}$ . ■

**Example 33.7**

What is the probability of rolling a 3 or a 4 with a fair die?

**Solution.**

Since the event of having a 3 or a 4 has two simple events  $\{3, 4\}$  then the

probability of rolling a 3 or a 4 is  $\frac{2}{6} = \frac{1}{3}$ . ■

It is important to keep in mind that the above definition of probability applies only to a sample space that has equally likely outcomes. Applying the definition to a space with outcomes that are not equally likely leads to incorrect conclusions. For example, the sample space for spinning the spinner in Figure 33.1 is given by  $S = \{\text{Red}, \text{Blue}\}$ , but the outcome Blue is more likely to occur than is the outcome Red. Indeed,  $P(\text{Blue}) = \frac{3}{4}$  whereas  $P(\text{Red}) = \frac{1}{4}$ .

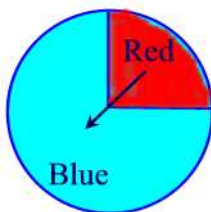


Figure 33.1

A widely used probability concept is the **experimental** probability which uses the relative frequency of an event and is given by the formula:

$$P(E) = \text{Relative frequency} = \frac{f}{n},$$

where  $f$  is the frequency of the event and  $n$  is the size of the sample space.

**Example 33.8**

Personality types are broadly defined according to four main preferences. Do married couples choose similar or different personality types in their mates? The following data give an indication:

Similarities and Differences in a Random Sample of 545 Married Couples

Number of Similar Preferences	Number of Married Couples
All four	108
Three	59
Two	146
One	141
None	91

Suppose that a married couple is selected at random. Use the data to estimate the probability (to two decimal places) that they will have no personality preferences in common.

**Solution.**

The probability they will have no personality preferences in common is  $P(0) = \frac{91}{545} = 0.17$ ■

Next, we define the probability of nonoccurrence of an event  $E$  (called its **failure**) to be the number  $P(E^c)$ . Not surprisingly, the probabilities of an event  $E$  and its complement  $E^c$  are related. The probability of the event  $E^c$  is easily found from the identity

$$\frac{\text{number of outcomes in } A}{\text{total number of outcomes}} + \frac{\text{number of outcomes not in } A}{\text{total number of outcomes}} = 1,$$

so that  $P(E) + P(E^c) = 1$ .

**Example 33.9**

The probability that a college student without a flu shot will get the flu is 0.45. What is the probability that a college student without the flu shot will not get the flu?

**Solution.**

Let  $E$  denote the event with outcomes those students without a flu shot. Then  $P(E) = 0.45$ . The probability that a student without the flu shot will not get the flu is then  $P(E^c) = 1 - P(E) = 1 - 0.45 = 0.55$ .■

**Practice Problems****Problem 33.1**

An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.

**Problem 33.2**

Three coins are thrown. List the outcomes which belong to each of the following events.

- (a) two tails   (b) at least two tails   (c) at most two tails.

**Problem 33.3**

For each of the following events A, B, C, list and count the number of outcomes it contains and hence calculate the probability of A, B or C occurring.

- (a)  $A =$  "throwing 3 or higher with one die",
- (b)  $B =$  "throwing exactly two heads with three coins",
- (c)  $C =$  "throwing a total score of 14 with two dice".

**Problem 33.4**

An experiment consists of throwing two four-faced dice.

- (a) Write down the sample space of this experiment.
- (b) If  $E$  is the event total score is at least 4 list the outcomes belonging to  $E^c$ .
- (c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?
- (d) What is the probability that a double: (i.e.  $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ ) will not be thrown?
- (e) What is the probability that a double is not thrown nor is the score greater than 6?

**Problem 33.5**

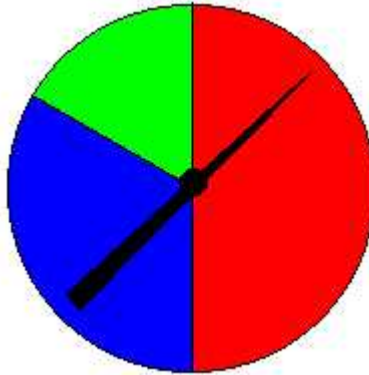
A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. One article is chosen at random. Find the probability that:

- (a) it has no defects,
- (b) it has no major defects,
- (c) it is either good or has major defects.

**Problem 33.6**

Consider the experiment of spinning the pointer on the game spinner pictured below. There are three possible outcomes, that is, when the pointer stops it must point to one of the three colors. (We rule out the possibility of landing on the border between two colors.)

- (a) What is the probability that the spinner is pointing to the red area?
- (b) What is the probability that the spinner is pointing to the blue area?
- (c) What is the probability that the spinner is pointing to the green area?



**Problem 33.7**

Consider the experiment of flipping a coin three times. If we denote a head by H and a tail by T, we can list the 8 possible ordered outcomes as (H,H,H), (H,H,T) each of which occurs with probability of  $1/8$ . Finish listing the remaining members of the sample space. Calculate the probability of the following events:

- (a) All three flips are heads.
- (b) Exactly two flips are heads.
- (c) The first flip is tail.
- (d) At least one flip is head.

**Problem 33.8**

Suppose an experiment consists of drawing one slip of paper from a jar containing 12 slips of paper, each with a different month of the year written on it. Find each of the following:

- (a) The sample space  $S$  of the experiment.
- (b) The event  $A$  consisting of the outcomes having a month beginning with J.
- (c) The event  $B$  consisting of outcomes having the name of a month that has exactly four letters.
- (d) The event  $C$  consisting of outcomes having a month that begins with M or N.

**Problem 33.9**

Let  $S = \{1, 2, 3, \dots, 25\}$ . If a number is chosen at **random**, that is, with the same chance of being drawn as all other numbers in the set, calculate each

of the following probabilities:

- (a) The event A that an even number is drawn.
- (b) The event B that a number less than 10 and greater than 20 is drawn.
- (c) The event C that a number less than 26 is drawn.
- (d) The event D that a prime number is drawn.
- (e) The event E that a number both even and prime is drawn.

**Problem 33.10**

Consider the experiment of drawing a single card from a standard deck of cards and determine which of the following are sample spaces with equally likely outcomes:

- (a) {face card, not face card}
- (b) {club, diamond, heart, spade}
- (c) {black, red}
- (d) {king, queen, jack, ace, even card, odd card}

**Problem 33.11**

An experiment consists of selecting the last digit of a telephone number. Assume that each of the 10 digits is equally likely to appear as a last digit. List each of the following:

- (a) The sample space
- (b) The event consisting of outcomes that the digit is less than 5
- (c) The event consisting of outcomes that the digit is odd
- (d) The event consisting of outcomes that the digit is not 2
- (e) Find the probability of each of the events in (b) - (d)

**Problem 33.12**

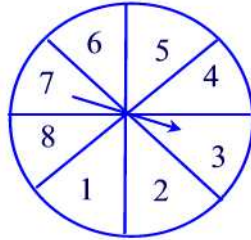
Each letter of the alphabet is written on a separate piece of paper and placed in a box and then one piece is drawn at random.

- (a) What is the probability that the selected piece of paper has a vowel written on it?
- (b) What is the probability that it has a consonant written on it?

**Problem 33.13**

The following spinner is spun:





Find the probabilities of obtaining each of the following:

- (a)  $P(\text{factor of } 35)$
- (b)  $P(\text{multiple of } 3)$
- (c)  $P(\text{even number})$
- (d)  $P(11)$
- (e)  $P(\text{composite number})$
- (f)  $P(\text{neither prime nor composite})$

**Problem 33.14**

An experiment consists of tossing four coins. List each of the following.

- (a) The sample space
- (b) The event of a head on the first coin
- (c) The event of three heads

**Problem 33.15**

Identify which of the following events are certain, impossible, or possible.

- (a) You throw a 2 on a die
- (b) A student in this class is less than 2 years old
- (c) Next week has only 5 days

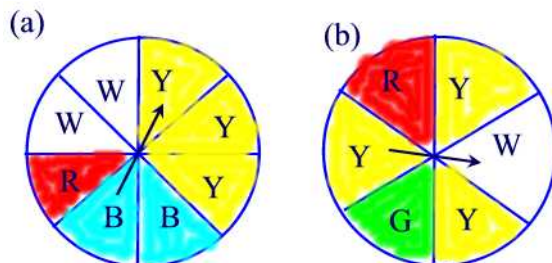
**Problem 33.16**

Two dice are thrown. If each face is equally likely to turn up, find the following probabilities.

- (a) The sum is even
- (b) The sum is not 10
- (c) The sum is a prime
- (d) The sum is less than 9
- (e) The sum is not less than 9

**Problem 33.17**

What is the probability of getting yellow on each of the following spinners?



**Problem 33.18**

A department store's records show that 782 of 920 women who entered the store on a Saturday afternoon made at least one purchase. Estimate the probability that a woman who enters the store on a Saturday afternoon will make at least one purchase.

**Problem 33.19**

Suppose that a set of 10 rock samples includes 3 that contain gold nuggets. If you were to pick up a sample at random, what is the probability that it includes a gold nugget?

**Problem 33.20**

When do creative people get their best ideas? A magazine did a survey of 414 inventors (who hold U.S. patents) and obtained the following information:

Time of Day When Best Ideas Occur

Time	Number of Inventors
6 A.M. - 12 noon	46
12 noon - 6 P.M.	188
6 P.M. - 12 midnight	63
12 midnight - 6 A.M.	117

Assuming that the time interval includes the left limit and all the times up to but not including the right limit, estimate the probability (to two decimal places) that an inventor has a best idea during the time interval 6 A.M. - 12 noon.

### Probability of Union and Intersection of Two Events

The **union** of two events  $A$  and  $B$  is the event  $A \cup B$  whose outcomes are either in  $A$  or in  $B$ . The **intersection** of two events  $A$  and  $B$  is the event  $A \cap B$  whose outcomes are outcomes of both events  $A$  and  $B$ . Two events  $A$  and  $B$  are said to be **mutually exclusive** if they have no outcomes in common. In this case  $A \cap B = \emptyset$ . Thus,  $P(A \cap B) = P(\emptyset) = 0$ .

#### Example 33.10

Consider the sample space of rolling a die. Let  $A$  be the event of rolling an even number,  $B$  the event of rolling an odd number, and  $C$  the event of rolling a 2. Find

- (a)  $A \cup B$ ,  $A \cup C$ , and  $B \cup C$ .
- (b)  $A \cap B$ ,  $A \cap C$ , and  $B \cap C$ .
- (c) Which events are mutually exclusive?

#### Solution.

- (a) We have

$$\begin{aligned}A \cup B &= \{1, 2, 3, 4, 5, 6\} \\A \cup C &= \{2, 4, 6\} \\B \cup C &= \{1, 2, 3, 5\}\end{aligned}$$

- (b)

$$\begin{aligned}A \cap B &= \emptyset \\A \cap C &= \{2\} \\B \cap C &= \emptyset\end{aligned}$$

- (c)  $A$  and  $B$  are mutually exclusive as well as  $B$  and  $C$ . ■

#### Example 33.11

Let  $A$  be the event of drawing a King from a well-shuffled standard deck of playing cards and  $B$  the event of drawing a "ten" card. Are  $A$  and  $B$  mutually exclusive?

#### Solution.

Since  $A = \{\text{king of diamonds, king of hearts, king of clubs, king of spades}\}$  and  $B = \{\text{ten of diamonds, ten of hearts, ten of clubs, ten of spades}\}$  then  $A$  and  $B$  are mutually exclusive since there are no cards common to both events. ■

The next result provides a relationship between the probabilities of the events  $A$ ,  $B$ , and  $A \cup B$  when  $A$  and  $B$  are mutually exclusive.

**Theorem 33.1**

If  $A$  and  $B$  are mutually exclusive events of a sample space  $S$  then  $P(A \cup B) = P(A) + P(B)$ .

**Proof.**

Since  $A$  and  $B$  are mutually exclusive then  $A$  and  $B$  have no elements in common so that  $n(A \cup B) = n(A) + n(B)$ . Thus,

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = P(A) + P(B). \blacksquare$$

Now, for any events  $A$  and  $B$ , not necessarily exclusive, the probability of  $A \cup B$  is given by the addition rule.

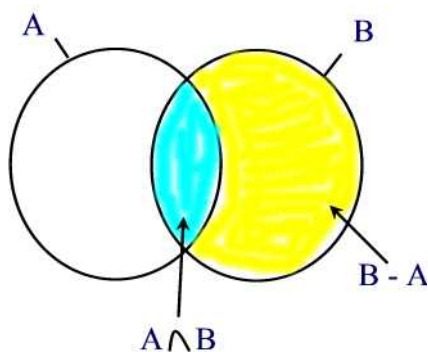
**Theorem 33.2**

Let  $A$  and  $B$  be two events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

**Proof.**

Let  $B - A$  denote the event whose outcomes are the outcomes in  $B$  that are not in  $A$ . Then using the Venn diagram below we see that  $B = (A \cap B) \cup (B - A)$  and  $A \cup B = A \cup (B - A)$ .



Since  $(A \cap B)$  and  $(B - A)$  are disjoint then

$$P(B) = P(A \cap B) + P(B - A).$$

Thus,

$$P(B - A) = P(B) - P(A \cap B).$$

Similarly, since  $A$  and  $B - A$  are disjoint then

$$P(A \cup B) = P(A) + P(B - A) = P(A) + P(B) - P(A \cap B). \blacksquare$$

**Example 33.12**

Let  $P(A) = 0.9$  and  $P(B) = 0.6$ . Find the minimum possible value for  $P(A \cap B)$ .

**Solution.**

Since  $P(A) + P(B) = 1.5$  and  $0 \leq P(A \cup B) \leq 1$  then by the previous theorem

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq 1.5 - 1 = 0.5.$$

So the minimum value of  $P(A \cap B)$  is 0.5 ■

**Example 33.13**

Suppose there's 40% chance of colder weather, 10% chance of rain and colder weather, 80% chance of rain or colder weather. Find the chance of rain.

**Solution.**

By the addition rule we have

$$P(R) = P(R \cup C) - P(C) + P(R \cap C) = 0.8 - 0.4 + 0.1 = 0.5. \blacksquare$$

## Practice Problem

**Problem 33.21**

Which of the following are mutually exclusive? Explain your answers.

- (a) A driver getting a ticket for speeding and a ticket for going through a red light.
- (b) Being foreign-born and being President of the United States.

**Problem 33.22**

If  $A$  and  $B$  are the events that a consumer testing service will rate a given stereo system very good or good,  $P(A) = 0.22$ ,  $P(B) = 0.35$ . Find

- (a)  $P(A^c)$ ;
- (b)  $P(A \cup B)$ ;
- (c)  $P(A \cap B)$ .

**Problem 33.23**

If the probabilities are 0.20, 0.15, and 0.03 that a student will get a failing grade in Statistics, in English, or in both, what is the probability that the student will get a failing grade in at least one of these subjects?

**Problem 33.24**

If  $A$  is the event "drawing an ace" from a deck of cards and  $B$  is the event "drawing a spade". Are  $A$  and  $B$  mutually exclusive? Find  $P(A \cup B)$ .

**Problem 33.25**

A bag contains 18 coloured marbles: 4 are coloured red, 8 are coloured yellow and 6 are coloured green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

**Problem 33.26**

Show that for any events  $A$  and  $B$ ,  $P(A \cap B) \geq P(A) + P(B) - 1$ .

**Problem 33.27**

A golf bag contains 2 red tees, 4 blue tees, and 5 white tees.

- (a) What is the probability of the event  $R$  that a tee drawn at random is red?
- (b) What is the probability of the event "not  $R$ " that is, that a tee drawn at random is not red?
- (c) What is the probability of the event that a tee drawn at random is either red or blue?

**Problem 33.28**

A fair pair of dice is rolled. Let  $E$  be the event of rolling a sum that is an even number and  $P$  the event of rolling a sum that is a prime number. Find the probability of rolling a sum that is even or prime?

**Problem 33.29**

If events  $A$  and  $B$  are from the same sample space, and if  $P(A)=0.8$  and  $P(B)=0.9$ , can events  $A$  and  $B$  be mutually exclusive?