A Second Course in Mathematics Concepts for Elementary Teachers: Theory, Problems, and Solutions

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Contents

25 Integers: Addition and Subtraction

Whole numbers and their operations were developed as a direct result of people's need to count. But nowadays many quantitative needs aside from counting require numbers other than whole numbers.
In this and the next section we study the set $\mathbb{Z}$ of integers which consists of:
• the set W of whole numbers, i.e. $W = \{0, 1, 2, 3, \cdots\}$,
• and the **negative integers** denoted by $N = \{-1, -2, -3, \cdots\}$.

That is,

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

In today's society these numbers are used to record debits and credits, profits and losses, changes in the stock markets, degrees above and below zero in measuring temperature, the altitude above and below sea level, etc.
As usual, we introduce the set of integers by means of pictures and diagrams with steadily increasing levels of abstraction to suggest, in turn, how these ideas might be presented to school children.

Representations of Integers
We will discuss two different ways for representing integers: set model and number line model.
• **Set Model:** In this model, we use signed counters. We represent $+1$ by $\oplus$ and -1 by $\ominus$. In this case, the number $+2$ is represented by

$$\oplus \quad \oplus$$

and the number -5 is represented by

$$\ominus \quad \ominus \quad \ominus \quad \ominus \quad \ominus$$

One plus counter neutralizes one minus counter. Thus, we can represent 0 by $\oplus \ominus$.
• **Number-Line Model:** The number line model is shown in Figure 25.1

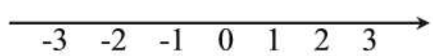


Figure 25.1

Example 25.1
Represent -4 on a number line and as a set of signed counters.

Solution.
Using signed counters we have

$$\ominus \ \ominus \ \ \ominus \ \ominus$$

Figure 25.2 shows the position of -4 on a number-line.■

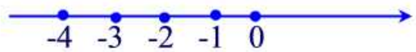


Figure 25.2

Looking at Figure 25.1, we see that -2 and 2 have the same distance from 0 but on opposite sides. Such numbers are called **opposite.** In general, the opposite of x is $-x$. Hence, the opposite of 5 is -5 and the opposite of -5 is $-(-5) = 5$.

Example 25.2
Find the opposite of each of these integers.
(a) 4 (b) -2 (c) 0

Solution.
(a) The opposite of 4 is -4.
(b) The opposite of -2 is $-(-2) = 2$.
(c) The opposite of 0 is 0 since 0 is neither positive nor negative.■

Absolute Value of an Integer
As an important application of the use of the number line model for integers is the concept of absolute value of an integer. We see from Figure 25.1 that the number -3 is represented by the point numbered -3 and the integer 3 is represented by the point numbered 3. Note that both -3 and 3 are three units from 0 on the number line. The **absolute value** of an integer x is the distance of the corresponding point on the number line from zero. We indicate the absolute value of x by $|x|$. It follows from the above discussion that $|-3| = |3| = 3$.

Example 25.3
Determine the absolute values of these integers.
(a) -11 (b) 11 (c) 0

Solution.
(a) $|-11| = 11$ (b) $|11| = 11$ (c) $|0| = 0.$■

Practice Problems

Problem 25.1
Which of the following are integers?
(a) -11 (b) 0 (c) $\frac{3}{4}$ (d) $\frac{-9}{3}$

Problem 25.2
Let $N = \{-1, -2, -3, \cdots\}$. Find
(a) $N \cup W$
(b) $N \cap \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers or positive integers and W is the set of whole numbers.

Problem 25.3
What number is 5 units to the left of -95?

Problem 25.4
A and B are 9 units apart on the number line. A is twice as far from 0 as B. What are A and B?

Problem 25.5
Represent -5 using signed counters and number line.

Problem 25.6
In terms of distance, explain why $|-4| = 4$.

Problem 25.7
Find the additive inverse (i.e. opposite) of each of the following integers.
(a) 2 (b) 0 (c) -5 (d) m (e) $-m$

Problem 25.8
Evaluate each of the following.
(a) $|-5|$ (b) $|10|$ (c) $-|-4|$ (d) $-|5|$

Problem 25.9
Find all possible integers x such that $|x| = 2$.

Problem 25.10
Let W be the set of whole numbers, $\mathbb{Z}^+$ the set of positive integers (i.e. $\mathbb{Z}^+ = \mathbb{N}$), $\mathbb{Z}^-$ the set of negative integers, and $\mathbb{Z}$ the set of all integers. Find each of the following.
(a) $W \cup \mathbb{Z}$ (b) $W \cap \mathbb{Z}$ (c) $\mathbb{Z}^+ \cup \mathbb{Z}^-$ (d) $\mathbb{Z}^+ \cap \mathbb{Z}^-$ (e) $W - \mathbb{Z}^+$

In the remainder of this section we discuss integer addition and subtraction. We will use the devices of signed counters and number lines to illustrate how these operations should be performed in the set of integers.

Addition of Integers
A football player loses 3 yards on one play and 2 yards on the next, what's the player's overall yardage on the two plays? The answer is $-3+(-2)$ which involves addition of integers. We compute this sum using the two models discussed above.
• **Using Signed Counters**
Since -3 is represented by 3 negative counters and -2 is represented by 2 negative counters then by combining these counters we obtain 5 negative counters which represent the integer -5. So $-3 + (-2) = -5$.
• **Using Number Line**
Move three units to the left of 0 and then 2 units to the left from -3 as shown in Figure 25.3.

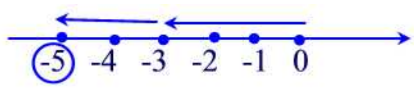
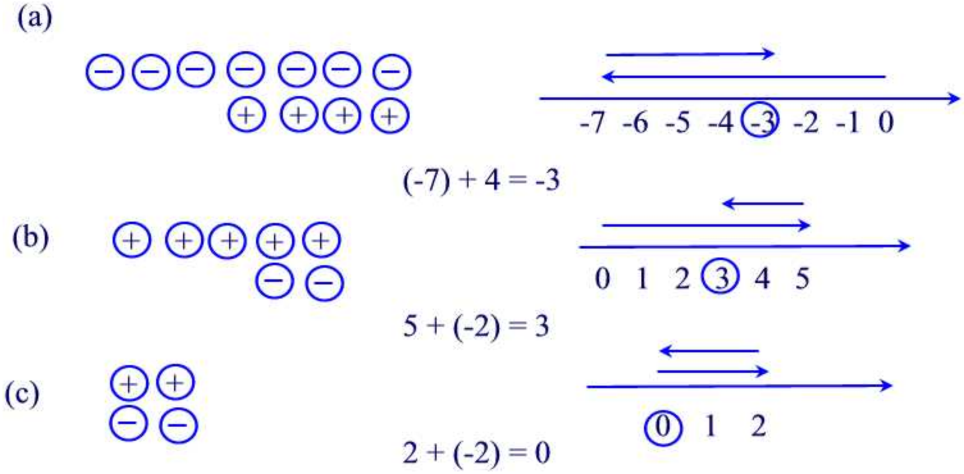


Figure 25.3

Example 25.4
Find the following sums using (i) signed counters, and (ii) number line.
(a) $(-7) + 4$ (b) $5 + (-2)$ (c) $2 + (-2)$

Solution.



The results of the above examples are entirely typical and we state them as a theorem.

Theorem 25.1
Let a and b be integers. Then the following are always true:
(i) $(-a) + (-b) = -(a + b)$
(ii) If $a > b$ then $a + (-b) = a - b$
(iii) If $a < b$ then $a + (-b) = -(b - a)$
(iv) $a + (-a) = 0$.

Example 25.5
Compute the following sums.
(a) $7 + 11$ (b) $(-6) + (-5)$ (c) $7 + (-3)$ (d) $4 + (-9)$ (e) $6 + (-6)$

Solution.
(a) $7 + 11 = 18$
(b) $(-6) + (-5) = -(6 + 5) = -11$
(c) $7 + (-3) = 7 - 3 = 4$
(d) $4 + (-9) = -(9 - 4) = -5$
(e) $6 + (-6) = 0$■

Integer addition has all the properties of whole number addition. These properties are summarized in the following theorem.

Theorem 25.2
Let a, b, and c be integers.
Closure $a + b$ is a unique integer.
Commutativity $a + b = b + a$.
Associativity $a + (b + c) = (a + b) + c$.
Identity Element 0 is the unique integer such that $a + 0 = 0 + a = a$.

A useful consequence of Theorem 25.1 (iv) is the **additive cancellation.**

Theorem 25.3
For any integers a, b, c, if $a + c = b + c$ then $a = b$.

Proof.

$$\begin{array}{rcll} a & = & a + 0 & \textit{identity element} \\ & = & a + (c + (-c)) & \textit{additive inverse} \\ & = & (a + c) + (-c) & \textit{associativity} \\ & = & (b + c) + c & \textit{given} \\ & = & b + (c + (-c)) & \textit{associativity} \\ & = & b + 0 & \textit{additive inverse} \\ & = & b & \textit{identity element}\blacksquare \end{array}$$

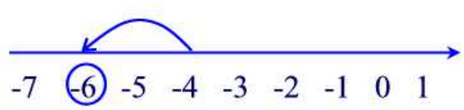
Example 25.6
Use the additive cancellation to show that $-(-a) = a$.

Solution.
Since $a + (-a) = (-(-a)) + (-a) = 0$ then by the additive cancellation property we have $a = -(-a)\blacksquare$

Practice Problems

Problem 25.11
What is the opposite or additive inverse of each of the following? (a and b are integers)
(a) $a + b$ (b) $a - b$

Problem 25.12
What integer addition problem is shown on the number line?



Problem 25.13
(a) Explain how to compute $-7+2$ with a number line.
(b) Explain how to compute $-7+2$ with signed counters.

Problem 25.14
In today's mail, you will receive a check for \$86, a bill for \$30, and another bill for \$20. Write an integer addition equation that gives the overall gain or loss.

Problem 25.15
Compute the following without a calculator.
(a) $-54+25$ (b) $(-8)+(-17)$ (c) $400+(-35)$

Problem 25.16
Show two ways to represent the integer 3 using signed counters.

Problem 25.17
Illustrate each of the following addition using the signed counters.
(a) $5+(-3)$ (b) $-2+3$ (c) $-3+2$ (d) $(-3)+(-2)$

Problem 25.18
Demonstrate each of the additions in the previous problem using number line model.

Problem 25.19
Write an addition problem that corresponds to each of the following sentences and then answer the question.
(a) The temperature was $-10°C$ and then it rose by $8°C$. What is the new temperature?
(b) A certain stock dropped 17 points and the following day gained 10 points. What was the net change in the stock's worth?
(c) A visitor to a casino lost \$200, won \$100, and then lost \$50. What is the change in the gambler's net worth?

Problem 25.20
Compute each of the following: (a) $|(-9)+(-5)|$ (b) $|7+(-5)|$ (c) $|(-7)+6|$

Problem 25.21
If a is an element of $\{-3, -2, -1, 0, 1, 2\}$ and b is an element of $\{-5, -4, -3, -2, -1, 0, 1\}$, find the smallest and largest values for the following expressions.
(a) $a + b$ (b) $|a + b|$

Subtraction of Integers

Like integer addition, integer subtraction can be illustrated with signed counters and number line.

• **Signed Counters**

Take Away Approach

As with the subtraction of whole numbers, one approach to integer subtraction is the notion of "take away." Figure 25.4 illustrates various examples of subtraction of integers.

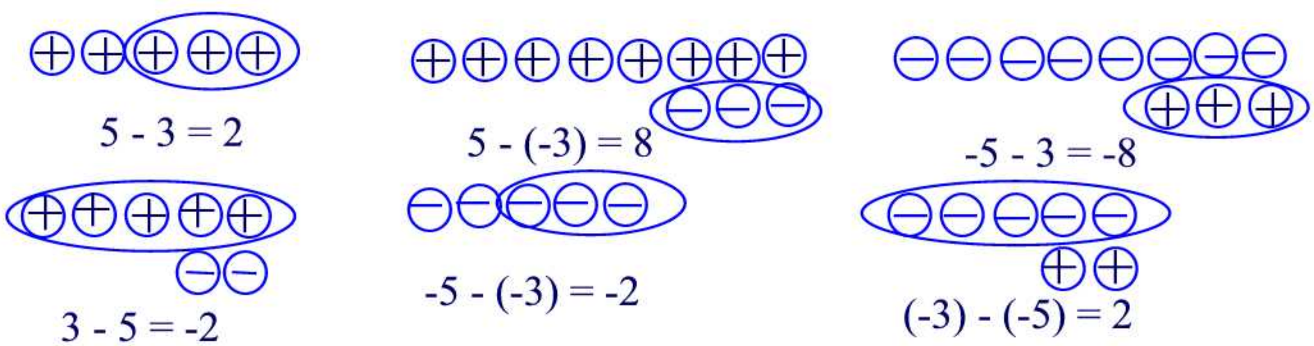


Figure 25.4

Another important fact about subtraction of integers is illustrated in Figure 25.5.

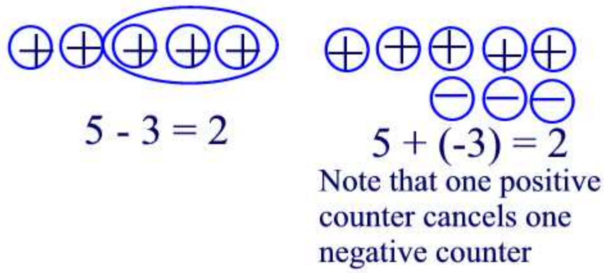


Figure 25.5

Note that $5 - 3 = 5 + (-3)$. In general, the following is true.

Adding the Opposite Approach
For all integers a and b, $a - b = a + (-b)$.

Adding the opposite is perhaps the most efficient method for subtracting integers because it replaces any subtraction problem with an equivalent addition problem.

Example 25.7
Find the following differences by adding the opposite.
(a) $-8 - 3$ (b) $4 - (-5)$

Solution.
(a) $-8 - 3 = (-8) + (-3) = -(8 + 3) = -11$.
(b) $4 - (-5) = 4 + [-(-5)] = 4 + 5 = 9$.■

Missing-Addend Approach
Signed counters model can be also used to illustrate the "missing-addend" approach for the subtraction of integers. Figure 25.6 illustrates both equations $3 - (-2) = 5$ and $3 = 5 + (-2)$.

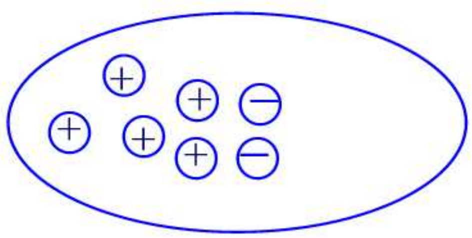


Figure 25.6

Since all subtraction diagrams can be interpreted in this way, we can introduce the "missing-addend" approach for subtraction:
If a, b, and c are any integers, then $a - b = c$ if and only if $a = c + b$.

• Number Line Model
The number line model used for integer addition can also be used to model integer subtraction. In this model, the operation of subtraction corresponds of starting from the tail of the first number facing the negative direction.

Thus, from the tail of the first number we move forward if we are subtracting a positive integer and backward if we are subtracting a negative integer. Figure 25.7 illustrates some examples.

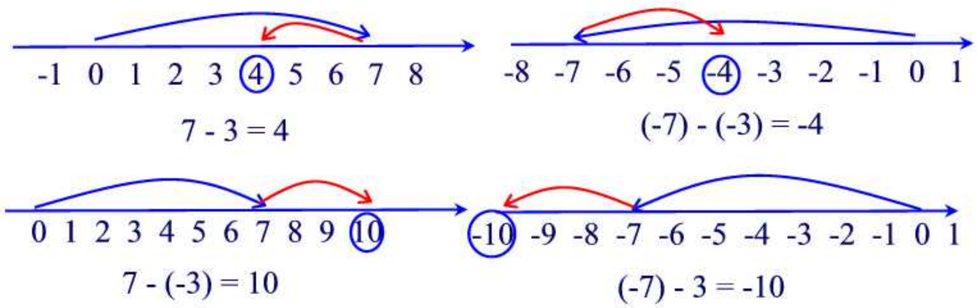


Figure 25.7

In summary there are three equivalent ways to view subtraction of integers.

1. Take away
2. Adding the opposite
3. Missing addend

Example 25.8
Find $4 - (-2)$ using all three methods of subtraction.

Solution.
Take away: See Figure 25.8

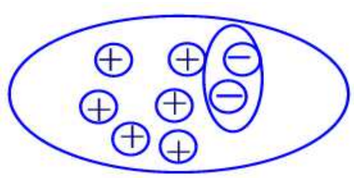


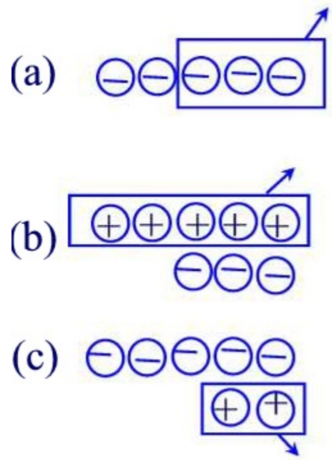
Figure 25.8

Adding the opposite: $4 - (-2) = 4 + [-(-2)] = 4 + 2 = 6$.
Missing-Addend: $4 - (-2) = c$ if and only if $4 = c + (-2)$. But $4 = 6 + (-2)$ so that $c = 6$.■

Practice Problems

Problem 25.22
Explain how to compute $5 - (-2)$ using a number line.

Problem 25.23
Tell what subtraction problem each picture illustrates.



Problem 25.24
Explain how to compute the following with signed counters.
(a) $(-6) - (-2)$ (b) $2 - 6$ (c) $-2 - 3$ (d) $2 - (-4)$

Problem 25.25
(a) On a number line, subtracting 3 is the same as moving _____ units to the _____.
(b) On a number line, adding -3 is the same as moving _____ units to the _____.

Problem 25.26
An elevetor is at an altitude of -10 feet. The elevator goes down 30 ft.
(a) Write an integer equation for this situation.
(b) What is the new altitude?

Problem 25.27
Compute each of the following using $a - b = a + (-b)$.
(a) $3 - (-2)$ (b) $-3 - 2$ (c) $-3 - (-2)$

Problem 25.28
Use number-line model to find the following.
(a) $-4 - (-1)$ (b) $-2 - 1$.

Problem 25.29
Compute each of the following.
(a) $|5-11|$ (b) $|(-4)-(-10)|$ (c) $|8-(-3)|$ (d) $|(-8)-2|$

Problem 25.30
Find x.
(a) $x+21=16$ (b) $(-5)+x=7$ (c) $x-6=-5$ (d) $x-(-8)=17$.

Problem 25.31
Which of the following properties hold for integer subtraction. If a property does not hold, disprove it by a counterexample.
(a) Closure (b) Commutative (c) Associative (d) Identity

26 Integers: Multiplication, Division, and Order

Integer multiplication and division are extensions of whole number multiplication and division. In multiplying and dividing integers, the one new issue is whether the result is positive or negative. This section shows how to explain the sign of an integer product or quotient using number line, patterns, and signed counters.

Multiplication of Integers Using Patterns

Consider the following pattern of equalities which are derived from using the repeated addition .

$$\begin{aligned} 4 \times 3 &= 12 \\ 4 \times 2 &= 8 \\ 4 \times 1 &= 4 \\ 4 \times 0 &= 0 \end{aligned}$$

Note that the second factors in the successive products decrease by 1 each time and that the successive results decrease by 4. If this pattern continues we obtain the following sequence of equalities.

$$\begin{aligned} 4 \times 3 &= 12 \\ 4 \times 2 &= 8 \\ 4 \times 1 &= 4 \\ 4 \times 0 &= 0 \\ 4 \times (-1) &= -4 \\ 4 \times (-2) &= -8 \\ 4 \times (-3) &= -12 \end{aligned}$$

Since these results agree with what would be obtained by repeated addition, the pattern of the product is an appropriate guide. In general, it suggests that

$$m \cdot (-n) = -(m \cdot n)$$

where m and n are positive integers. This says that the product of a positive integer times a negative integer is always a negative integer.

Using this result we can write the following pattern of equalities

$$\begin{aligned} 3 \times (-3) &= -9 \\ 2 \times (-3) &= -6 \\ 1 \times (-3) &= -3 \\ 0 \times (-3) &= 0 \end{aligned}$$

So we notice that when the first factors decrease by 1 the successive results increase by 3. Continuing the pattern we find

$$\begin{aligned} 3 \times (-3) &= -9 \\ 2 \times (-3) &= -6 \\ 1 \times (-3) &= -3 \\ 0 \times (-3) &= 0 \\ (-1) \times (-3) &= 3 \\ (-2) \times (-3) &= 6 \end{aligned}$$

In general, this suggests that

$$(-m) \cdot (-n) = m \cdot n,$$

where m and n are positive integers. Hence the product of two negative integers is always positive.
What about the product of a negative integer times a positive integer such as $(-3) \times 2$? Using the results just derived above we see that

$$\begin{aligned} (-3) \times (-3) &= 9 \\ (-3) \times (-2) &= 6 \\ (-3) \times (-1) &= 3 \\ (-3) \times 0 &= 0 \end{aligned}$$

Note that the second factors in the successive products increase by 1 each time and the successive results decrease by 3. Hence, continuing the pattern we obtain

$$\begin{aligned} (-3) \times (-3) &= 9 \\ (-3) \times (-2) &= 6 \\ (-3) \times (-1) &= 3 \\ (-3) \times 0 &= 0 \\ (-3) \times 1 &= -3 \\ (-3) \times 2 &= -6 \end{aligned}$$

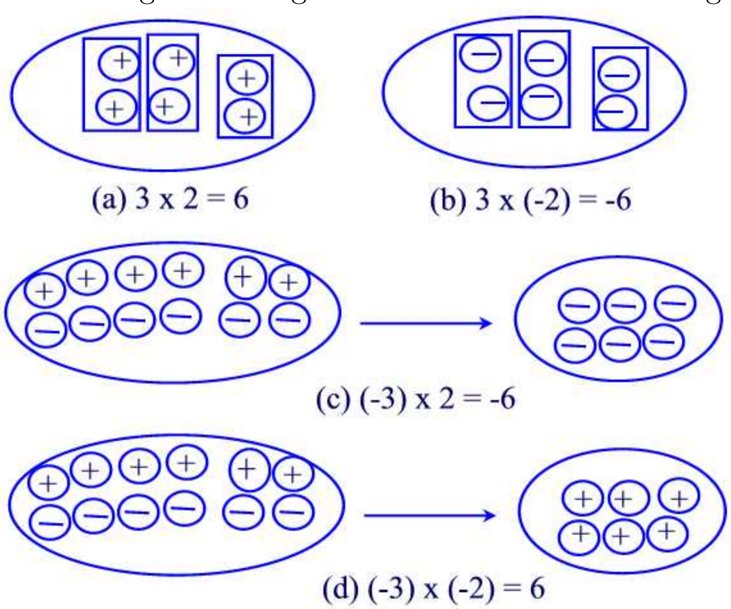
This suggests the general rule

$$(-n) \cdot m = -(n \cdot m),$$

where m and n are positive integers. That is, a negative number times a positive number is always negative.
Summarizing these results into a theorem we have

Theorem 26.1
Let m and n be two positive integers. Then the following are true:
(1) $m(-n) = -mn$
(2) $(-m)n = -mn$
(3) $(-m)(-n) = mn$.

Multiplication of Integers Using Signed Counters
The signed counters can be used to illustrate multiplication of integers, although an interpretation must be given to the sign. See Figure 26.1(a). The product 3×2 illustrates three groups each having 2 positive counters. A product such as $3 \times (-2)$ represents three groups each having 2 negative counters as shown in Figure 26.1(b). A product like $(-3) \times 2$ is interpreted as taking away 3 groups each having 2 positive counters as shown in Figure 26.1(c), and the product $(-3) \times (-2)$ is interpreted as taking away three groups each consisting of two negative counters as shown in Figure 26.1(d).



(a) 3 x 2 = 6
(b) 3 x (-2) = -6
(c) (-3) x 2 = -6
(d) (-3) x (-2) = 6

Figure 26.1

Multiplication of Integers Using Number Line
Since a product has a first factor and a second factor we can identify the first factor as the velocity of a moving object who starts from 0 and the second factor is time. A positive velocity means that the object is moving east and a negative velocity means that the object is moving west. Time in future is denoted by a positive integer and time in the past is denoted by a negative integer.

Example 26.1
Use a number-line model to answer the following questions.
(a) If you are now at 0 and travel east at a speed of 50km/hr, where will you be 3 hours from now?
(b) If you are now at 0 and travel east at a speed of 50km/hr, where were you 3 hours ago?
(c) If you are now at 0 and travel west at a speed of 50km/hr, where will you be 3 hours from now?
(d) If you are now at 0 and travel west at a speed of 50km/hr, where were you 3 hours ago?

Solution.
Figure 26.2 illustrates the various products using a number-line model.■

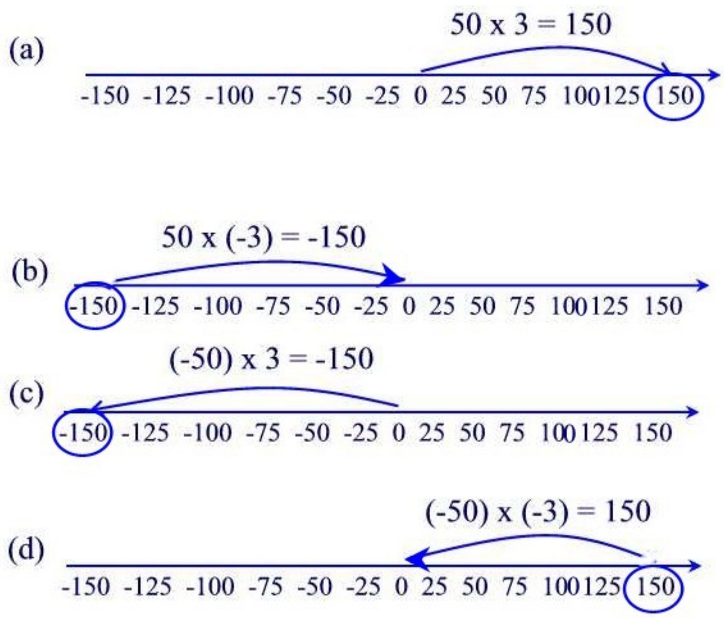


Figure 26.2

The set of integers has properties under multiplication analogous to those of the set of whole numbers. We summarize these properties in the following theorem.

Theorem 26.2
Let a, b, and c be any integers. Then the following are true.

Closure $a \cdot b$ is a unique integer.
Commutativity $a \cdot b = b \cdot a$
Associativity $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Identity Element $a \cdot 1 = 1 \cdot a = a$
Distributivity $a \cdot (b + c) = a \cdot b + a \cdot c$.
Zero Product $a \cdot 0 = 0 \cdot a = 0$.

Using the previous theorem one can derive some important results that are useful in computations.

Theorem 26.3
For any integer a we have $(-1) \cdot a = -a$.

Proof.
First, note that $0 = 0 \cdot a = [1 + (-1)] \cdot a = a + (-1) \cdot a$. But $a + (-a) = 0$ so that $a+(-a) = a+(-1)a$. By the additive cancellation property we conclude that $-a = (-1) \cdot a$ ■

We have seen that $(-a)b = -ab$ and $(-a)(-b) = ab$ where a and b are positive integer. The following theorem shows that those two equations are true for any integer not just positive integers.

Theorem 26.4
Let a and b be any integers. Then

$$(-a)b = -ab$$

and

$$(-a)(-b) = ab.$$

Proof.
To prove the first equation, we proceed as follows.

$$(-a)b = [(-1)a]b = (-1)(ab) = -ab.$$

To prove the second property, we have

$$(-a)(-b) = (-1)[a(-b)] = (-1)[(-a)b] = (-1)[(-1)(ab)] = (-1)(-1)(ab) = ab \blacksquare$$

Another important property is the so-called the **multiplicative cancellation property.**

Theorem 26.5
Let a, b, and c be integers with $c \neq 0$. If $ac = bc$ then $a = b$.

Proof.
Suppose $c \neq 0$. Then either $c > 0$ or $c < 0$. If $\frac{c}{1} = c > 0$ then we know from Theorem 20.1 that c has a multiplicative inverse $\frac{1}{c}$, i.e. $cc^{-1} = 1$. In this case, we have

$$\begin{aligned} a &= a \cdot 1 \\ &= a(cc^{-1}) \\ &= (ac)c^{-1} \\ &= (bc)c^{-1} \\ &= b(cc^{-1}) \\ &= b \cdot 1 = b \end{aligned}$$

If $c < 0$ then $-c > 0$ and we can use the previous argument to obtain

$$\begin{aligned} a &= a \cdot 1 \\ &= a[(-c)(-c)^{-1}] \\ &= [a(-c)](-c)^{-1} \\ &= -(ac)(-c)^{-1} \\ &= -(bc)(-c)^{-1} \\ &= [b(-c)](-c)^{-1} \\ &= b[(-c)(-c)^{-1}] \\ &= b \cdot 1 = b \blacksquare \end{aligned}$$

The following result follows from the multiplicative cancellation property.

Theorem 26.6 *(Zero Divisor)*
Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$.

Proof.
Suppose that $b \neq 0$. Then $ab = 0 \cdot b$. By the previous theorem we must have $a = 0$ $\blacksquare$

Practice Problems

Problem 26.1
Use patterns to show that $(-1)(-1) = 1$.

Problem 26.2
Use signed counters to show that $(-4)(-2) = 8$.

Problem 26.3
Use number line to show that $(-4) \times 2 = -8$.

Problem 26.4
Change $3 \times (-2)$ into a repeated addition and then compute the answer.

Problem 26.5
(a) Compute $4 \times (-3)$ with repeated addition.
(b) Compute $4 \times (-3)$ using signed counters.
(c) Compute $4 \times (-3)$ using number line.

Problem 26.6
Show how $(-2) \times 4$ can be found by extending a pattern in a whole-number multiplication.

Problem 26.7
Mike lost 3 pounds each week for 4 weeks.
(a) What was the total change in his weight?
(b) Write an integer equation for this situation.

Problem 26.8
Compute the following without a calculator.
(a) $3 \times (-8)$
(b) $(-5) \times (-8) \times (-2) \times (-3)$

Problem 26.9
Extend the following pattern by writing the next three equations.

$$\begin{array}{rcl} 6 \times 3 & = & 18 \\ 6 \times 2 & = & 12 \\ 6 \times 1 & = & 6 \\ 6 \times 0 & = & 0 \end{array}$$

Problem 26.10
Find the following products.
(a) $6(-5)$ (b) $(-2)(-16)$ (c) $-(-3)(-5)$ (d) $-3(-7-6)$.

Problem 26.11
Represent the following products using signed counters and give the results.
(a) $3 \times (-2)$ (b) $(-3) \times (-4)$

Problem 26.12
In each of the following steps state the property used in the equations.

$$\begin{array}{rcl} a(b-c) & = & a[b+(-c)] \\ & = & ab + a(-c) \\ & = & ab + [-(ac)] \\ & = & ab - ac \end{array}$$

Problem 26.13
Extend the meaning of a whole number exponent

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\ factors}$$

where a is any integer and n is a positive integer. Use this definition to find the following values.
(a) $(-2)^4$ (b) -2^4 (c) $(-3)^5$ (d) -3^5

Problem 26.14
Illustrate the following products on an integer number line.
(a) $2 \times (-5)$ (b) $3 \times (-4)$ (c) $5 \times (-2)$

Problem 26.15
Expand each of the following products.
(a) $-6(x+2)$ (b) $-5(x-11)$ (c) $(x-3)(x+2)$

Problem 26.16
Name the property of multiplication of integers that is used to justify each of the following equations.
(a) $(-3)(-4) = (-4)(-3)$
(b) $(-5)[(-2)(-7)] = [(-5)(-2)](-7)$
(c) $(-5)(-7)$ is a unique integer
(d) $(-8) \times 1 = -8$
(e) $4 \cdot [(-8) + 7] = 4 \cdot (-8) + 4 \cdot 7$

Problem 26.17
If $3x = 0$ what can you conclude about the value of x?

Problem 26.18
If a and b are negative integers and c is positive integer, determine whether the following are positive or negative.
(a) $(-a)(-c)$ (b) $(-a)(b)$ (c) $(c-b)(c-a)$ (d) $a(b-c)$

Problem 26.19
Is the following equation true? $a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$.

Division of Integers

Suppose that a town's population drops by 400 people in 5 years. What is the average population change per year? The answer is given by $-400 \div 5$, where the negative sign indicates a decrease in population. Such a problem requires division of integers which we dicuss next.

You recall the missing-factor approach for the division of whole numbers: we write $a \div b, b \neq 0$ to mean a unique whole number c such that $a = bc$. Division on the set of integers is defined analogously:

Let a and b be any integers with $b \neq 0$. Then $a \div b$ is the unique integer c, if it exists, such that $a = bc$.

Example 26.2

Find the following quotients, if possible.

(a) $12 \div (-4)$ (b) $(-12) \div 4$ (c) $(-12) \div (-4)$ (d) $7 \div (-2)$

Solution.

(a) Let $c = 12 \div (-4)$. Then $-4c = 12$ and consequently, $c = -3$.

(b) If $c = (-12) \div 4$ then $4c = -12$ and solving for c we find $c = -3$.

(c) Letting $c = (-12) \div (-4)$ we obtain $-4c = -12$ and consequently $c = 3$.

(d) Let $c = 7 \div (-2)$. Then $-2c = 7$ Since there is no integer when multiplied by -2 yields 7 then we say that $7 \div (-2)$ is undefined in the integers ■

The previous example suggests the following rules of division of integers.

(i) The quotient, if it exists, of two integers with the same sign is always positive.

(ii) The quotient, if it exists, of two integers with different signs is always negative.

Negative Exponents and Scientific Notation

Recall that for any whole number a and n, a positive integer we have

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\ factors}.$$

We would like to extend this definition to include negative exponents. This is done by using "looking for a pattern" strategy. Consider the following sequence of equalities where a is a non zero integer.

$$\begin{array}{rcl} a^3 & = & a \cdot a \cdot a \\ a^2 & = & a \cdot a \\ a^1 & = & a \\ a^0 & = & 1 \end{array}$$

We see that each time the exponent is decreased by 1 the result is being divided by a. If this pattern continues we will get, $a^{-1} = \frac{1}{a}, a^{-2} = \frac{1}{a^2}$, etc. In general, we have the following definition.
Let $a \neq 0$ be any integer and n be a positive integer then

$$a^{-n} = \frac{1}{a^n}.$$

It can be shown that the theorems on whole-number exponents discussed in Section 10 can be extended to integer exponents. We summarize these properties in the following theorem.

Theorem 26.7
Let a, b be integers and m, n be positive integers. Then
(a) $a^m \cdot a^n = a^{m+n}$
(b) $\frac{a^m}{a^n} = a^{m-n}$
(c) $(a^m)^n = a^{mn}$
(d) $(a \cdot b)^m = a^m \cdot b^m$
(e) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, assuming $b \neq 0$
(f) $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$.

Example 26.3
Write each of the following in simplest form using positive exponents in the final answer.
(a) $16^2 \cdot 8^{-3}$ (b) $20^2 \div 2^4$ (c) $(10^{-1} + 5 \cdot 10^{-2} + 3 \cdot 10^{-3}) \cdot 10^3$

Solution.
(a) $16^2 \cdot 8^{-3} = (2^4)^2 \cdot (2^3)^{-3} = 2^8 \cdot 2^{-9} = 2^{8-9} = 2^{-1} = \frac{1}{2}$.
(b) $20^2 \div 2^4 = (5 \cdot 2^2)^2 \div 2^4 = (5^2 \cdot (2^2)^2) \div 2^4 = (25 \cdot 2^4) \div 2^4 = 5^2 = 25$.
(c)

$$\begin{array}{rcl} (10^{-1} + 5 \cdot 10^{-2} + 3 \cdot 10^{-3}) \cdot 10^3 & = & 10^{-1} \cdot 10^3 + 5 \cdot 10^{-2} \cdot 10^3 + 3 \cdot 10^{-3} \cdot 10^3 \\ & = & 10^{-1+3} + 5 \cdot 10^{-2+3} + 3 \cdot 10^{-3+3} \\ & = & 10^2 + 5 \cdot 10 + 3 \\ & = & 153 \blacksquare \end{array}$$

In application problems that involve very large or very small numbers, we use scientific notation to represent these numbers. A number is said to be in **scientific notation** when it is expressed in the form $a \times 10^n$ where $1 \leq a < 10$ is called the **mantissa** and n is an integer called the **characteristic**. For example, the diameter of Jupiter in standard notation is 143,800,000 meters. Using scientific notation this can be written in the form 1.438×10^8 meters. When converting from standard notation to scientific notation and vice versa recall the following rules of multiplying by powers of 10: When you multiply by 10^n move the decimal point n positions to the right. When you multiply by 10^{-n} move the decimal point n positions to the left.

Example 26.4
Convert as indicated.
(a) 38,500,000 to scientific notation
(b) 4.135×10^{11} to standard notation
(c) 7.2×10^{-14} to standard notation
(d) 0.0000961 to scientific notation.

Solution.
(a) $38,500,000 = 3.85 \times 10^7$
(b) $4.135 \times 10^{11} = 413,500,000,000$
(c) $7.2 \times 10^{-14} = 0.000000000000072$
(d) $0.0000961 = 9.61 \times 10^{-5}$ ■

Order of Operations on Integers
When addition, subtraction, multiplication, division, and exponentiation appear without parentheses, exponentiation is done first, then multiplication and division are done from left to right, and finally addition and subtraction from left to right. Arithmetic operations that appear inside parentheses must be done first.

Example 26.5
Evaluate each of the following.
(a) $(2 - 5) \cdot 4 + 3$
(b) $2 + 16 \div 4 \cdot 2 + 8$
(c) $2 - 3 \cdot 4 + 5 \cdot 2 - 1 + 5$.

Solution.
(a) $(2 - 5) \cdot 4 + 3 = (-3) \cdot 4 + 3 = -12 + 3 = -9$.

(b) $2 + 16 \div 4 \cdot 2 + 8 = 2 + 4 \cdot 2 + 8 = 2 + 8 + 8 = 18.$
(c) $2 - 3 \cdot 4 + 5 \cdot 2 - 1 + 5 = 2 - 12 + 10 - 1 + 5 = 4.$ ■

Practice Problems

Problem 26.20
Perform these divisions.
(a) $36 \div 9$ (b) $(-36) \div 9$ (c) $36 \div (-9)$ (d) $(-36) \div (-9)$ (e) $165 \div (-11)$
(f) $275 \div 11$

Problem 26.21
Write another multiplication equation and two division equations that are equivalent to the equation

$$(-11) \cdot (-25,753) = 283,283.$$

Problem 26.22
Write two multiplication equations and another division equation that are equivalent to the equation

$$(-1001) \div 11 = -91.$$

Problem 26.23
Use the definition of division to find each quotient, if possible. If a quotient is not defined, explain why.
(a) $(-40) \div (-8)$ (b) $(-143) \div 11$ (c) $0 \div (-5)$ (d) $(-5) \div 0$ (e) $0 \div 0$

Problem 26.24
Find all integers x (if possible) that make each of the following true.
(a) $x \div 3 = -12$ (b) $x \div (-3) = -2$ (c) $x \div (-x) = -1$ (d) $0 \div x = 0$
(e) $x \div 0 = 1$

Problem 26.25
Write two division equations that are equivalent to $3 \times (-2) = -6$.

Problem 26.26
Explain how to compute $-10 \div 2$ using signed counters.

Problem 26.27
Rewrite each of the following as an equivalent multiplication problem, and give the solution.
(a) $(-54) \div (-6)$ (b) $32 \div (-4)$

Problem 26.28
A store lost $480,000 last year.
(a) What was the average net change per month?
(b) Write an integer equation for this situation.

Problem 26.29
Compute the following, using the correct rules for order of operations.
(a) $-2^2 - 3$ (b) $-5 + (-4)^2 \times (-2)$

Problem 26.30
A stock change as follows for 5 days:-2,4,6,3,-1. What is the average daily change in price?

Problem 26.31
Compute: $-2 \div (-2) + (-2) - (-2)$

Problem 26.32
For what integers a and b does $a \div b = b \div a$?

Problem 26.33
Find each quotient, if possible.
(a) $[144 \div (-12)] \div (-3)$ (b) $144 \div [-12 \div (-3)]$

Problem 26.34
Compute the following writing the final answer in terms of positive exponents.
(a) $4^{-2} \cdot 4^6$
(b) $\frac{6^3}{6^{-7}}$
(c) $(3^{-4})^{-2}$

Problem 26.35
Express each of the following in scientific notation.
(a) 0.0004 (b) 0.0000016 (c) 0.000000000000071

Problem 26.36
Hair on the human body can grow as fast as 0.0000000043 meter per second.
(a) At this rate, how much would a strand of hair grow in one month of 30 days? Express your answer in scientific notation.
(b) About how long would it take for a strand of hair to grow to be 1 meter in length?

Problem 26.37
Compute each of these to three significant figures using scientific notation.
(a) $(2.47 \times 10^{-5}) \cdot (8.15 \times 10^{-9})$
(b) $(2.47 \times 10^{-5}) \div (8.15 \times 10^{-9})$

Problem 26.38
Convert each of the following to standard notation.
(a) 6.84×10^{-5} (b) 3.12×10^{7}

Problem 26.39
Write each of the following in scientific notation.
(a) 413,682,000 (b) 0.000000231 (c) 100,000,000

Problem 26.40
Evaluate each of the following.
(a) $-5^2 + 3(-2)^2$
(b) $-2 + 3 \cdot 5 - 1$
(c) $10 - 3 \cdot 7 - 4(-2) \div 2 + 3$

Problem 26.41
Evaluate each of the following, if possible.
(a) $(-10 \div (-2))(-2)$
(b) $(-10 \cdot 5) \div 5$
(c) $-8 \div (-8 + 8)$
(d) $(-6 + 6) \div (-2 + 2)$
(e) $|-24| \div 4 \cdot (3 - 15)$

Comparing and Ordering Integers

In this section we extend the notion of "less than" to the set of all integers. We describe two equivalent ways for viewing the meaning of less than: a number line approach and an addition (or algebraic) approach. In what follows, a and b denote any two integers.

Number-Line Approach
We say that a is **less than** b, and we write $a < b$, if the point representing a on the number-line is to the left of b. For example, Figure 26.3 shows that $-4 < 2$.

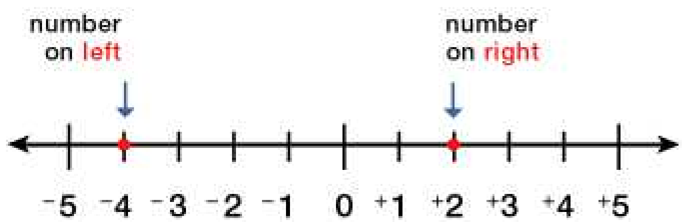


Figure 26.3

Example 26.6
Order the integers $-1, 0, 4, 3, -2, 2, -4, -3, 1$ from smallest to largest using the number line approach.

Solution.
The given numbers are ordered as shown in Figure 26.4■

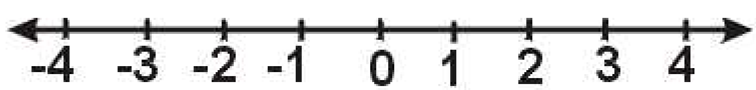


Figure 26.4

Addition Approach
Note that from Figure 26.3 we have $2 = (-4) + 6$ so we say that 2 is 6 more than -2. In general, if $a < b$ then we can find a unique integer c such that $b = a + c$.

Example 26.7
Use the addition approach to show that $-7 < -3$.

Solution.
Since $-3 = (-7) + 4$ then -3 is 4 more than -7, that is $-7 < -3$■

Notions similar to less than are included in the following table.

Inequality Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal
$\geq$	greater than or equal

The following rules are valid for any of the inequality listed in the above table.

Rules for Inequalities
• **Trichotomy Law:** For any integers a and b exactly one of the following is true:

$$a < b, a > b, a = b.$$

• **Transitivity:** For any integers $a, b,$ and c if $a < b$ and $b < c$ then $a < c$.
• **Addition Property:** For any integers $a, b,$ and c if $a < b$ then $a+c < b+c$.
• **Multiplication Property:** For any integers $a, b,$ and c if $a < b$ then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

The first three properties are extensions of similar statements in the whole number system. Note that the fourth property asserts that an inequality symbol must be reversed when multiplying by a negative integer. This is illustrated in Figure 26.5 when $c = -1$.

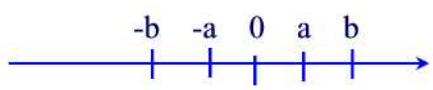


Figure 26.5

Practice Problems

Problem 26.42
Use the number-line approach to verify each of the following.
(a) $-4 < 1$ (b) $-4 < -2$ (c) $-1 > -5$

Problem 26.43
Order each of the following lists from smallest to largest.
(a) $\{-4, 4, -1, 1, 0\}$
(b) $\{23, -36, 45, -72, -108\}$

Problem 26.44
Replace the blank by the appropriate symbol.
(a) If $x > 2$ then $x + 4$____6
(b) If $x < -3$ then $x - 6$____-9

Problem 26.45
Determine whether each of the following statements is correct.
(a) $-3 < 5$ (b) $6 < 0$ (c) $3 \leq 3$ (d) $-6 > -5$ (e) $2 \times 4 - 6 \leq -3 \times 5 + 1$

Problem 26.46
What different looking inequality means the same as $a < b$?

Problem 26.47
Use symbols of inequalities to represent "at most" and "at least".

Problem 26.48
For each inequality, determine which of the numbers $-5, 0, 5$ satisfies the inequality.
(a) $x > -5$ (b) $5 < x$ (c) $-5 > x$

Problem 26.49
Write the appropriate inequality symbol in the blank so that the two inequalities are true.
(a) If $x \leq -3$ then $-2x$____6
(b) If $x + 3 > 9$ then x____6

Problem 26.50
How do you know when to reverse the direction of an inequality symbol?

Problem 26.51
Show that each of the following inequality is true by using the addition approach.
(a) $-4 < -2$ (b) $-5 < 3$ (c) $-17 > -23$

Problem 26.52
A student makes a connection between debts and negative numbers. So the number -2 represents a debt of \$2. Since a debt of \$10 is larger than a debt of \$5 then the student writes $-10 > -5$. How would you convince him/her that this inequality is false?

Problem 26.53
At an 8% sales tax rate, Susan paid more than $1500 sales tax when she purchased her new Camaro. Describe this situation using an inequality with p denoting the price of the car.

Problem 26.54
Show that if a and b are positive integers with $a < b$ then $a^2 < b^2$. Does this result hold for any integers?

Problem 26.55
Elka is planning a rectangular garden that is twice as long as it is wide. If she can afford to buy at most 180 feet of fencing, then what are the possible values for the width?

27 Rational Numbers

Integers such as -5 were important when solving the equation $x+5=0$. In a similar way, fractions are important for solving equations like $2x=1$. What about equations like $2x+1=0$? Equations of this type require numbers like $-\frac{1}{2}$. In general, numbers of the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$ are solutions to the equation $bx=a$. The set of all such numbers is the set of **rational numbers**, denoted by $\mathbb{Q}$:

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}.$$

That is, the set of rational numbers consists of all fractions with their opposites. In the notation $\frac{a}{b}$ we call a the **numerator** and b the **denominator**. Note that every fraction is a rational number. Also, every integer is a rational number for if a is an integer then we can write $a = \frac{a}{1}$. Thus, $\mathbb{Z} \subset \mathbb{Q}$.

Example 27.1
Draw a Venn diagram to show the relationship between counting numbers, whole numbers, integers, and rational numbers.

Solution.
The relationship is shown in Figure 27.1■

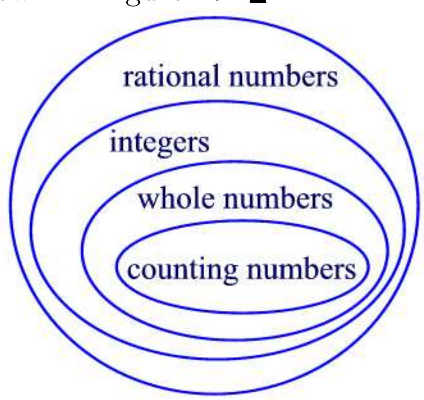


Figure 27.1

All properties that hold for fractions apply as well for rational numbers.

Equality of Rational Numbers: Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers. Then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ (Cross-multiplication).

Example 27.2
Determine if the following pairs are equal.
(a) $\frac{3}{-12}$ and $\frac{-36}{144}$.
(b) $\frac{-21}{86}$ and $\frac{-51}{215}$.

Solution.
(a) Since $3(144) = (-12)(-36)$ then $\frac{3}{-12} = \frac{-36}{144}$.
(b) Since $(-21)(215) \neq (86)(-51)$ then $\frac{-21}{86} \neq \frac{-51}{215}$ ■

The Fundamental Law of Fractions: Let $\frac{a}{b}$ be any rational number and n be a nonzero integer then

$$\frac{a}{b} = \frac{an}{bn} = \frac{a \div n}{b \div n}.$$

As an important application of the Fundamental Law of Fractions we have

$$\frac{a}{-b} = \frac{(-1)a}{(-1)(-b)} = \frac{-a}{b}.$$

We also use the notation $-\frac{a}{b}$ for either $\frac{a}{-b}$ or $\frac{-a}{b}$.

Example 27.3
Write three rational numbers equal to $-\frac{2}{5}$.

Solution.
By the Fundamental Law of Fractions we have

$$-\frac{2}{5} = \frac{4}{-10} = -\frac{6}{15} = \frac{-8}{20} \blacksquare$$

Rational Numbers in Simplest Form: A rational number $\frac{a}{b}$ is in **simplest form** if a and b have no common factors greater than 1.
The methods of reducing fractions into simplest form apply as well with rational numbers.

Example 27.4
Find the simplest form of the rational number $\frac{294}{-84}$.

Solution.
Using the prime factorizations of 294 and 84 we find

$$\frac{294}{-84} = \frac{2 \cdot 3 \cdot 7^2}{(-2) \cdot 2 \cdot 3 \cdot 7} = \frac{7}{-2} = \frac{-2}{7} \blacksquare$$

Practice Problems

Problem 27.1
Show that each of the following numbers is a rational number.
(a) -3 (b) $4\frac{1}{2}$ (c) -5.6 (d) 25%

Problem 27.2
Which of the following are equal to -3?

$$\frac{-3}{1}, \frac{3}{-1}, \frac{3}{1}, -\frac{3}{1}, \frac{-3}{-1}, -\frac{-3}{1}, -\frac{-3}{-1}.$$

Problem 27.3
Determine which of the following pairs of rational numbers are equal.
(a) $\frac{-3}{5}$ and $\frac{63}{-105}$
(b) $\frac{-18}{-24}$ and $\frac{45}{60}$.

Problem 27.4
Rewrite each of the following rational numbers in simplest form.
(a) $\frac{5}{-7}$ (b) $\frac{21}{-35}$ (c) $\frac{-8}{-20}$ (d) $\frac{-144}{180}$

Problem 27.5
How many different rational numbers are given in the following list?

$$\frac{2}{5}, 3, \frac{-4}{-10}, \frac{39}{13}, \frac{7}{4}$$

Problem 27.6
Find the value of x to make the statement a true one.

(a) $\frac{-7}{25} = \frac{x}{500}$ (b) $\frac{18}{3} = \frac{-5}{x}$

Problem 27.7
Find the prime factorizations of the numerator and the denominator and use them to express the fraction $\frac{247}{-77}$ in simplest form.

Problem 27.8
(a) If $\frac{a}{b} = \frac{a}{c}$, what must be true?

(b) If $\frac{a}{c} = \frac{b}{c}$, what must be true?

<u>Addition of Rational Numbers</u>
The definition of adding fractions extends to rational numbers.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Example 27.5
Find each of the following sums.
(a) $\frac{2}{-3}+\frac{1}{5}$ (b) $\frac{-2}{5}+\frac{4}{-7}$ (c) $\frac{3}{7}+\frac{-5}{7}$

Solution.
(a) $\frac{2}{-3}+\frac{1}{5}=\frac{2\cdot(-5)}{(-3)\cdot(-5)}+\frac{1\cdot 3}{5\cdot 3}=\frac{-10}{15}+\frac{3}{15}=\frac{(-10)+3}{15}=\frac{-7}{15}$
(b) $\frac{-2}{5}+\frac{4}{-7}=\frac{(-2)\cdot 7}{5\cdot 7}+\frac{4\cdot(-5)}{(-5)\cdot(-7)}=\frac{-14}{35}+\frac{-20}{35}=\frac{(-14)+(-20)}{35}=\frac{-34}{35}.$
(c) $\frac{3}{7}+\frac{-5}{7}=\frac{3+(-5)}{7}=\frac{-2}{7}$ ■

Rational numbers have the following properties for addition.

Theorem 27.1
Closure: $\frac{a}{b}+\frac{c}{d}$ is a unique rational number.
Commutative: $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$
Associative: $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$
Identity Element: $\frac{a}{b}+0=\frac{a}{b}$
Additive inverse: $\frac{a}{b}+\left(-\frac{a}{b}\right)=0$

Example 27.6
Find the additive inverse for each of the following:
(a) $\frac{3}{5}$ (b) $\frac{-5}{11}$ (c) $\frac{2}{-3}$ (d) $-\frac{2}{5}$

Solution.
(a) $-\frac{3}{5}=\frac{-3}{5}=\frac{3}{-5}$

(b) $\frac{5}{11}$

(c) $\frac{2}{3}$

(d) $\frac{2}{5}$■

Subtraction of Rational Numbers

Subtraction of rational numbers like subtraction of fractions can be defined in terms of addition as follows.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right).$$

Using the above result we obtain the following:

$$\begin{array}{rcl} \frac{a}{b} - \frac{c}{d} & = & \frac{a}{b} + \left(-\frac{c}{d}\right) \\ & = & \frac{a}{b} + \frac{-c}{d} \\ & = & \frac{ad+b(-c)}{bd} \\ & = & \frac{ad-bc}{bd} \end{array}$$

Example 27.7

Compute $\frac{103}{24} - \frac{-35}{16}$.

Solution.

$$\begin{array}{rcl} \frac{103}{24} - \frac{-35}{16} & = & \frac{206}{48} - \frac{-105}{48} \\ & = & \frac{206-(-105)}{48} \\ & = & \frac{206+105}{48} = \frac{311}{48} \blacksquare \end{array}$$

Practice Problems

Problem 27.9

Use number line model to illustrate each of the following sums.

(a) $\frac{3}{4} + \frac{-2}{4}$ (b) $\frac{-3}{4} + \frac{2}{4}$ (c) $\frac{-3}{4} + \frac{-1}{4}$

Problem 27.10

Perform the following additions. Express your answer in simplest form.

(a) $\frac{6}{8} - \frac{-25}{100}$ (b) $\frac{-57}{100} + \frac{13}{10}$

Problem 27.11

Perform the following subtractions. Express your answer in simplest form.

(a) $\frac{137}{214} - \frac{-1}{3}$ (b) $\frac{-23}{100} - \frac{198}{1000}$

Problem 27.12

Compute the following differences.

(a) $\frac{2}{3} - \frac{-9}{8}$ (b) $\left(-2\frac{1}{4}\right) - 4\frac{2}{3}$

Multiplication of Rational Numbers

The multiplication of fractions is extended to rational numbers. That is, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Multiplication of rational numbers has properties analogous to the properties of multiplication of fractions. These properties are summarized in the following theorem.

Theorem 27.2

Let $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then we have the following:

Closure: The product of two rational numbers is a unique rational number.

Commutativity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.

Associativity: $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$.

Identity: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$.

Inverse: $\frac{a}{b} \cdot \frac{b}{a} = 1$. We call $\frac{b}{a}$ the **reciprocal** of $\frac{a}{b}$ or the **multiplicative inverse** of $\frac{a}{b}$.

Distributivity: $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$.

Example 27.8

Perform each of the following multiplications. Express your answer in simplest form.

(a) $\frac{-5}{6} \cdot \frac{7}{3}$ (b) $\frac{-3}{10} \cdot \frac{-25}{27}$

Solution.

(a) We have

$$\frac{-5}{6} \cdot \frac{7}{3} = \frac{(-5) \cdot 7}{6 \cdot 3} = \frac{-35}{18}.$$

(b)

$$\frac{-3}{10} \cdot \frac{-25}{27} = \frac{-1}{2} \cdot \frac{-5}{9} = \frac{(-1)(-5)}{2(9)} = \frac{5}{18} \blacksquare$$

Example 27.9

Use the properties of multiplication of rational numbers to compute the following.

(a) $-\frac{3}{5}\cdot\left(\frac{11}{17}\cdot\frac{5}{3}\right)$
(b) $\frac{2}{3}\cdot\left(\frac{3}{2}+\frac{5}{7}\right)$
(c) $\frac{5}{9}\cdot\frac{2}{7}+\frac{2}{7}\cdot\frac{4}{9}$

Solution.
(a) $-\frac{3}{5}\cdot\left(\frac{11}{17}\cdot\frac{5}{3}\right)=\frac{-3}{5}\cdot\frac{11\cdot 5}{17\cdot 3}=\frac{-3}{5}\cdot\frac{55}{51}=\frac{-1}{1}\cdot\frac{11}{17}=\frac{-11}{17}$

(b) $\frac{2}{3}\cdot\left(\frac{3}{2}+\frac{5}{7}\right)=\frac{2}{3}\cdot\frac{31}{14}=\frac{1}{3}\cdot\frac{31}{7}=\frac{31}{21}$

(c) $\frac{5}{9}\cdot\frac{2}{7}+\frac{2}{7}\cdot\frac{4}{9}=\frac{5}{9}\cdot\frac{2}{7}+\frac{4}{9}\cdot\frac{2}{7}=\left(\frac{5}{9}+\frac{4}{9}\right)\cdot\frac{2}{7}=\frac{2}{7}$ ■

Division of Rational Numbers

We define the division of rational numbers as an extension of the division of fractions. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with $\frac{c}{d}\neq 0$. Then

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}.$$

Using words, to find $\frac{a}{b}\div\frac{c}{d}$ multiply $\frac{a}{b}$ by the reciprocal of $\frac{c}{d}$.
By the above definition one gets the following two results.

$$\frac{a}{b}\div\frac{c}{b}=\frac{a}{c}$$

and

$$\frac{a}{b}\div\frac{c}{d}=\frac{a\div c}{b\div d}.$$

Remark 27.1

After inverting, it is often simplest to "cancel" before doing the multiplication. Cancelling is dividing one factor of the numerator and one factor of the denominator by the same number. For example: $\frac{2}{9}\div\frac{3}{12}=\frac{2}{9}\times\frac{12}{3}=\frac{2\times 12}{9\times 3}=\frac{2\times 4}{3\times 3}=\frac{8}{9}$.

Remark 27.2

Exponents and their properties are extended to rational numbers in a natural way. For example, if a is any rational number and n is a positive integer then

$$a^n=\underbrace{a\cdot a\cdots a}_{n\ factors}\quad and\quad a^{-n}=\frac{1}{a^n}$$

Example 27.10
Compute the following and express the answers in simplest form.

(a) $\frac{-7}{4} \div \frac{2}{3}$ (b) $\frac{13}{17} \cdot \frac{-4}{9}$ (c) $\frac{-18}{23} \cdot \frac{-6}{23}$

Solution.
(a) $\frac{-7}{4} \div \frac{2}{3} = \frac{-7}{4} \cdot \frac{3}{2} = \frac{(-7)(3)}{(4)(2)} = \frac{-21}{8}$.

(b) $\frac{13}{17} \cdot \frac{-4}{9} = \frac{13}{17} \cdot \frac{9}{-4} = \frac{13 \cdot 9}{17 \cdot (-4)} = \frac{-117}{68}$.

(c) $\frac{-18}{23} \cdot \frac{-6}{23} = \frac{-18}{23} \cdot \frac{23}{-6} = \frac{3}{1} \cdot \frac{1}{1} = 3$ ■

Practice Problems

Problem 27.13
Multiply the following rational numbers. Write your answers in simplest form.

(a) $\frac{3}{5} \cdot \frac{-10}{21}$ (b) $\frac{-6}{11} \cdot \frac{-33}{18}$ (c) $\frac{5}{12} \cdot \frac{48}{-15} \cdot \frac{-9}{8}$

Problem 27.14
Find the following quotients. Write your answers in simplest form.

(a) $\frac{-8}{9} \div \frac{2}{9}$ (b) $\frac{12}{15} \div \frac{-4}{3}$ (c) $\frac{-13}{24} \div \frac{-39}{-48}$

Problem 27.15
State the property that justifies each statement.

(a) $\left(\frac{5}{7} \cdot \frac{7}{8}\right) \cdot \frac{-8}{3} = \frac{5}{7} \cdot \left(\frac{7}{8} \cdot \frac{-8}{3}\right)$
(b) $\frac{1}{4}\left(\frac{8}{3} + \frac{-5}{4}\right) = \frac{1}{4} \cdot \frac{8}{3} + \frac{1}{4} \cdot \frac{-5}{4}$

Problem 27.16
Compute the following and write your answers in simplest form.

(a) $\frac{-40}{27} \div \frac{-10}{9}$ (b) $\frac{21}{25} \div \frac{-3}{5}$ (c) $\frac{-10}{9} \div \frac{-9}{8}$

Problem 27.17
Find the reciprocals of the following rational numbers.

(a) $\frac{4}{-9}$ (b) 0 (c) $\frac{-3}{2}$ (d) $\frac{-4}{-9}$

Problem 27.18
Compute: $\left(\frac{-4}{7} \cdot \frac{2}{-5}\right) \div \frac{2}{-7}$.

Problem 27.19
If $\frac{a}{b} \cdot \frac{-4}{7} = \frac{2}{3}$ what is $\frac{a}{b}$?

Problem 27.20
Compute $-4\frac{1}{2} \times -5\frac{2}{3}$

Problem 27.21
Compute $-17\frac{8}{9} \div 5\frac{10}{11}$

Problem 27.22
Compute each of the following:

(a) $-\left(\frac{3}{4}\right)^2$ (b) $\left(-\frac{3}{4}\right)^2$ (c) $\left(\frac{3}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^7$

Comparing and Ordering Rational Numbers

In this section we extend the notion of "less than" to the set of all rationals. We describe two equivalent ways for viewing the meaning of less than: a number line approach and an addition (or algebraic) approach. In what follows, $\frac{a}{b}$ and $\frac{c}{d}$ denote any two rationals.

Number-Line Approach

We say that $\frac{a}{b}$ is **less than** $\frac{c}{d}$, and we write $\frac{a}{b} < \frac{c}{d}$, if the point representing $\frac{a}{b}$ on the number-line is to the left of $\frac{c}{d}$. For example, Figure 27.2 shows that $\frac{1}{2} < \frac{2}{3}$.

Figure 27.2

Example 27.11
Use the number line approach to order the pair of numbers $\frac{3}{7}$ and $\frac{5}{2}$.

Solution.
When the two numbers have unlike denominators then we find the least common denominator and then we order the numbers. Thus, $\frac{3}{7} = \frac{6}{14}$ and $\frac{5}{2} = \frac{35}{14}$.

Hence, on a number line, $\frac{3}{7}$ is to the left of $\frac{5}{2}$.■

Addition Approach
As in the case of ordering integers, we say that $\frac{a}{b} < \frac{c}{d}$ if there is a unique fraction $\frac{e}{f}$ such that $\frac{a}{b} + \frac{e}{f} = \frac{c}{d}$.

Example 27.12
Use the addition approach to show that $\frac{-3}{7} < \frac{5}{2}$.

Solution.
Since $\frac{5}{2} = \frac{-3}{7} + \frac{41}{14}$ then $\frac{-3}{7} < \frac{5}{2}$■

Notions similar to less than are included in the following table.

Inequality Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal
$\geq$	greater than or equal

The following rules are valid for any of the inequality listed in the above table.

Rules for Inequalities
- **Trichotomy Law:** For any rationals $\frac{a}{b}$ and $\frac{c}{d}$ exactly one of the following is true:
$$\frac{a}{b} < \frac{c}{d}, \frac{a}{b} > \frac{c}{d}, \frac{a}{b} = \frac{c}{d}.$$
- **Transitivity:** For any rationals $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$ then $\frac{a}{b} < \frac{e}{f}$.
- **Addition Property:** For any rationals $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} + \frac{e}{f} < \frac{c}{d} + \frac{e}{f}$.
- **Multiplication Property:** For any rationals $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}$ if $\frac{e}{f} > 0$ and $\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}$ if $\frac{e}{f} < 0$
- **Density Property:** For any rationals $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{a}{b} < \frac{c}{d}$ then
$$\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}.$$

Practice Problems

Problem 27.23
True or false: (a) $\frac{-2}{3} < \frac{-3}{7}$ (b) $\frac{15}{-9} > \frac{-13}{4}$.

Problem 27.24
Show that $\frac{-3}{4} < \frac{-1}{4}$ using the addition approach.

Problem 27.25
Show that $\frac{-3}{4} < \frac{-1}{4}$ by using a number line.

Problem 27.26
Put the appropriate symbol, $<, =, >$ between each pair of numbers to make a true statement.

(a) $-\frac{5}{6}$_____$-\frac{11}{12}$

(b) $-\frac{1}{3}$_____$\frac{5}{4}$

(c) $\frac{-12}{15}$_____$\frac{36}{-45}$

(d) $-\frac{3}{12}$_____$\frac{-4}{20}$

Problem 27.27
Find three rational numbers between $\frac{1}{4}$ and $\frac{2}{5}$.

Problem 27.28
The properties of rational numbers are used to solve inequalities. For example,

$$\begin{array}{ccc} x + \frac{3}{5} & < & \frac{-7}{10} \\ x + \frac{3}{5} + \left(-\frac{3}{5}\right) & < & \frac{-7}{10} + \left(-\frac{3}{5}\right) \\ x & < & -\frac{13}{10} \end{array}$$

Solve the inequality

$$-\frac{2}{5}x + \frac{1}{5} > -1.$$

Problem 27.29
Solve each of the following inequalities.

(a) $x - \frac{6}{5} < \frac{-12}{7}$

(b) $\frac{2}{5}x < -\frac{7}{8}$

(c) $\frac{-3}{7}x > \frac{8}{5}$

Problem 27.30
Verify the following inequalities.

(a) $\frac{-4}{5} < \frac{-3}{4}$ (b) $\frac{1}{10} < \frac{1}{4}$ (c) $\frac{19}{-60} > \frac{-1}{3}$

Problem 27.31
Use the number-line approach to arrange the following rational numbers in increasing order:

(a) $\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}$
(b) $\frac{-7}{12}, \frac{-2}{3}, \frac{3}{-4}$

Problem 27.32
Find a rational number between $\frac{5}{12}$ and $\frac{3}{8}$.

Problem 27.33
Complete the following, and name the property that is used as a justification.

(a) If $\frac{-2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{5}$ then $\frac{-2}{3}$____$\frac{7}{5}$.

(b) If $\frac{-3}{5} < \frac{-6}{11}$ then $\left(\frac{-3}{5}\right) \cdot \left(\frac{2}{3}\right)$ ____ $\left(\frac{-6}{11}\right) \cdot \left(\frac{2}{3}\right)$

(c) If $\frac{-4}{7} < \frac{7}{4}$ then $\frac{-4}{7} + \frac{5}{8} < \frac{7}{4} +$ ____

(d) If $\frac{-3}{4} < \frac{11}{3}$ then $\left(\frac{-3}{4}\right) \cdot \left(\frac{-5}{7}\right)$ ____ $\left(\frac{11}{3}\right) \cdot \left(\frac{-5}{7}\right)$

(e) There is a rational number ____ any two unequal rational numbers.

28 Real Numbers

In the previous section we introduced the set of rational numbers. We have seen that integers and fractions are rational numbers. Now, what about decimal numbers? To answer this question we first mention that a decimal number can be terminating, nonterminating and repeating, nonterminating and nonrepeating.

If the number is terminating then we can use properties of powers of 10 to write the number as a rational number. For example, $-0.123 = -\frac{123}{1000}$ and $15.34 = \frac{1534}{100}$. Thus, every terminating decimal number is a rational number.

Now, suppose that the decimal number is nonterminating and repeating. For the sake of argument let's take the number $12.341341\cdots$ where 341 repeats indefinitely. Then $12.341341\cdots = 12 + 0.341341\cdots$. Let $x = 0.341341\cdots$. Then $1000x = 341.341341\cdots = 341 + x$. Thus, $999x = 341$ and therefore $x = \frac{341}{999}$. It follows that

$$12.341341\cdots = 12 + \frac{341}{999} = \frac{12329}{999}$$

Hence, every nonterminating repeating decimal is a rational number.

Next, what about nonterminating and non repeating decimals. Such numbers are not rationals and the collection of all such numbers is called the set of **irrational numbers**. As an example of irrational numbers, let's consider finding the hypotenuse c of a right triangle where each leg has length 1. Then by the Pythagorean formula we have

$$c^2 = 1^2 + 1^2 = 2.$$

We will show that c is irrational. Suppose the contrary. That is, suppose that c is rational so that it can be written as

$$\frac{a}{b} = c$$

where a and $b \neq 0$ are integers. Squaring both sides we find $\frac{a^2}{b^2} = c^2 = 2$ or $a^2 = 2 \cdot b^2$. If a has an even number of prime factors then a^2 has an even number of prime factors. If a has an odd number of prime factors then a^2 has an even number of prime factors. So, a^2 and b^2 have both even number of prime factors. But $2 \cdot b^2$ has an odd number of prime factors. So we have that a^2 has an even number and an odd number of prime factors. This cannot happen by the Fundamental Theorem of Arithmetic which says that every positive integer has a unique number of prime factors. In conclusion, c cannot be written in the form $\frac{a}{b}$ so it is not rational. Hence, its decimal form is nonterminating and nonrepeating.

Remark 28.1
It follows from the above discussion that the equation $c^2 = 2$ has no answers in the set of rationals. This is true for the equation $c^2 = p$ where p is a prime number. Since there are infinite numbers of primes then the set of irrational numbers is infinite.

We define the set of **real numbers** to be the union of the set of rationals and the set of irrationals. We denote it by the letter $\mathbb{R}$. The relationships among the sets of all numbers discussed in this book is summarized in Figure 28.1

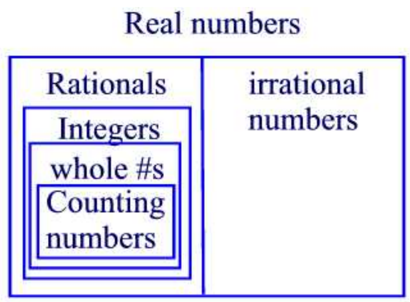


Figure 28.1

Now, in the set of real numbers, the equation $c^2 = 2$ has a solution denoted by $\sqrt{2}$. Thus, $(\sqrt{2})^2 = 2$. In general, we say that a positive number b is the **square root** of a positive number a if $b^2 = a$. We write

$$\sqrt{a} = b.$$

Representing Irrational Numbers on a Number Line
Figure 28.2 illustrates how we can accurately plot the length $\sqrt{2}$ on the number line.

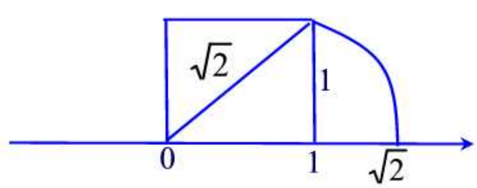


Figure 28.2

Practice Problems

Problem 28.1
Given a decimal number, how can you tell whether the number is rational or irrational?

Problem 28.2
Write each of the following repeating decimal numbers as a fraction.
(a) $0.\overline{8}$ (b) $0.\overline{37}$ (c) $0.\overline{02714}$

Problem 28.3
Which of the following describe 2.6?
(a) A whole number
(b) An integer
(c) A rational number
(d) An irrational number
(e) A real number

Problem 28.4
Classify the following numbers as rational or irrational.
(a) $\sqrt{11}$ (b) $\frac{3}{7}$ (c) π (d) $\sqrt{16}$

Problem 28.5
Classify the following numbers as rational or irrational.
(a) $0.34938661\cdots$ (b) $0.\overline{26}$ (c) $0.565665666\cdots$

Problem 28.6
Match each word in column A with a word in column B.

A	B
Terminating	Rational
Repeating	Irrational
Infinite nonrepeating	

Problem 28.7
(a) How many whole numbers are there between -3 and 3 (not including 3 and -3)?
(b) How many integers are there between -3 and 3?
(c) How many real numbers are there between -3 and 3?

Problem 28.8
Prove that $\sqrt{3}$ is irrational.

Problem 28.9
Show that $1 + \sqrt{3}$ is irrational.

Problem 28.10
Find an irrational number between $0.\overline{37}$ and $0.\overline{38}$

Problem 28.11
Write an irrational number whose digits are twos and threes.

Problem 28.12
Classify each of the following statement as true or false. If false, give a counter example.

(a) The sum of any rational number and any irrational number is a rational number.
(b) The sum of any two irrational numbers is an irrational number.
(c) The product of any two irrational numbers is an irrational number.
(d) The difference of any two irrational numbers is an irrational number.

Arithmetic of Real Numbers
Addition, subtraction, multiplication, division, and "less than" are defined on the set of real numbers in such a way that all the properties on the rationals still hold. These properties are summarized next.

Closure: For any real numbers a and b, $a + b$ and ab are unique real numbers.
Commutative: For any real numbers a and b, $a + b = b + a$ and $ab = ba$.
Associative: For any real numbers, a, b, and c we have $a+(b+c) = (a+b)+c$ and $a(bc) = (ab)c$.
Identity element: For any real numbers a we have $a + 0 = a$ and $a \cdot 1 = a$.
Inverse Elements: For any real numbers a and $b \neq 0$ we have $a + (-a) = 0$ and $b \cdot \frac{1}{b} = 1$.
Distributive: For any real numbers a, b, and c we have $a(b + c) = ab + ac$.
Transitivity: If $a < b$ and $b < c$ then $a < c$.
Addition Property: If $a < b$ then $a + c < b + c$.
Multiplication Property: If $a < b$ then $ac < bc$ if $c > 0$ and $ac > bc$ if

$c < 0$.
Density: If $a < b$ then $a < \frac{a+b}{2} < b$.

Practice Problems

Problem 28.13
Construct the lengths $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \cdots$ as follows.
(a) First construct a right triangle with both legs of length 1. What is the length of the hypotenuse?
(b) This hypotenuse is the leg of the next right triangle. The other leg has length 1. What is the length of the hypotenuse of this triangle?
(c) Continue drawing right triangles, using the hypotenuse of the proceeding triangle as a leg of the next triangle until you have constructed one with length $\sqrt{6}$.

Problem 28.14
Arrange the following real numbers in increasing order.
$0.56,\ 0.\overline{56},\ 0.5\overline{66},\ 0.56565556\cdots,\ 0.\overline{566}$

Problem 28.15
Which property of real numbers justify the following statement

$$2\sqrt{3} + 5\sqrt{3} = (2+5)\sqrt{3} = 7\sqrt{3}$$

Problem 28.16
Find $x : x + 2\sqrt{2} = 5\sqrt{2}$.

Problem 28.17
Solve the following equation: $2x - 3\sqrt{6} = 3x + \sqrt{6}$

Problem 28.18
Solve the inequality: $\frac{3}{2}x - 2 < \frac{5}{6}x + \frac{1}{3}$

Problem 28.19
Determine for what real number x, if any, each of the following is true:
(a) $\sqrt{x} = 8$ (b) $\sqrt{x} = -8$ (c)$\sqrt{-x} = 8$ (d) $\sqrt{-x} = -8$

Radical and Rational Exponents
By rational exponents we mean exponents of the form

$$a^{\frac{m}{n}}$$

where a is any real number and m and n are integers. First we consider the simple case $a^{\frac{1}{n}}$ (where n is a positive integer) which is defined as follows

$$a^{\frac{1}{n}} = b \text{ is equivalent to } b^n = a.$$

We call b the **nth root** of a and we write $b = \sqrt[n]{a}$. We call a the **radicand** and n the **index**. The symbol $\sqrt[n]{\ }$ is called a **radical**. It follows that

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Note that the above definition requires $a \geq 0$ for n even since b^n is always greater than or equal to 0.

Example 28.1
Compute each of the following.
(a) $25^{\frac{1}{2}}$ (b) $(-8)^{\frac{1}{3}}$ (c) $(-16)^{\frac{1}{4}}$ (d) $-16^{\frac{1}{4}}$

Solution.
(a) Since $5^2 = 25$ then $25^{\frac{1}{2}} = 5$.
(b) Since $(-2)^3 = -8$ then $(-8)^{\frac{1}{3}} = -2$.
(c) Since the radicand is negative and the index is even then $(-16)^{\frac{1}{4}}$ is undefined.
(d) Since $2^4 = 16$ then $-16^{\frac{1}{4}} = -2$ ■

As the last two parts of the previous example have once again shown, we really need to be careful with parentheses. In this case parenthesis makes the difference between being able to get an answer or not.

For a negative exponent we define

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}}.$$

Now, for a general rational exponent we define

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

That is,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

Now, if $b = (a^{\frac{1}{n}})^m$ then $b^n = ((a^{\frac{1}{n}})^m)^n = (a^{\frac{1}{n}})^{mn} = ((a^{\frac{1}{n}})^n)^m = (a^{\frac{n}{n}})^m = a^m$. This shows that $b = (a^m)^{\frac{1}{n}}$. Hence, we have established that

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

or

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

Example 28.2
Rewrite each expression using rational exponents.

(a) $\sqrt[4]{5xy}$ (b) $\sqrt[3]{4a^2b^5}$

Solution.
(a) $\sqrt[4]{5xy} = (5xy)^{\frac{1}{4}}$
(b) $\sqrt[3]{4a^2b^5} = (4a^2b^5)^{\frac{1}{3}}$ ■

Example 28.3
Express the following values without exponents.

(a) $9^{\frac{3}{2}}$ (b) $125^{-\frac{4}{3}}$.

Solution.
(a) $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$.
(b) $125^{-\frac{4}{3}} = (125^{\frac{1}{3}})^{-4} = 5^{-4} = \frac{1}{625}$.■

The properties of integer exponents also hold for rational exponents. These properties are equivalent to the corresponding properties of radicals if the expressions involving radicals are meaningful.

Theorem 28.1
Let a and b real numbers and n a nonzero integer. Then

$$\begin{array}{cc} (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}} & \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \\ \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} & \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \end{array}$$

Practice Problems

Problem 28.20
Express the following values without exponents.
(a)$25^{\frac{1}{2}}$ (b) $32^{\frac{1}{5}}$ (c) $9^{\frac{5}{2}}$ (d) $(-27)^{\frac{4}{3}}$

Problem 28.21
Write the following radicals in simplest form if they are real numbers.
(a) $\sqrt[3]{-27}$ (b) $\sqrt[4]{-16}$ (c) $\sqrt[5]{32}$

Problem 28.22
A student uses the formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$ to show that -1 = 1 as follows:

$$-1 = (\sqrt{-1})^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

What's wrong with this argument?

Problem 28.23
Give an example where the following statement is true: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Problem 28.24
Give an example where the following statement is false: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Problem 28.25
Find an example where the following statement is false: $\sqrt{a^2+b^2} = a + b$.

Problem 28.26
Express the following values without using exponents:
(a) $(3^{10})^{\frac{3}{5}}$ (b) $81^{-\frac{5}{4}}$

Problem 28.27
If possible, find the square root of each of the following without using a calculator.
(a) 225 (b) 169 (c) -81 (d) 625

Problem 28.28
Write each of the following in the form $a\sqrt{b}$ where a and b are integers and b has the least value possible.
(a) $\sqrt{180}$ (b) $\sqrt{363}$ (c) $\sqrt{252}$

29 Functions and their Graphs

The concept of a function was introduced and studied in Section 7 of these notes. In this section we explore the graphs of functions. Of particular interest, we consider the graphs of linear functions, quadratic functions, cubic functions, square root functions, and exponential functions. These graphs are represented in a coordinate system known as the **Cartesian coordinate system** which we explore next.

The Cartesian Plane
The Cartesian coordinate system was developed by the mathematician René Descartes in 1637. The **Cartesian coordinate system**, also known as the rectangular coordinate system or the xy-plane, consists of two number scales, called the **x-axis** and the **y-axis**, that are perpendicular to each other at point O called the **origin.** Any point in the system is associated with an **ordered pair** of numbers (x, y) called the **coordinates** of the point. The number x is called the **abscissa** or the **x-coordinate** and the number y is called the **ordinate** or the **y-coordinate.** The abscissa measures the distance from the point to the y-axis whereas the ordinate measures the distance of the point to the x-axis. Positive values of the x-coordinate are measured to the right, negative values to the left. Positive values of the y-coordinate are measured up, negative values down. The origin is denoted as $(0, 0)$.
The axes divide the coordinate system into four regions called **quadrants** and are numbered counterclockwise as shown in Figure 29.1
To **plot a point** $P(a, b)$ means to draw a dot at its location in the xy-plane.

Example 29.1
Plot the point P with coordinates $(5, 2)$.

Solution.
Figure 29.1 shows the location of the point $P(5, 2)$ in the xy-plane. ■

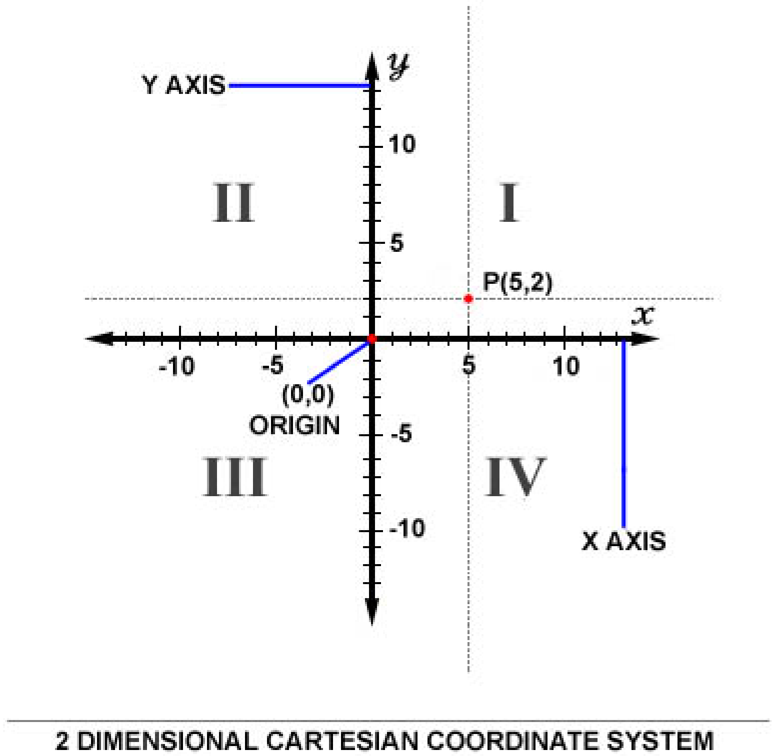


Figure 29.1

Example 29.2
Complete the following table of signs of the coordinates of a point $P(x, y)$.

	x	y
Quadrant I		
Quadrant II		
Quadrant III		
Quadrant IV		
Positive x-axis		
Negative x-axis		
Positive y-axis		
Negative y-axis		

Solution.

	x	y
Quadrant I	+	+
Quadrant II	-	+
Quadrant III	-	-
Quadrant IV	+	-
Positive x-axis	+	0
Negative x-axis	-	0
Positive y-axis	0	+
Negative y-axis	0	-

When Does a Graph Represent a Function

To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following simple visual test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 29.3

Which of the graphs (a), (b), (c) in Figure 29.2 represent y as a function of x?

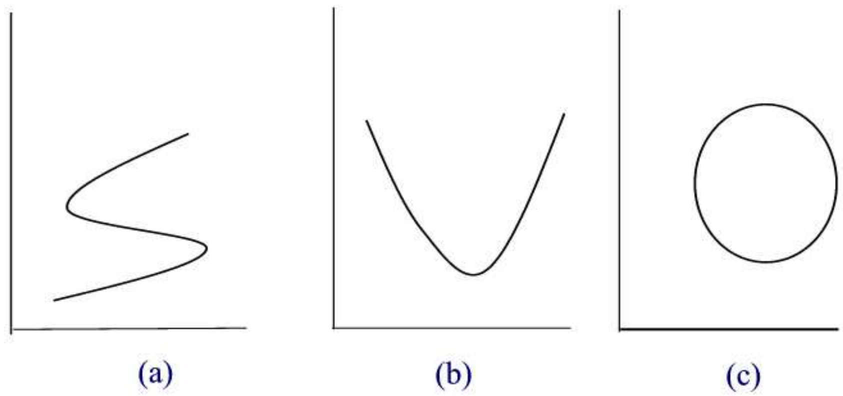


Figure 29.2

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

The **domain** of a function is the collection of all possible x-coordinates that can be used in the formula of the function. For example, $x = 1$ is in the domain of $f(x) = x + 1$ since $f(1) = 1 + 1 = 2$ whereas $x = 1$ is not in the domain of $f(x) = \frac{1}{x-1}$ since $f(1) = \frac{1}{0}$ which is undefined.
The collection of all values of y-coordinates that correspond to the x-coordinates is called the **range** of the function. For example, the range of $f(x) = \sqrt{x-1}$ is the interval $[0, \infty)$ whereas that of the function $f(x) = \frac{1}{x-1}$ is the set $\mathbb{R} - \{0\}$.

Practice Problems

Problem 29.1
Plot the points whose coordinates are given on a Cartesian coordinate system.

(a) $(2, 4), (0, -3), (-2, 1), (-5, -3)$.
(b) $(-3, -5), (-4, 3), (0, 2), (-2, 0)$.

Problem 29.2
Plot the following points using graph papers.
(a) (3,2) (b) (5,0) (c) (0,-3) (d) (-3,4) (e) (-2,-3) (f) (2,-3)

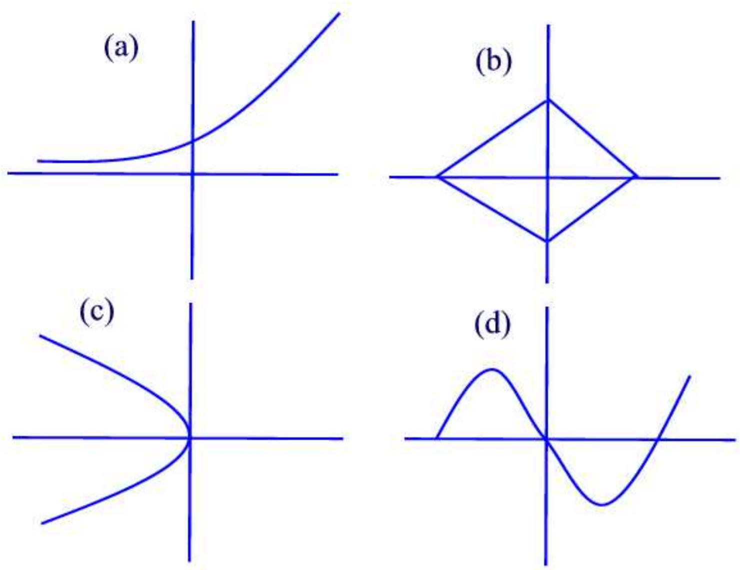
Problem 29.3
Complete the following table.

(x,y)	$x > 0, y > 0$	$x < 0, y > 0$	$x > 0, y < 0$	$x < 0, y < 0$
Quadrant				

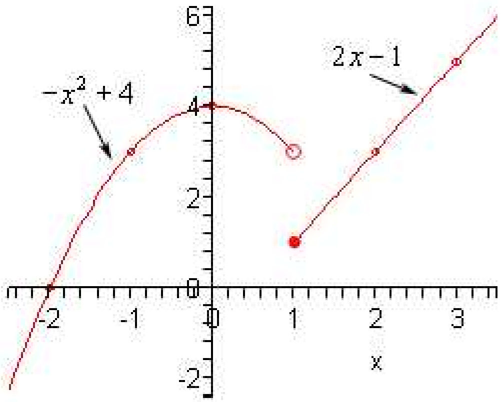
Problem 29.4
In the Cartesian plane, shade the region consisting of all points (x, y) that satisfy the two conditions

$$-3 \leq x \leq 2 \text{ and } 2 \leq y \leq 4$$

Problem 29.5
Determine which of the following graphs represent a function.



Problem 29.6
Consider the function f whose graph is given below.



(a) Complete the following table

x	-2	-1	0	1	2	3
f(x)						

(b) Find the domain and range of f.
(c) For which values of x is $f(x) = 2.5$?

Graphs of Linear Functions
A **linear** function is any function that can be written in the form $f(x) = mx + b$. As the name suggests, the graph of such a function is a straight line.

Example 29.4
The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus we have a relationship between the quantities C and p such that each value of p determines exactly one value of C. In this case, we say that C is a function of p. Find a formula for p and graph.

Solution.
The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

The graph of the function C is obtained by plotting the data in the above table. See Figure 29.3.
The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p. \blacksquare$$

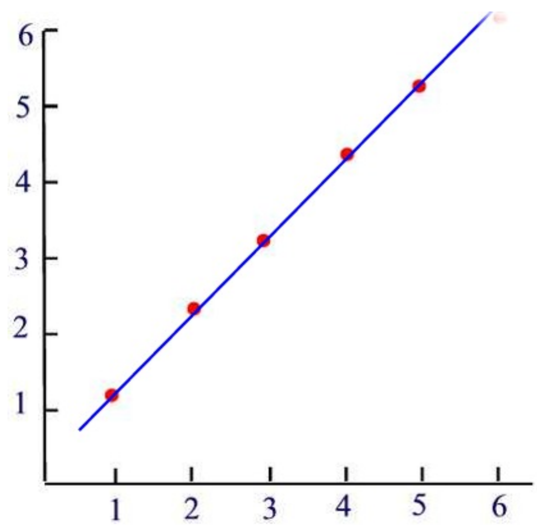


Figure 29.3

Graphs of Quadratic Functions
You recall that a linear function is a function that involves a first power of x. A function of the form

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a **quadratic function**. The word "quadratus" is the latin word for a square.
Quadratic functions are useful in many applications in mathematics when a linear function is not sufficient. For example, the motion of an object thrown either upward or downward is modeled by a quadratic function.
The graph of a quadratic function is a curve called a **parabola.** Parabolas may open upward or downward and vary in "width" or "steepness", but they all have the same basic "U" shape.
All parabolas are symmetric with respect to a line called the **axis of symmetry.** A parabola intersects its axis of symmetry at a point called the **vertex** of the parabola.

Many quadratic functions can be graphed easily by hand using the techniques of stretching/shrinking and shifting (translation) the parabola $y = x^2$.

Example 29.5
Sketch the graph of $y = \frac{x^2}{2}$.

Solution.
Starting with the graph of $y = x^2$, we shrink by a factor of one half. This means that for each point on the graph of $y = x^2$, we draw a new point that is one half of the way from the x-axis to that point. See Figure 29.4 ■

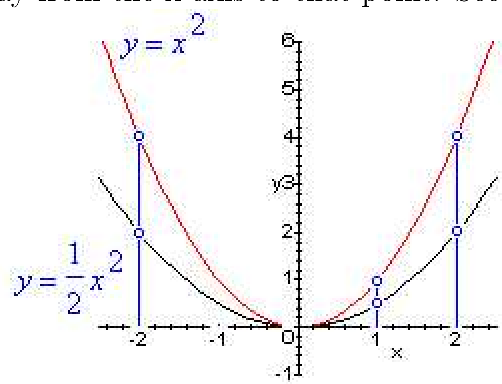


Figure 29.4

When a quadratic function is in standard form, then it is easy to sketch its graph by reflecting, shifting, and stretching/shrinking the parabola $y = x^2$.

The quadratic function $f(x) = a(x - h)^2 + k$, a not equal to zero, is said to be in **standard form.** If a is positive, the graph opens upward, and if a is negative, then it opens downward. The line of symmetry is the vertical line $x = h$, and the vertex is the point (h, k).
Any quadratic function can be rewritten in standard form by completing the square. Note that when a quadratic function is in standard form it is also easy to find its zeros by the square root principle.

Example 29.6
Write the function $f(x) = x^2 - 6x + 7$ in standard form. Sketch the graph of f and find its zeros and vertex.

Solution.
Using completing the square method we find

$$\begin{array}{rcl} f(x) & = & x^2 - 6x + 7 \\ & = & (x^2 - 6x) + 7 \\ & = & (x^2 - 6x + 9 - 9) + 7 \;\; (Just\ square\ \frac{6}{2}) \\ & = & (x^2 - 6x + 9) - 9 + 7 \\ & = & (x-3)^2 - 2 \end{array}$$

From this result, one easily finds the vertex of the graph of f is $(3, -2)$. To find the zeros of f, we set f equal to 0 and solve for x.

$$\begin{array}{rcl} (x-3)^2 - 2 & = & 0 \\ (x-3)^2 & = & 2 \\ x - 3 & = & \pm\sqrt{2} \\ x & = & 3 \pm \sqrt{2} \end{array}$$

Finally, the graph of f is given in Figure 29.5 ■

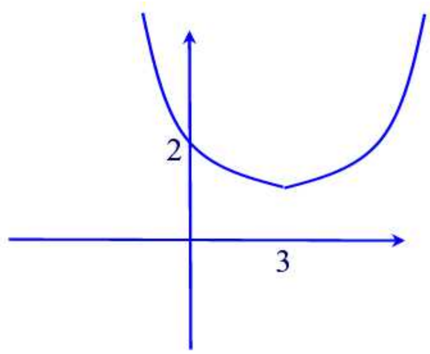


Figure 29.5

Graphs of Exponential Functions
An **exponential function** is a function that can be written in the form

$$f(t) = b \cdot a^t$$

where a is positive and different from 1. We call a the **base** of the function. Figure 29.6 shows the graph of an exponential function.

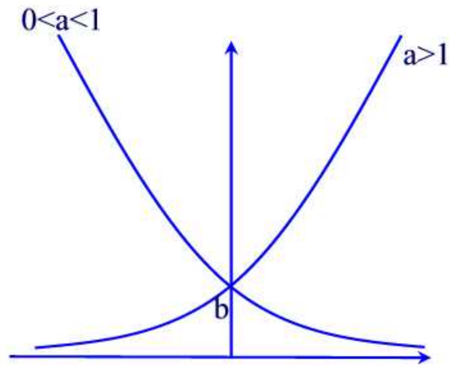


Figure 29.6

For $a > 1$ the function is increasing. In this case, we say that the function represents an exponential **growth.** If $0 < a < 1$ then the function represents an exponential **decay**.

Remark 29.1
Why a is restricted to $a > 0$ and $a \neq 1$? Since t is allowed to have any value then a negative a will create meaningless expressions such as $\sqrt{a}$ (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

Miscellaneous Functions
We consider some of the graphs of some important functions.

Square Root Function
The square root function is the function $f(x) = \sqrt{x}$. To get the graph well just plug in some values of x and then plot the points.

x	0	1	4	9
f(x)	0	1	2	3

The graph is given in Figure 29.7

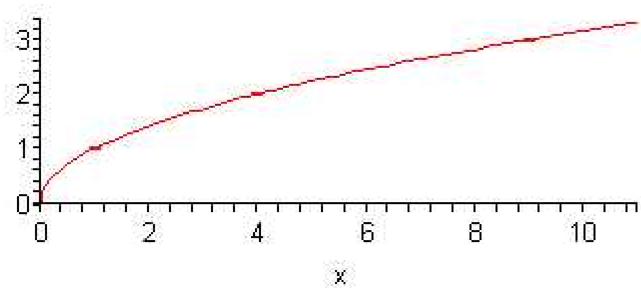


Figure 29.7

Absolute Value Function

We've dealt with this function several times already. It's now time to graph it. First, let's remind ourselves of the definition of the absolute value function.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Finding some points to plot we get

x	-2	-1	0	1	2
f(x)	2	1	0	1	2

The graph is given in Figure 29.8

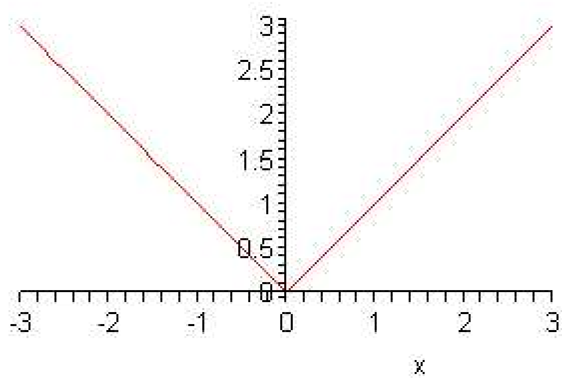


Figure 29.8

Cubic Function
We will consider the following simple form of a cubic function $f(x) = x^3$. First, we find some points on the graph.

x	-2	-1	0	1	2
f(x)	-8	-1	0	1	8

The graph is given in Figure 29.9

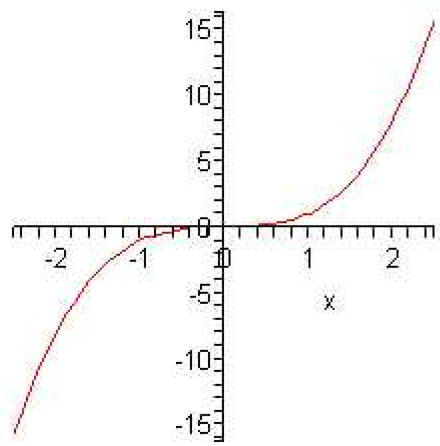


Figure 29.9

Step Functions
The far we have considered functions whose graphs are "continuous", that is, the graphs have no holes or jumps. Step functions are functions that are not continuous.

Example 29.7
The charge for a taxi ride is \$1.50 for the first $\frac{1}{5}$ of a mile, and \$0.25 for each additional $\frac{1}{5}$ of a mile (rounded up to the nearest $\frac{1}{5}$ mile).

(a) Sketch a graph of the cost function C as a function of the distance traveled x, assuming that $0 \le x \le 1$.
(b) Find a formula for C in terms of x on the interval $[0, 1]$.
(c) What is the cost for a $\frac{4}{5} - mile$ ride?

Solution.
(a) The graph is given in Figure 29.10.

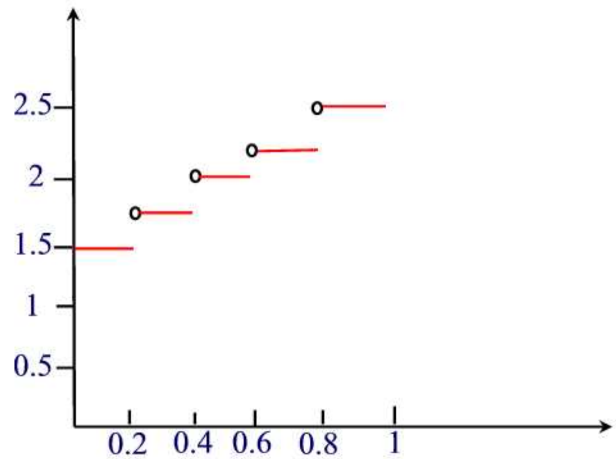


Figure 29.10

(b) A formula of $C(x)$ is

$$C(x) = \begin{cases} 1.50 & \text{if } 0 \leq x \leq \frac{1}{5} \\ 1.75 & \text{if } \frac{1}{5} < x \leq \frac{2}{5} \\ 2.00 & \text{if } \frac{2}{5} < x \leq \frac{3}{5} \\ 2.25 & \text{if } \frac{3}{5} < x \leq \frac{4}{5} \\ 2.50 & \text{if } \frac{4}{5} < x \leq 1. \end{cases}$$

(c) The cost for a $\frac{4}{5}$ ride is $C(\frac{4}{5}) = \$2.25.$ ■

Practice Problems

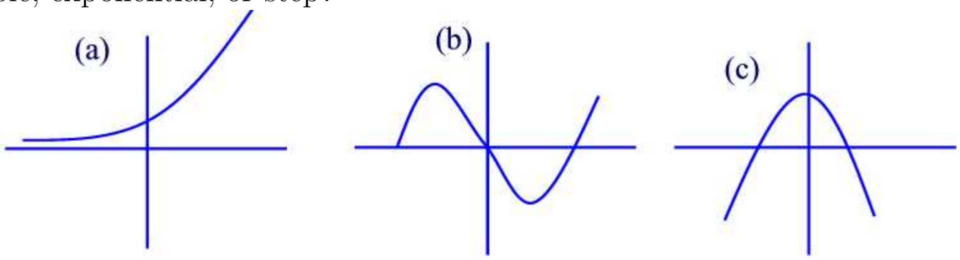
Problem 29.7
Make a table of five values of the function $f(x) = 2x + 3$ and then use the points to sketch the graph of $f(x)$.

Problem 29.8
Make a table of five values of the function $f(x) = \frac{1}{2}x^2 + x$ and then use the points to sketch the graph of $f(x)$.

Problem 29.9
Make a table of five values of the function $f(x) = 3^x$ and then use the points to sketch the graph of $f(x)$.

Problem 29.10
Make a table of five values of the function $f(x) = 2 - x^3$ and then use the points to sketch the graph of $f(x)$.

Problem 29.11
Which type of function best fits each of the following graphs: linear, quadratic, cubic, exponential, or step?



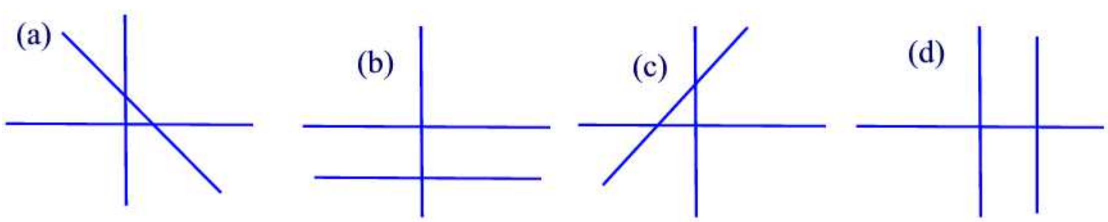
Problem 29.12
Suppose that a function f is given by a table. If the output changes by a fixed amount each time the input changes by a constant then the function is linear. Determine whether each of the following functions below are linear.

x	0	2	4	6
f(x)	20	40	80	160

x	10	20	30	40
g(x)	6	12	18	24

Problem 29.13
Show how to solve the equation $2x + 3 = 11$ using a calculator.

Problem 29.14
In the linear function $f(x) = mx + b$ the parameter m is called the **slope.** The slope of the line determines whether the line rises, falls, is vertical or horizontal. Classify the slope of each line as positive, negative, zero, or undefined.



Problem 29.15
Algebraically, one finds the slope of a line given two points (x_1, y_1) and (x_2, y_2) on the line by using the formula

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Would the ratio

$$\frac{y_1 - y_2}{x_1 - x_2}$$

give the same answer? Explain.

Problem 29.16
Water is being pumped into a tank. Reading are taken every minutes.

Time(min)	0	3	6	9	12
Quarts of water	0	90	180	270	360

(a) Plot the 5 points.
(b) What is the slope of the line joining the 5 points?
(c) Estimate how much water is in the tank after 1 minute.
(d) At what rate is the water being pumped in?

Problem 29.17
(a) Graph $f(x) = x^2 - 2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Problem 29.18
(a) Graph $f(x) = x^2 + 3$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Problem 29.19
(a) Graph $f(x) = (x - 1)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Problem 29.20
(a) Graph $f(x) = (x + 2)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Problem 29.21
Graph each equation by plotting points that satisfy the equation.

(a) $x - y = 4$.
(b) $y = -2|x - 3|$.
(c) $y = \frac{1}{2}(x - 1)^2$.

Problem 29.22
Find the x- and y-intercepts of each equation.

(a) $2x + 5y = 12$.
(b) $x = |y| - 4$.
(c) $|x| + |y| = 4$.

30 Graphical Representations of Data

Visualization techniques are ways of creating and manipulating graphical representations of data. We use these representations in order to gain better insight and understanding of the problem we are studying - pictures can convey an overall message much better than a list of numbers.
In this section we describe some graphical presentations of data.

Line or Dot Plots
Line plots are graphical representations of numerical data. A **line plot** is a number line with x's placed above specific numbers to show their frequency. By the **frequency** of a number we mean the number of occurrence of that number. Line plots are used to represent one group of data with fewer than 50 values.

Example 30.1
Suppose thirty people live in an apartment building. These are the following ages:

58	30	37	36	34	49	35	40	47	47
39	54	47	48	54	50	35	40	38	47
48	34	40	46	49	47	35	48	47	46

Make a line plot of the ages.

Solution.
The dot plot is given in Figure 30.1

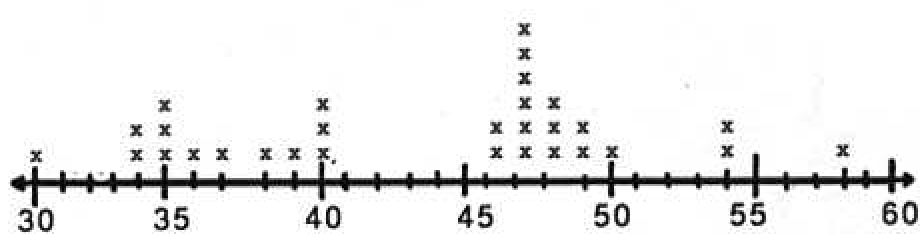


Figure 30.1

This graph shows all the ages of the people who live in the apartment building. It shows the youngest person is 30, and the oldest is 58. Most people in the building are over 46 years of age. The most common age is 47.■

Line plots allow several features of the data to become more obvious. For example, outliers, clusters, and gaps are apparent.

• **Outliers** are data points whose values are significantly larger or smaller than other values, such as the ages of 30, and 58.
• **Clusters** are isolated groups of points, such as the ages of 46 through 50.
• **Gaps** are large spaces between points, such as 41 and 45.

Practice Problems

Problem 30.1
Following are the ages of the 30 students at Washington High School who participated in the city track meet. Draw a dot plot to represent these data.

10	10	11	10	13	8	10	13	14	9
14	13	10	14	11	9	13	10	11	12
11	12	14	13	12	8	13	14	9	14

Problem 30.2
The height (in inches) of the players on a professional basketball team are 70, 72, 75, 77, 78, 78, 80, 81,81,82, and 83. Make a line plot of the heights.

Problem 30.3
Draw a Dot Plot for the following dataset.

50	35	70	55	50	30	40
65	50	75	60	45	35	75
60	55	55	50	40	55	50

Stem and Leaf Plots
Another type of graph is the **stem-and-leaf plot**. It is closely related to the line plot except that the number line is usually vertical, and digits are used instead of x's. To illustrate the method, consider the following scores which twenty students got in a history test:

69	84	52	93	61	74	79	65	88	63
57	64	67	72	74	55	82	61	68	77

We divide each data value into two parts. The left group is called a **stem** and the remaining group of digits on the right is called a **leaf**. We display horizontal rows of leaves attached to a vertical column of stems. we can construct the following table

5	2	7	5					
6	9	1	5	3	4	7	1	8
7	4	9	2	4	7			
8	4	8	2					
9	3							

where the stems are the ten digits of the scores and the leaves are the one digits.
The disadvantage of the stem-and-leaf plots is that data must be grouped according to place value. What if one wants to use different groupings? In this case histograms are more suited.

If you are comparing two sets of data, you can use a **back-to-back stem-and-leaf plot** where the leaves of sets are listed on either side of the stem as shown in the table below.

										9	6	0	5	7									
							8	7	6	4	1	1											
8	8	7	6	5	5	3	2	2	2	2	1	2	2	5	6	7	8	8	9				
					9	9	6	4	4	3	2	3	1	2	3	4	4	4	5	6	7	8	9
								9	6	5	1	4	2	3	5	6	7	8	9	9			

Practice Problems

Problem 30.4
Given below the scores of a class of 26 fourth graders.

64	82	85	99	96	81	97	80	81	80	84	87	98
75	86	88	82	78	81	86	80	50	84	88	83	82

Make a stem-and-leaf display of the scores.

Problem 30.5
Each morning, a teacher quizzed his class with 20 geography questions. The class marked them together and everyone kept a record of their personal

scores. As the year passed, each student tried to improve his or her quiz marks. Every day, Elliot recorded his quiz marks on a stem and leaf plot. This is what his marks looked like plotted out:

0	3	6	5									
1	0	1	4	3	5	6	5	6	8	9	7	9
2	0	0	0	0								

What is his most common score on the geography quizzes? What is his highest score? His lowest score? Are most of Elliot's scores in the 10s, 20s or under 10?

Problem 30.6
A teacher asked 10 of her students how many books they had read in the last 12 months. Their answers were as follows:

$$12, 23, 19, 6, 10, 7, 15, 25, 21, 12$$

Prepare a stem and leaf plot for these data.

Problem 30.7
Make a back-to-back stem and leaf plot for the following test scores:

Class 1:

100	96	93	92	92	92	90	90
89	89	85	82	79	75	74	73
73	73	70	69	68	68	65	61
35							

Class 2:

79	85	56	79	84	64	44	57
69	85	65	81	73	51	61	67
71	89	69	77	82	75	89	92
74	70	75	88	46			

Frequency Distributions and Histograms
When we deal with large sets of data, a good overall picture and sufficient information can be often conveyed by distributing the data into a number of **classes** or **class intervals** and to determine the number of elements belonging to each class, called **class frequency.** For instance, the following table shows some test scores from a math class.

65	91	85	76	85	87	79	93
82	75	100	70	88	78	83	59
87	69	89	54	74	89	83	80
94	67	77	92	82	70	94	84
96	98	46	70	90	96	88	72

It's hard to get a feel for this data in this format because it is unorganized. To construct a frequency distribution,

- Compute the class width $CW = \frac{Largest\ data\ value - smallest\ data\ value}{Desirable\ number\ of\ classes}$.

- Round CW to the next highest whole number so that the classes cover the whole data.

Thus, if we want to have 6 class intervals then $CW = \frac{100-46}{6} = 9$. The low number in each class is called the **lower class limit**, and the high number is called the **upper class limit.**
With the above information we can construct the following table called **frequency distribution**.

Class	Frequency
41-50	1
51-60	2
61-70	6
71-80	8
81-90	14
91-100	9

Once frequency distributions are constructed, it is usually advisable to present them graphically. The most common form of graphical representation is the **histogram**.
In a histogram, each of the classes in the frequency distribution is represented by a vertical bar whose height is the class frequency of the interval. The horizontal endpoints of each vertical bar correspond to the class endpoints.
A histogram of the math scores is given in Figure 30.2

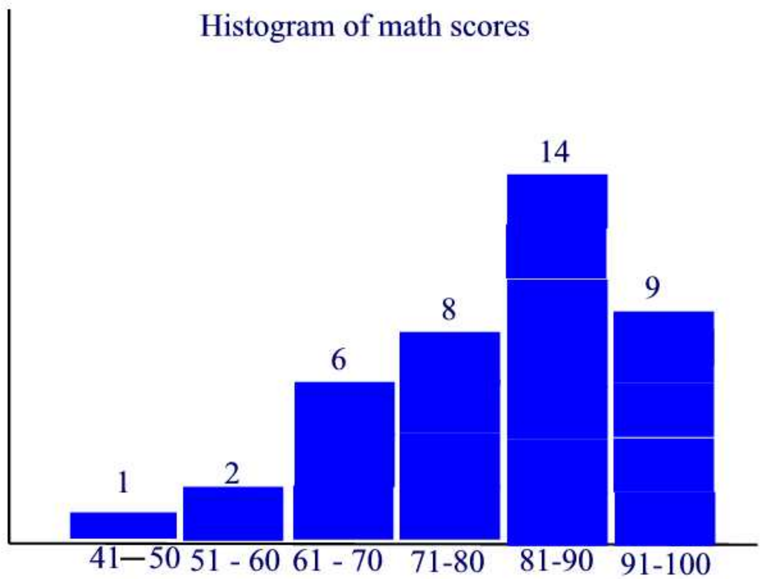


Figure 30.2

One advantage to the stem-and-leaf plot over the histogram is that the stem-and-leaf plot displays not only the frequency for each interval, but also displays all of the individual values within that interval.

Practice Problems

Problem 30.8
Suppose a sample of 38 female university students was asked their weights in pounds. This was actually done, with the following results:

130	108	135	120	97	110
130	112	123	117	170	124
120	133	87	130	160	128
110	135	115	127	102	130
89	135	87	135	115	110
105	130	115	100	125	120
120	120				

(a) Suppose we want 9 class intervals. Find CW.
(b) Construct a frequency distribution.
(c) Construct the corresponding histogram.

Problem 30.9
The table below shows the response times of calls for police service measured in minutes.

34	10	4	3	9	18	4
3	14	8	15	19	24	9
36	5	7	13	17	22	27
3	6	11	16	21	26	31
32	38	40	30	47	53	14
6	12	18	23	28	33	
3	4	62	24	35	54	
15	6	13	19	3	4	
4	20	5	4	5	5	
10	25	7	7	42	44	

Construct a frequency distribution and the corresponding histogram.

Problem 30.10
A nutritionist is interested in knowing the percent of calories from fat which Americans intake on a daily basis. To study this, the nutritionist randomly selects 25 Americans and evaluates the percent of calories from fat consumed in a typical day. The results of the study are as follows

34%	18%	33%	25%	30%
42%	40%	33%	39%	40%
45%	35%	45%	25%	27%
23%	32%	33%	47%	23%
27%	32%	30%	28%	36%

Construct a frequency distribution and the corresponding histogram.

Bar Graphs
Bar Graphs, similar to histograms, are often useful in conveying information about categorical data where the horizontal scale represents some nonnumerical attribute. In a bar graph, the bars are nonoverlapping rectangles of equal width and they are equally spaced. The bars can be vertical or horizontal. The length of a bar represents the quantity we wish to compare.

Example 30.2
The areas of the various continents of the world (in millions of square miles)

are as follows:11.7 for Africa; 10.4 for Asia; 1.9 for Europe; 9.4 for North America; 3.3 Oceania; 6.9 South America; 7.9 Soviet Union. Draw a bar chart representing the above data and where the bars are horizontal.

Solution.
The bar graph is shown in Figure 30.3.

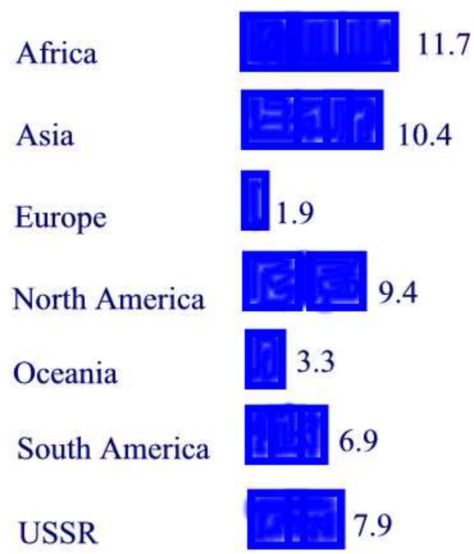


Figure 30.3: Areas (in millions of square miles) of the variuos continents of the world

A double bar graph is similar to a regular bar graph, but gives 2 pieces of information for each item on the vertical axis, rather than just 1. The bar chart in Figure 30.4 shows the weight in kilograms of some fruit sold on two different days by a local market. This lets us compare the sales of each fruit over a 2 day period, not just the sales of one fruit compared to another. We can see that the sales of star fruit and apples stayed most nearly the same. The sales of oranges increased from day 1 to day 2 by 10 kilograms. The same amount of apples and oranges was sold on the second day.

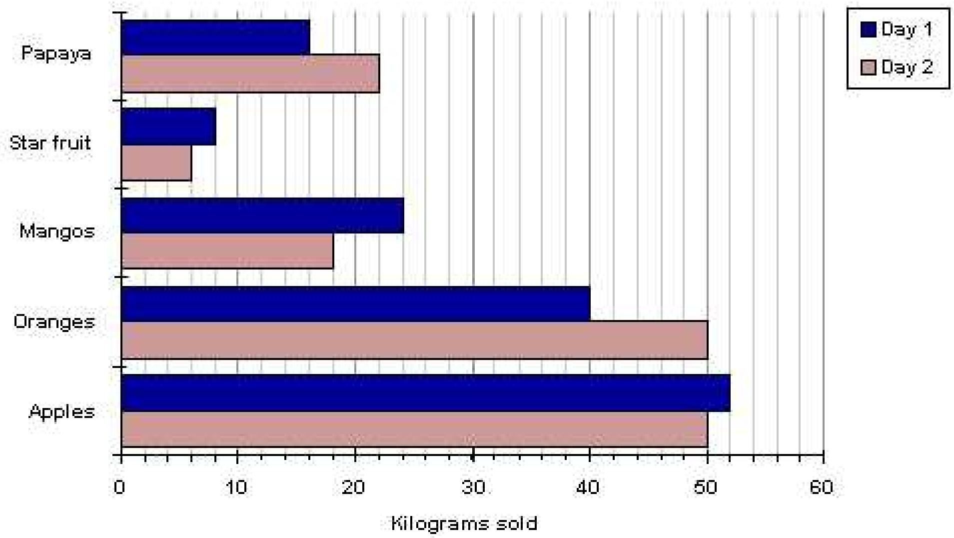


Figure 30.4

Example 30.3
The figures for total population, male and female population of the UK at decade intervals since 1959 are given below:

Year	*Total UK Resident Population*	*Male*	*Female*
1959	51,956,000	25,043,000	26,913,000
1969	55,461,000	26,908,000	28,553,000
1979	56,240,000	27,373,000	28,867,000
1989	57,365,000	27,988,000	29,377,000
1999	59,501,000	29,299,000	30,202,000

Construct a bar chart representing the data.

Solution.
The bar graph is shown in Figure 30.5.

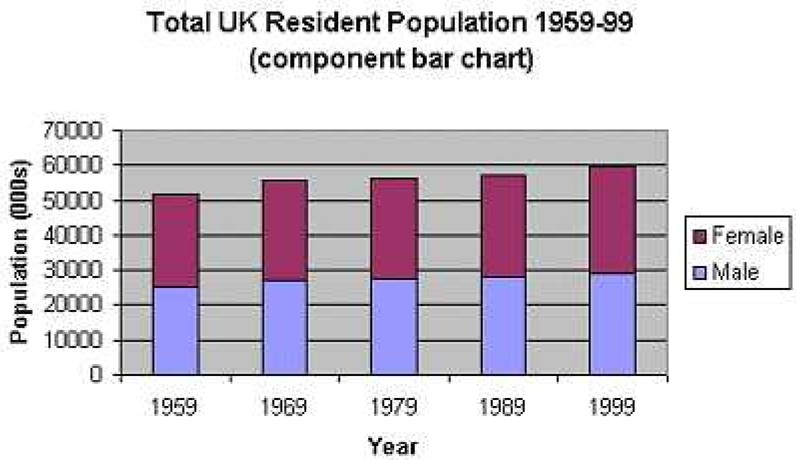


Figure 30.5

Line Graphs

A **Line graph** (or **time series plot**)is particularly appropriate for representing data that vary continuously. A line graph typically shows the trend of a variable over time. To construct a time series plot, we put time on the horizontal scale and the variable being measured on the vertical scale and then we connect the points using line segments.

Example 30.4

The population (in millions) of the US for the years 1860-1950 is as follows: 31.4 in 1860; 39.8 in 1870; 50.2 in 1880; 62.9 in 1890; 76.0 in 1900; 92.0 in 1910; 105.7 in 1920; 122.8 in 1930; 131.7 in 1940; and 151.1 in 1950. Make a time plot showing this information.

Solution.

The line graph is shown in Figure 30.6.

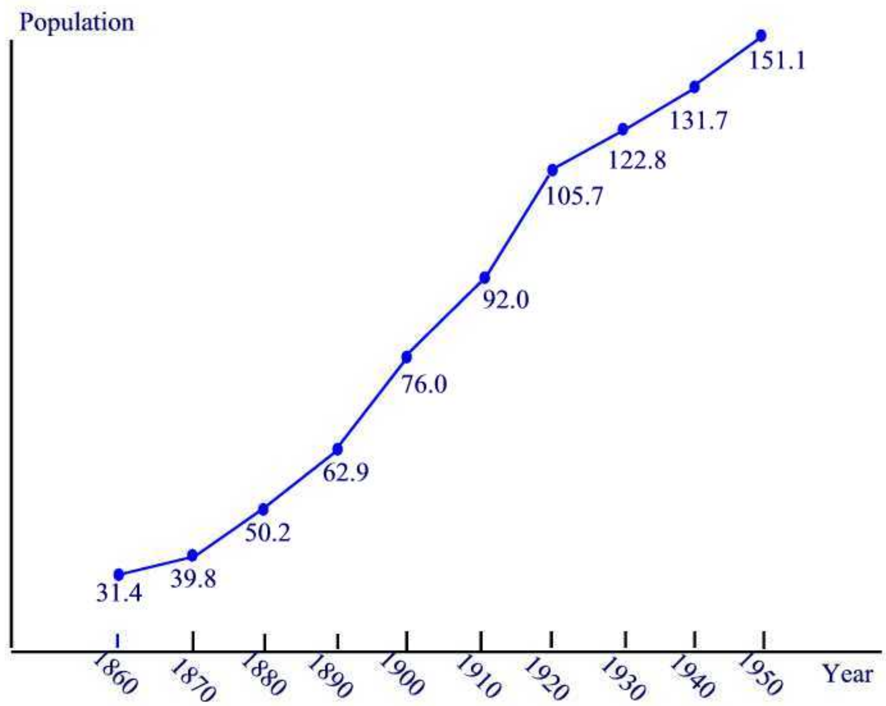


Figure 30.6: US population (in millions) from 1860 - 1950

Practice Problems

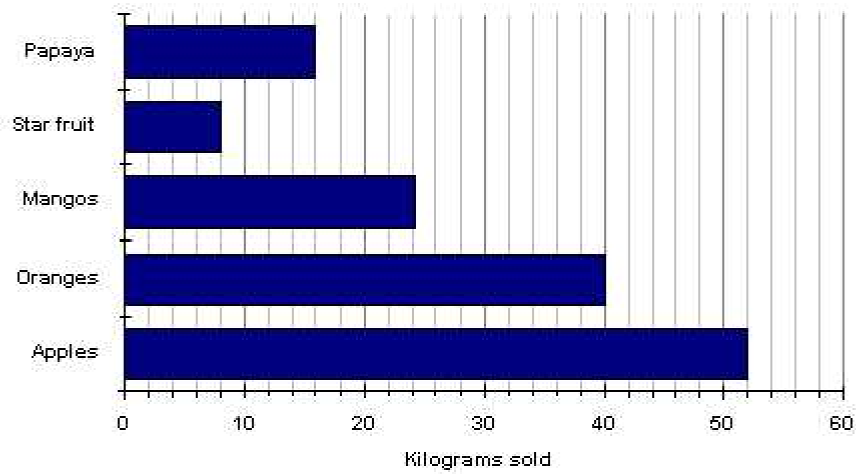
Problem 30.11
Given are several gasoline vehicles and their fuel consumption averages.

Buick	27 mpg
BMW	28 mpg
Honda Civic	35 mpg
Geo	46 mpg
Neon	38 mpg
Land Rover	16 mpg

(a) Draw a bar graph to represent these data.
(b) Which model gets the least miles per gallon? the most?

Problem 30.12
The bar chart below shows the weight in kilograms of some fruit sold one

day by a local market.



How many kg of apples were sold? How many kg of oranges were sold?

Problem 30.13
The figures for total population at decade intervals since 1959 are given below:

Year	*Total UK Resident Population*
1959	51,956,000
1969	55,461,000
1979	56,240,000
1989	57,365,000
1999	59,501,000

Construct a bar chart for this data.

Problem 30.14
The following data gives the number of murder victims in the U.S in 1978 classified by the type of weapon used on them. Gun, 11,910; cutting/stabbing, 3,526; blunt object, 896; strangulation/beating, 1,422; arson, 255; all others 705. Construct a bar chart for this data. Use vertical bars.

Circle Graphs or Pie Charts

Another type of graph used to represent data is the circle graph. A **circle graph** or **pie chart**, consists of a circular region partitioned into disjoint sections, with each section representing a part or percentage of a whole. To construct a pie chart we first convert the distribution into a percentage distribution. Then, since a complete circle corresponds to 360 degrees, we obtain the central angles of the various sectors by multiplying the percentages by 3.6. We illustrate this method in the next example.

Example 30.5

A survey of 1000 adults uncovered some interesting housekeeping secrets. When unexpected company comes, where do we hide the mess? The survey showed that 68% of the respondents toss their mess in the closet, 23% shove things under the bed, 6% put things in the bath tub, and 3% put the mess in the freezer. Make a circle graph to display this information.

Solution.

We first find the central angle corresponding to each case:

$$\begin{array}{lrcr} \textit{in closet} & 68 \times 3.6 & = & 244.8 \\ \textit{under bed} & 23 \times 3.6 & = & 82.8 \\ \textit{in bathtub} & 6 \times 3.6 & = & 21.6 \\ \textit{in freezer} & 3 \times 3.6 & = & 10.8 \end{array}$$

Note that

$$244.8 + 82.8 + 21.6 + 10.8 = 360.$$

The pie chart is given in Figure 30.7.

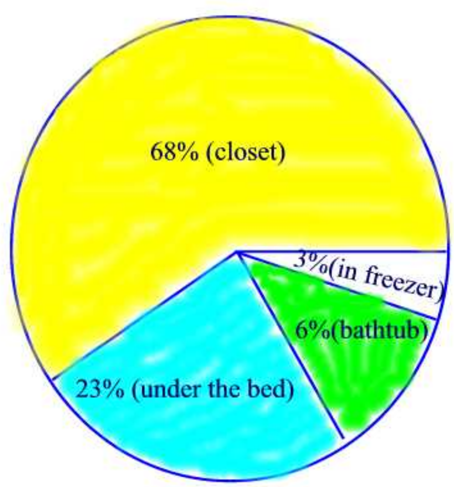


Figure 30.7: Where to hide the mess?

Practice Problems

Problem 30.15
The table below shows the ingredients used to make a sausage and mushroom pizza.

Ingredient	%
Sausage	7.5
Cheese	25
Crust	50
TomatoSause	12.5
Mushroom	5

Plot a pie chart for this data.

Problem 30.16
A newly qualified teacher was given the following information about the ethnic origins of the pupils in a class.

Ethnic origin	*No. of pupils*
White	12
Indian	7
BlackAfrican	2
Pakistani	3
Bangladeshi	6
TOTAL	30

Plot a pie chart representing the data.

Problem 30.17
The following table represent a survey of people's favorite ice cream flavor

Flovor	*Number of people*
Vanilla	21.0%
Chocolate	33.0%
Strawberry	12.0%
Raspberry	4.0%
Peach	7.0%
Neopolitan	17.0%
Other	6.0%

Plot a pie chart to represent this data.

Problem 30.18
In the United States, approximately 45% of the population has blood type O; 40% type A; 11% type B; and 4% type AB. Illustrate this distribution of blood types with a pie chart.

Pictographs
One type of graph seen in newspapers and magazines is a **pictograph**. In a pictograph, a symbol or icon is used to represent a quantity of items. A pictograph needs a title to describe what is being presented and how the data are classified as well as the time period and the source of the data. Example of a pictograph is given in Figure 30.8.

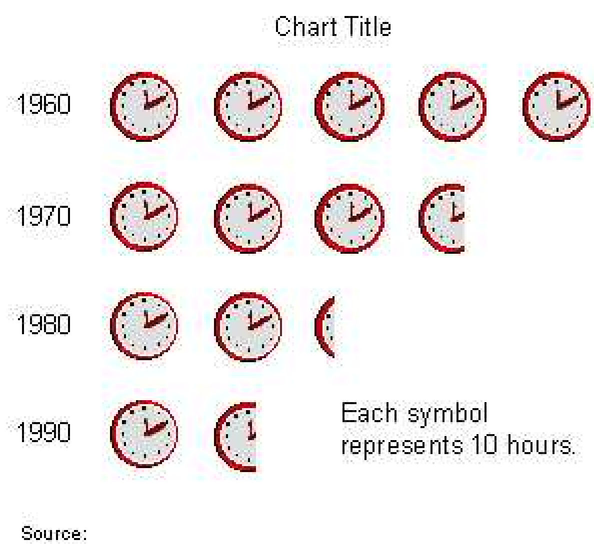


Figure 30.8

A disadvantage of a pictograph is that it is hard to quantify partial icons.

Practice Problems

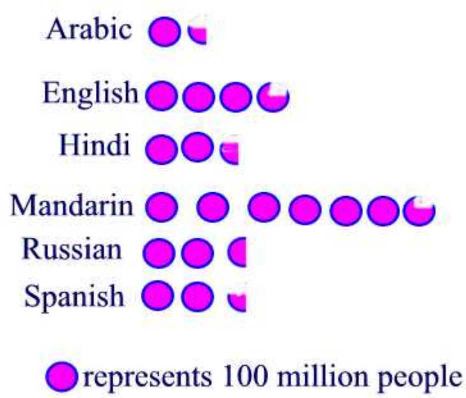
Problem 30.19

Make a pictograph to represent the data in the following table. Use  to represent 10 glasses of lemonade.

Day	Frequency
Monday	15
Tuesday	20
Wednesday	30
Thursday	5
Friday	10

Problem 30.20

The following pictograph shows the approximate number of people who speak the six common languages on earth.
(a) About how many people speak Spanish?
(b) About how many people speak English?
(c) About how many more people speak Mandarin than Arabic?



Problem 30.21

Twenty people were surveyed about their favorite pets and the result is shown in the table below.

Pet	Frequency
Dog	2
Cat	5
Hamster	3

Make a pictograph for the following table of data. Let  stand for 2 votes.

Scatterplots

A relationship between two sets of data is sometimes determined by using a **scatterplot**. Let's consider the question of whether studying longer for a test will lead to better scores. A collection of data is given below

Study Hours	3	5	2	6	7	1	2	7	1	7
Score	80	90	75	80	90	50	65	85	40	100

Based on these data, a scatterplot has been prepared and is given in Figure 30.9.(Remember when making a scatterplot, do NOT connect the dots.)

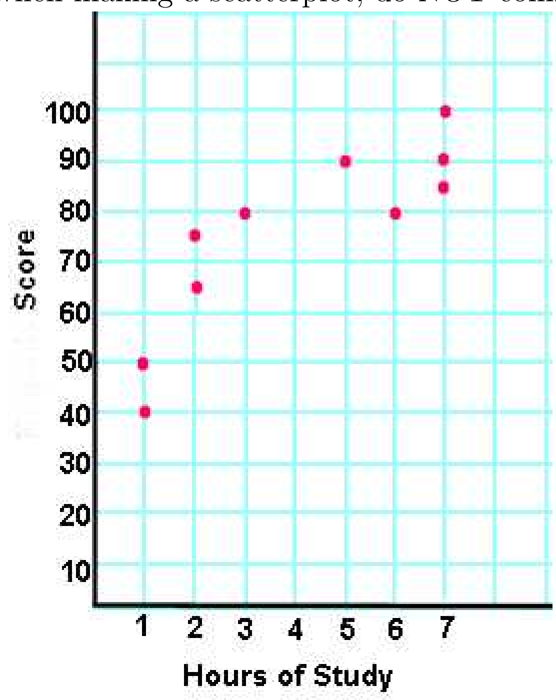


Figure 30.9

The data displayed on the graph resembles a line rising from left to right. Since the slope of the line is positive, there is a **positive correlation** between the two sets of data. This means that according to this set of data, the longer I study, the better grade I will get on my exam score.
If the slope of the line had been negative (falling from left to right), a **negative correlation** would exist. Under a negative correlation, the longer I study, the worse grade I would get on my exam.
If the plot on the graph is scattered in such a way that it does not approximate a line (it does not appear to rise or fall), there is **no correlation** between the sets of data. No correlation means that the data just doesn't show if studying longer has any affect on my exam score.

Practice Problems

Problem 30.22
Coach Lewis kept track of the basketball team's jumping records for a 10-year period, as follows:

Year	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02
Record(nearest in)	65	67	67	68	70	74	77	78	80	81

(a) Draw a scatterplot for the data.
(b) What kind of corrolation is there for these data?

Problem 30.23
The gas tank of a National Motors Titan holds 20 gallons of gas. The following data are collected during a week.

Fuel in tank (gal.)	20	18	16	14	12	10
Dist. traveled (mi)	0	75	157	229	306	379

(a) Draw a scatterplot for the data.
(b) What kind of corrolation is there for these data?

31 Misleading Graphs and Statistics

It is a well known fact that statistics can be misleading. They are often used to prove a point, and can easily be twisted in favour of that point! The purpose of this section is to learn how to recognize common statisitcal deception so that to avoid being mislead.

Bad Sampling
When you use a sample to represent a larger group, you must make sure that the people in the sample are fairly representative of the larger group.

Example 31.1
Decide whether a mall is a good place to find a sample for a survey about the amount of allowance received by people ages 10 to 15.

Solution.
The mall is probably not a representative place to find a fair sample of people in this age range. Taking a sample at the mall might not represent fairly those people who receive a small allowance, or none■

Misleading Graphs
Good graphs are extremely powerful tools for displaying large quantities of complex data; they help turn the realms of information available today into knowledge. But, unfortunately, some graphs deceive or mislead. This may happen because the designer chooses to give readers the impression of better performance or results than is actually the situation. In other cases, the person who prepares the graph may want to be accurate and honest, but may mislead the reader by a poor choice of a graph form or poor graph construction.
The following things are important to consider when looking at a graph:

1. Title
2. Labels on both axes of a line or bar chart and on all sections of a pie chart
3. Source of the data
4. Key to a pictograph
5. Uniform size of a symbol in a pictograph
6. Scale: Does it start with zero? If not, is there a break shown
7. Scale: Are the numbers equally spaced?

Scaling and Axis Manipulation

A graph can be altered by changing the scale of the graph. For example, data in the two graphs of Figure 31.1 are identical, but scaling of the Y-axis changes the impression of the magnitude of differences.

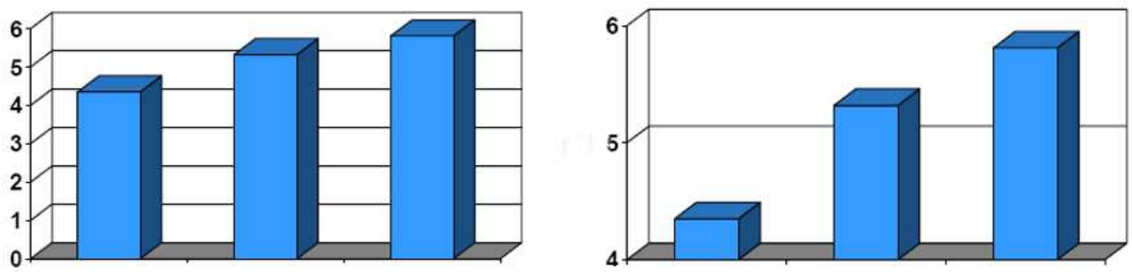
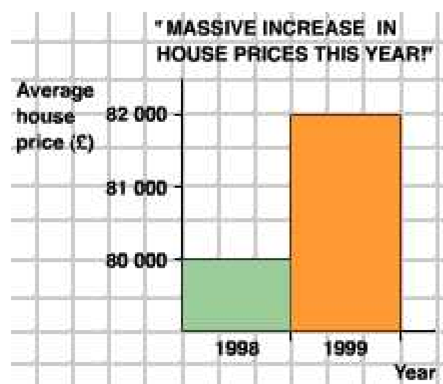


Figure 31.1

Example 31.2

Why does the bar chart below misleading? How should the information be represented?



Solution.

The bar chart indicates that house prices have tripled in one year. The scale of vertical must start at 0 and that's not the case. A less misleading graph would look like the one in Figure 31.2. This gives a much more accurate picture of what has happened.

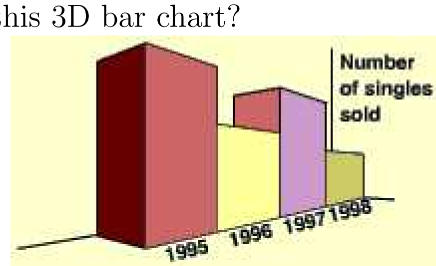
Figure 31.2

Example 31.3
What is wrong with the information represented on this graph?

Solution.
Although the vertical scale starts at 0, it does not go up in even steps. This distorts the graph, and makes it look as though the biggest jump is between 1 and 2 rather than 3 and 4. Also, there are no labels on the axes so we have no idea what this graph represents! ■

Three Dimensional Effects

Example 31.4
What is wrong with this 3D bar chart?



Solution.
This 3D bar chart might look very attractive, but it is also very misleading. There is no scale on the vertical axis, and because of the perspective it looks as though the sales for 1995 were far greater than those for any other year. In fact they were identical to those for 1997. It would be much better to draw a 2D bar chart like the one shown in Figure 31.3 with the appropriate labelling on each axis:

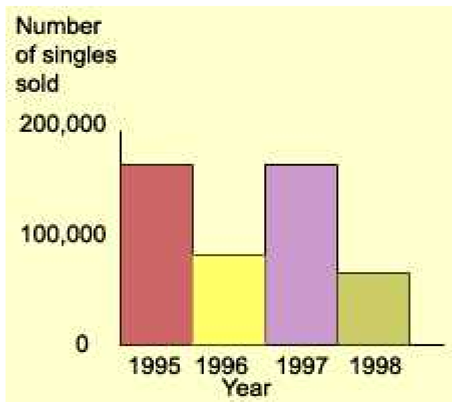
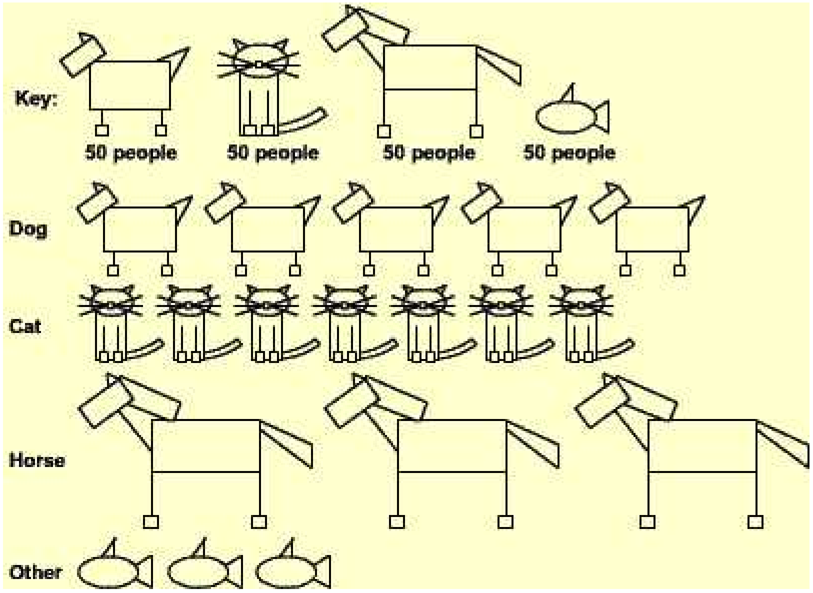


Figure 31.3

Deceptive Pictographs

Example 31.5
What is wrong with this pictogram showing the number of people who own different types of pets?



Solution.
On this pictogram there isn't a category for those people who do not own a pet. The pictures are different sizes and it appears that more people own a horse than any other animal.
An improvement would be to redraw the pictogram with each of the animals the same size and aligned with one another as shown in Figure 31.4.

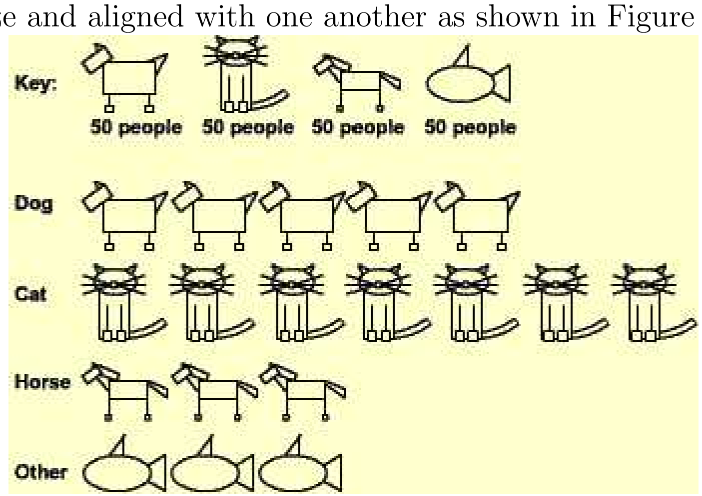
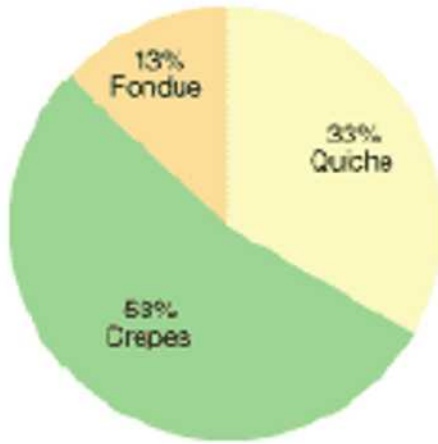


Figure 31.4

Example 31.6
A survey was conducted to determine what food would be served at the

French club party. Explain how the graph misrepresents the data.

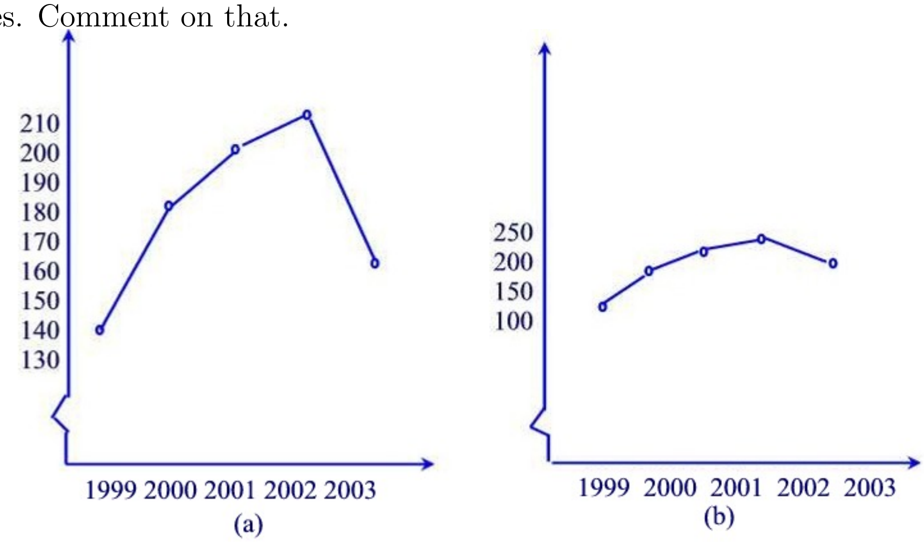


Solution.
The percents on the circle graph do not sum to 100.■

Example 31.7
The number of graduates from a community college for the years 1999 through 2003 is given in the following table:

Year	1999	2000	2001	2002	2003
# of Graduates	140	180	200	210	160

The figure below shows the line graphs of the same data but with different scales. Comment on that.

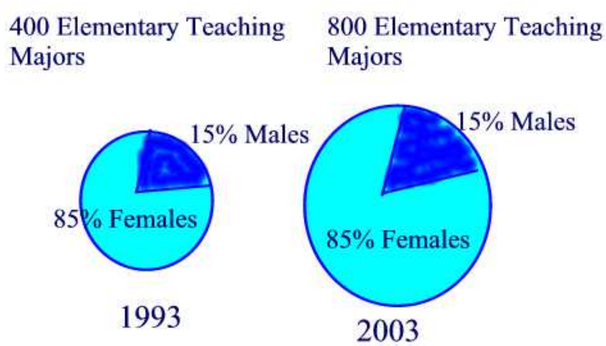


Solution.
The two graphs do not convey the same message. In Figure (b) the spacing of the years on the horizontal axis is more spread out and that for the numbers on the vertical axis is more condensed than Figure (a). A college administrator might use a graph like Figure (b) to convince people that the college was not in serious enrollment trouble.■

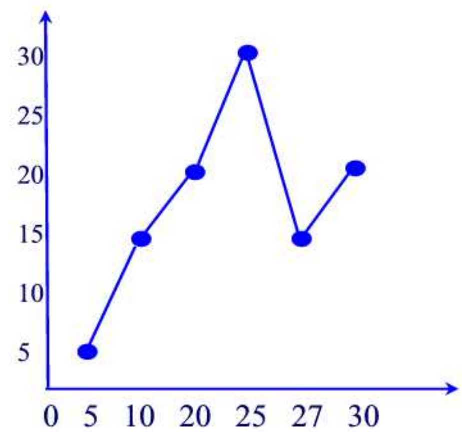
Practice Problems

Problem 31.1
Jenny averaged 70 on her quizzes during the first part of the quarter and 80 on her quizzes during the second part of the quarter. When she found out that her final average for the quarter was not 75, she went to argue with her teacher. Give a possible explanation for Jenny's misunderstanding.

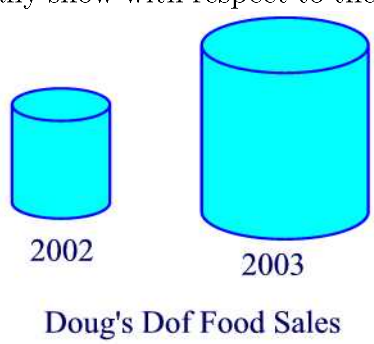
Problem 31.2
Suppose the following circle graphs are used to illustrate the fact that the number of elementary teaching majors at teachers' colleges has doubled between 1993 and 2003, while the percent of male elementary teaching majors has stayed the same. What is misleading about the way the graphs are constructed?



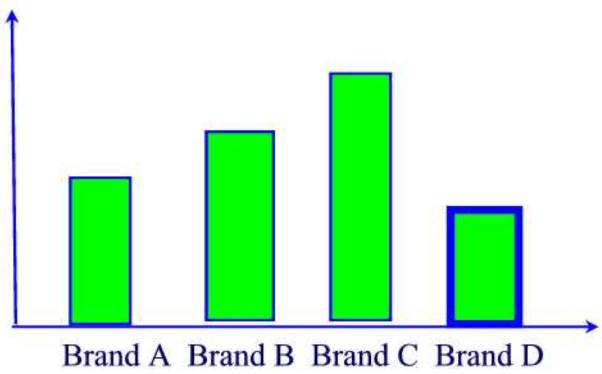
Problem 31.3
What is wrong with the following line graph?



Problem 31.4
Doug's Dog Food Company wanted to impress the public with the magnitude of the company's growth. Sales of Doug's Dog Food had doubled from 2002 to 2003, so the company displayed the following graph, in which the radius of the base and the height of the 2003 can are double those of the 2002 can. What does the graph really show with respect to the growth of the company?



Problem 31.5
What's wrong with the following graph?



Problem 31.6
Refer to the following pictograph:

Ms McNulty claims that on the basis of this information, we can conclude that men are worse drivers than women. Discuss whether you can reach that conclusion from the pictograph or you need more information. If more information is needed, what would you like to know?

Problem 31.7
Larry and Marc took the same courses last quarter. Each bet that he would receive the better grades. Their courses and grades are as follows:

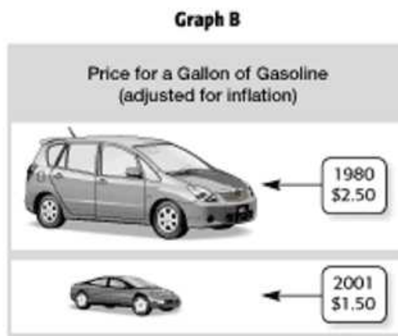
Course	Larry's Grades	Marc's Grades
Math(4 credits)	A	C
Chemistry(4 credits)	A	C
English(3 credits)	B	B
Psychology(3 credits)	C	A
Tennis(1 credit)	C	A

Marc claimed that the results constituted a tie, since both received 2 A's, 1 B, and 2 C's. Larry said that he won the bet because he had the higher GPA for the quarter. Who is correct?(Allow 4 points for A, 3 points for B, 2 points for C, 1 point for D, and 0 point for F.)

Problem 31.8
Oil prices went up 20% one year and 30% the next. Is it true that over the two years, prices went up 50%?

Problem 31.9
True or false? My rent went down 10% last year and then rose 20% this year. Over the two years my rent went up by 10%.

Problem 31.10
Which graph could be used to indicate a greater decrease in the price of gasoline? Explain.



32 Measures of Central Tendency and Dispersion

In this section we discuss two important aspects of data which are its center and its spread. The mean, median, and the mode are **measures of central tendency** that describe where data are centered. The range, variance, and standard deviation are **measures of dispersion** that describe the spread of data.

Measures of Central Tendency: Mode, Median, Mean

Mode
The first measure of the center is the **mode**. It is defined as the value which occurs with the highest frequency in the data. Thus, it is used when one is interested in the most common value in a distribution. A mode may not exist and usually this happens when no data value occurs more frequently than all the others. Also a mode may not be unique.

Example 32.1
The final grades of a class of six graduate students were A, C, B, C, A, B. What is the mode?

Solution.
Since the grades occur at the same frequency then there is no mode for this data.■

Example 32.2
A sample of the records of motor vehicle bureau shows that 18 drivers in a certain age group received $3, 2, 0, 0, 2, 3, 3, 1, 0, 1, 0, 3, 4, 0, 3, 2, 3, 0$ traffic tickets during the last three year. Find the mode?

Solution.
The mode consists of 0 and 3 since they occur six times on the list.■

Median
Another measure of the center is the **median**. The median usually is found when we have an ordered distribution. It is computed as follows. We arrange

the numerical data from smallest to largest. If n denotes the size of the set of data then the median can be found by using the **median rank**

$$MR = \frac{n+1}{2}.$$

If MR is a whole number then the median is the value in that position. If MR ends in .5, we take the sum of the adjacent positions and divide by 2. Unlike the mode, the median always exists and is unique. But it may or may not be one of the given data values. Note that extreme values (smallest or largest) do not affect the median.

Example 32.3
Among groups of 40 students interviewed at each of 10 different colleges, $18, 13, 15, 12, 8, 3, 7, 14, 16, 3$ said that they jog regularly. Find the median.

Solution.
First, arrange the numbers from smallest to largest to obtain

$$3 \quad 3 \quad 7 \quad 8 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 18$$

Next, compute the median rank $MR = \frac{10+1}{2} = 5.5$. Hence, the median is $\frac{12+13}{2} = 12.5$■

Example 32.4
Nine corporations reported that in 1982 they made cash donations to $9, 16, 11, 10, 13, 12, 6, 9,$ and 12 colleges. Find the median number.

Solution.
Arranging the numbers from smallest to largest to obtain

$$6 \quad 9 \quad 9 \quad 10 \quad 11 \quad 12 \quad 12 \quad 13 \quad 16$$

The median rank is $MR = \frac{9+1}{2} = 5$. The median is 11.■

Arithmetic Mean
Another most widely used measure of the center is the **arithmetic mean** or simply **mean**. The mean of a set of N numbers $x_1, x_2, ..., x_N$, denoted by $\overline{x}$, is defined as

$$\overline{x} = \frac{x_1 + x_2 + ... + x_N}{N}.$$

Unlike the median, the mean can be affected by extreme values since it uses the exact value of each data.

Example 32.5
If nine school juniors averaged 41 on the verbal portion of the PSAT test, at most how many of them can have scored 65 or more?

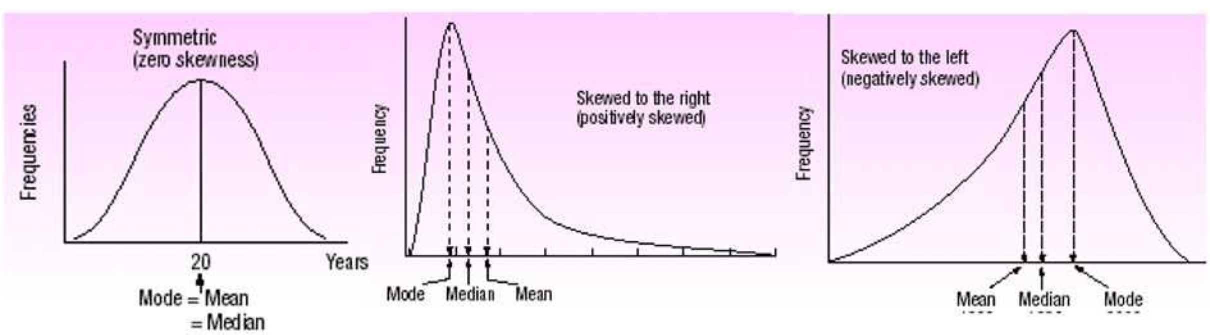
Solution.
We have that $\overline{x} = 41$ and $N = 9$ so that $x_1 + x_2 + \cdots + x_9 = 41 \times 9 = 369$. Since $6 \times 65 = 390 > 369$ and $5 \times 65 = 325$ then at most 5 students can score more than 65.■

Example 32.6
If the numbers $x_1, x_2, ..., x_N$ occur with frequencies $m_1, m_2, ..., m_N$ respectively then what is the mean in this case?

Solution.
The mean is given by

$$\overline{x} = \frac{m_1x_1 + m_2x_2 + ... + m_Nx_N}{m_1 + m_2 + \cdots + m_N}.■$$

The figure below gives the relationships among the measures of central tendency.



Practice Problems

Problem 32.1
Find (a) the mean, (b) median, and (c) the mode for the following collection of data:

60 60 70 95 95 100

Problem 32.2
Suppose a company employs 20 people. The president of the company earns \$200,000, the vice president earns \$75,000, and 18 employees earn \$10,000 each. Is the mean the best number to choose to represent the "average" salary of the company?

Problem 32.3
Suppose nine students make the following scores on a test:

$$30, 35, 40, 40, 92, 92, 93, 98, 99$$

Is the median the best "average" to represent the set of scores?

Problem 32.4
Is the mode an appropriate "average" for the following test scores?

$$40, 42, 50, 62, 63, 65, 98, 98$$

Problem 32.5
The 20 meetings of a square dance club were attended by 26, 25, 28, 23, 25, 24, 24, 23, 26, 26, 28, 26, 24, 32, 25, 27, 24, 23, 24, and 22 of its members. Find the mode, median, and mean.

Problem 32.6
If the mean annual salary paid to the top of three executives of a firm is $\$96,000$, can one of them receive an annual salary of $\$300,000$?

Problem 32.7
An instructor counts the final examination in a course four times as much as of the four one-hour examinations. What is the average grade of a student who received grades of $74, 80, 61$, and 77 in the four one- hour examinations and 83 in the final examination?

Problem 32.8
In 1980 a college paid its 52 instructors a mean salary of $\$13,200$, its 96 assistant professors a mean salary of $\$15,800$, its 67 associate professors a mean salary of $\$18,900$, and its 35 full professors a mean salary of $\$23,500$. What was the mean salary paid to all the teaching staff of this college?

Measures of Dispersion: Range, Variance, and Standard Deviation
While mean and median tell you about the center of your observations, they say nothing about how data are scattered. **variability** or **dispersion** measures the extent to which data are spread out.
The measures of variability for data that we look at are: the range, the interquartile range, the variance, and the standard deviation.

The Range
To measure the variability between extreme values (i.e. smallest and largest values) one uses the **range**. The range is the difference between the largest and smallest values of a distribution.

Example 32.7
Find the range of each of the following samples:

Sample 1: 6,18,18,18,18,18,18,18,18,18.
Sample 2: 6,6,6,6,6,6,18,18,18,18,18.
Sample 3: 6,7,9,11,12,14,15,16,17,18.

Solution.
Sample 1: 18 - 6 = 12
Sample 2: 18 - 6 = 12
Sample 3: 18 - 6 = 12.■

As you can see from this example, each sample has a range $18 - 6 = 12$ but the spread out of values is quite different in each case. In Sample 1, the spread is uniform whereas it is not in Sample 3. This is a disadvantage of this kind of measure. The range tells us nothing about the dispersion of the values between the extreme (smallest and largest) values. A better understanding is obtained by determining **quartiles**.

Quartiles and Percentiles
Recall that if a set of data is arranged from smallest to largest, the middle value Q_2 (or the arithmetic mean of the two middle values) that divides the set of observations into two equal parts I_1 and I_2 is called the **median** of the distribution. That is, 50 percent of the observations are larger than the median and 50 percent are smaller.

Now, the median of I_1 is denoted by Q_1 (called the **lower quartile**) and that of I_2 by Q_3 (called the **upper quartile**). Thus, Q_1, Q_2 and Q_3 divide the set of data into four equal parts. We call Q_1, Q_2 and Q_3 the three **quartiles** of the distribution.
In a similar manner, one could construct other measures of variation by considering percentiles rather than quartiles. For a whole number P, where $1 \leq P \leq 99$, the Pth **percentile** of a distribution is a value such that $P\%$ of the data fall at or below it. Thus, there are 99 percentiles. The percentiles locations are found using the formula

$$L_P = (n+1) \cdot \frac{P}{100}.$$

Thus, the median Q_2 is the 50th percentile so that $P = 50$ and $L_{50} = \frac{n+1}{2}$ which is the median rank. Note that Q_1 is the 25th percentile and Q_3 is the 75th percentile.
An example will help to explain further.

Example 32.8
Listed below are the commissions earned, in dollars, last month by a sample of 15 brokers at Salomon Smith Barneys office.

2038	1758	1721	1637	2097
2047	2205	1787	2287	1940
2311	2054	2406	1471	1460

(a) Rank the data from smallest to largest.
(b) Find the median rank and then the median.
(c) Find the quartiles Q_1 and Q_3.

Solution.
(a) Arranging the data from smallest to largest we find

1460	1471	1637	1721	1758
1787	1940	2038	2047	2054
2097	2205	2287	2311	2406

(b) The median Q_2 is the 50th percentile so that $P = 50$ and $L_{50} = \frac{15+1}{2} = 8$. Thus, $Q_2 = 2038$.

(c) Q_1 is the 25th percentile so that $L_{25} = (15+1)\cdot\frac{25}{100} = 4$ and so $Q_1 = 1721$. Similarly, Q_3 is the 75th percentile so that $P = 75$ and $L_{75} = (15+1)\cdot\frac{75}{100} = 12$. Thus, $Q_3 = 2205$. ■

Percentiles are extensively used in such fields as educational testing. Undoubtedly some of you have had the experience of being told at what percentile you rated on a scholastic aptitude test.

Example 32.9
You took the English achievement test to obtain college credit in freshman English by examination. If your score was in the 89th percentile, what does this mean?

Solution.
This means 89% of the scores were at or below yours.■

Remark 32.1
In Example 32.8, L_P was found to be a whole number. That is not always the case. For example, if $n = 20$ observations then $L_{25} = (20+1)\cdot\frac{25}{100} = 5.25$. In this case, to find Q_1 we locate the fifth value in the ordered array and then move .25 of the distance between the fifth and sixth values and report that as the first quartile. Like the median, the quartile does not need to be one of the actual values in the data set.
To explain further, suppose a data set contained the six values: 91, 75, 61, 101, 43, and 104. We want to locate the first quartile. We order the values from smallest to largest: 43, 61, 75, 91, 101, and 104. The first quartile is located at

$$L_{25} = (6+1)\cdot\frac{25}{100} = 1.75.$$

The position formula tells us that the first quartile is located between the first and the second value and that it is .75 of the distance between the first and the second values. The first value is 43 and the second is 61. So the distance between these two values is 18. To locate the first quartile, we need to move .75 of the distance between the first and second values, so $.75(18) = 13.5$. To complete the procedure, we add 13.5 to the first value and report that the first quartile is 56.5.

Box-and-Whisker Plot
A **box-and-whisker plot** is a graphical display, based on quartiles, that

helps us picture a set of data. The steps for making such a box are as follows:

- Draw a vertical (or horizontal) scale to include the lowest and highest data values.
- To the right (or upper for the horizontal scale) of the scale draw a box from Q_1 to Q_3.
- Include a solid line through the box at the median level.
- Draw solid lines, called **whiskers**, from Q_1 to the lowest value and from Q_3 to the largest value.

An example will help to explain.

Example 32.10
Alexanders Pizza offers free delivery of its pizza within 15 km. Alex, the owner, wants some information on the time it takes for delivery. How long does a typical delivery take? Within what range of times will most deliveries be completed? For a sample of 20 deliveries, he determined the following information:

$$\begin{array}{lcl} \text{Minimum value} & = & 13 \text{ minutes} \\ Q_1 & = & 15 \text{ minutes} \\ \text{Median} & = & 18 \text{ minutes} \\ Q_3 & = & 22 \text{ minutes} \\ \text{Maximum value} & = & 30 \text{ minutes} \end{array}$$

Develop a box plot for the delivery times. What conclusions can you make about the delivery times?

Solution.
The box plot is shown in Figure 32.1.

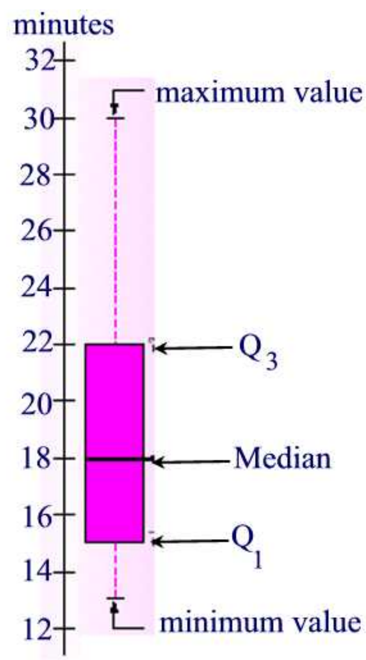


Figure 32.1

The box plot shows that the middle 50 percent of the deliveries take between 15 minutes and 22 minutes. The distance between the ends of the box, 7 minutes, is the **interquartile range**. The interquartile range is the distance between the first and the third quartile. It tells us the spread of the middle half of the data.
The box plot also reveals that the distribution of delivery times is positively skewed since the median is not in the center of the box and the distance from the first quartile to the median is smaller than the distance from the median to the third quartile.

Practice Problems

Problem 32.9

The following table gives the average costs of a single-lens reflex camera:

800	650	300	430	560	470	640	830
400	280	800	410	360	600	310	370

(a) Rank the data from smallest to largest.
(b) Find the quartiles $Q_1, Q_2,$ and Q_3.
(c) Make a box-and-whisker plot.

Problem 32.10
Mr. Eyha took a general aptitude test and scored in the 82nd percentile for aptitude in accounting. What percentage of the scores were at or below his score? What percentage were above?

Problem 32.11
At Center Hospital there is a concern about the high turnover of nurses. A survey was done to determine how long (in months) nurses had been in their current positions. The responses of 20 nurses were:

23	2	5	14	25	36	27	42	12	8
7	23	29	26	28	11	20	31	8	36

(a) Rank the data.
(b) Make a box-and-whisker plot of the data.
(c) What are your conclusions from the plot?

Variance and Standard Deviation
We have seen that a remedy for the deficiency of the range is the use of box and whisker plot. However, an even better measure of variability is the standard deviation. Unlike the range, the variance combines all the values in a data set to produce a measure of spread. The variance and the standard deviation are both measures of the spread of the distribution about the mean. If μ is the population mean of set of data then the quantity $(x-\mu)$ is called the **deviation from the mean**. The **variance** of a data set is the arithmetic average of squared deviation from the mean. It is denoted by s^2 and is given by the formula

$$s^2 = \frac{\sum_{i=1}^n (x_i-\mu)^2}{n}.$$

Note that the variance is nonnegative, and it is zero only if all observations are the same.
The **standard deviation** is the square root of the variance:

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i-\mu)^2}{n}}.$$

The variance is the nicer of the two measures of spread from a mathematical point of view, but as you can see from the algebraic formula, the physical unit of the variance is the square of the physical unit of the data. For example, if our variable represents the weight of a person in pounds, the variance measures spread about the mean in squared pounds. On the other hand, standard deviation measures spread in the same physical unit as the original data, but because of the square root, is not as nice mathematically. Both measures of spread are useful.

A step by step approach to finding the standard deviation is:

1. Calculate the mean.
2. Subtract the mean from each observation.
3. Square each result.
4. Add these squares.
5. Divide this sum by the number of observations.
6. Take the positive square root.

The variance and standard deviation introduced above are for a population. We define the **sample variance** by the formula

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

and the **sample standard deviation** by the formula

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}.$$

The reason that for s we use $n-1$ instead of n is because usually a sample does not contain extreme values and we want s to be an estimate of σ therefore by using $n-1$ we make s a little larger than if we divide by n.

Remark 32.2
Note that if the standard deviation is large than the data are more spread out whereas when the standard deviation is small then the data are more concentrated near the mean.

Example 32.11
The owner of the Ches Tahoe restaurant is interested in how much people spend at the restaurant. He examines 10 randomly selected receipts for parties of four and writes down the following data.

44 50 38 96 42 47 40 39 46 50

(a) Find the arithmetic mean.
(b) Find the variance and the standard deviation.

Solution.
(a) The arithmetic mean is the sum of the above values divided by 10,i.e., $\overline{x} = 49.2$
Below is the table for getting the standard deviation:

x	$x - 49.2$	$(x - 49.2)^2$
44	-5.2	27.04
50	0.8	0.64
38	11.2	125.44
96	46.8	2190.24
42	-7.2	51.84
47	-2.2	4.84
40	-9.2	84.64
39	-10.2	104.04
46	-3.2	10.24
50	0.8	0.64
Total		2600.4

Hence the variance is $s^2 = \frac{2600.4}{9} \approx 289$ and the standard deviation is $s = \sqrt{289} = 17$. What this means is that most of the patrons probably spend between 49.20 - 17 =\$32.20 and 49.20+17=\$66.20. ■

Practice Problems

Problem 32.12
The following are the wind velocities reported at 6 P.M.on six consecutive days: 13,8,15,11,3 and 10. Find the range, sample mean, sample variance, and sample standard deviation.

Problem 32.13
An airline's records show that the flights between two cities arrive on the average 4.6 minutes late with a standard deviation of 1.4 minutes. At least what percentage of its flights between these two cities arrive anywhere between 1.8 minutes late and 7.4 minutes late?

Problem 32.14
One patient's blood pressure, measured daily over several weeks, averaged 182 with a standard deviation of 5.3, while that of another patient averaged 124 with a standard deviation of 9.4. Which patient's blood pressure is relatively more variable?

Problem 32.15
By sampling different landscapes in a national park over a 2-year period, the number of deer per square kilometer was determined. The results were (deer per square kilometer)

30	20	5	29	58	7
20	18	4	29	22	9

Compute the range, sample mean, sample variance, and sample standard deviation.

Problem 32.16
A researcher wants to find the number of pets per household. The researcher conducts a survey of 35 households. Find the sample variance and standard deviation.

0	2	3	1	0
1	2	3	1	0
1	2	1	1	0
3	2	1	1	1
4	1	2	2	4
3	2	1	2	3
2	3	4	0	2

Problem 32.17
Suppose two machines produce nails which are on average 10 inches long. A sample of 11 nails is selected from each machine.

Machine A: 6, 8, 8, 10, 10, 10, 10, 10, 12, 12, 14.
Machine B: 6, 6, 6, 8, 8, 10, 12, 12, 14, 14, 14.

Which machine is better than the other?

Problem 32.18
Find the missing age in the following set of four student ages.

Student	Age	Deviation from the Mean
A	19	-4
B	20	-3
C	?	1
D	29	6

Problem 32.19
The maximum heart rates achieved while performing a particular aerobic exercise routine are measured (in beats per minute) for 9 randomly selected individuals.

145 155 130 185 170 165 150 160 125

(a) Calculate the range of the time until failure.
(b) Calculate the sample variance of the time until failure.
(c) Calculate the sample standard variation of the time until failure.

Problem 32.20
The following data gives the number of home runs that Babe Ruth hit in each of his 15 years with the New York Yankees baseball team from 1920 to 1934:

54 59 35 41 46 25 47 60 54 46 49 46 41 34 22.

The following are the number of home runs that Roger Maris hit in each of the ten years he played in the major leagues from 1957 on:

8 13 14 16 23 26 28 33 39 61

Calculate the mean and standard deviation for each player's data and comment on the consistency of performance of each player.

Problem 32.21
An office of Price Waterhouse Coopers LLP hired five accounting trainees this year. Their monthly starting salaries were: \$2536; \$2173; \$2448; \$2121; and \$2622.

(a) Compute the population mean.
(b) Compute the population variance.
(c) Compute the population standard deviation.

Normal Distribution

To better understand how standard deviations are used as measures of dispersion, we next consider normal distributions. The graph of a normal distribution is called a **normal curve** or a **bell-shaped curve.** The curve has the following properties:

1. The curve is bell-shaped with the highest point over the mean μ.
2. It is symmetrical about the line through μ.
3. The curve approaches the horizontal axis but never touches or crosses it.
4. The points where the curve changes concavity occur at $\mu+\sigma$ and $\mu-\sigma$.
5. The total area under the curve is assumed to be 1.(0.5 to the left of the mean and 0.5 to the right).

The data for a normal distribution are spread according to the following rules (See Figure 32.2):

- About 68 percent of the data values will lie within one standard deviation of the mean.
- About 95 percent of the data values will lie within two standard deviations of the mean.
- About 99.7 percent of the data values will lie within three standard deviations of the mean.

This result is sometimes referred to as the **emperical rule**, because the given percentages are observed in practice.

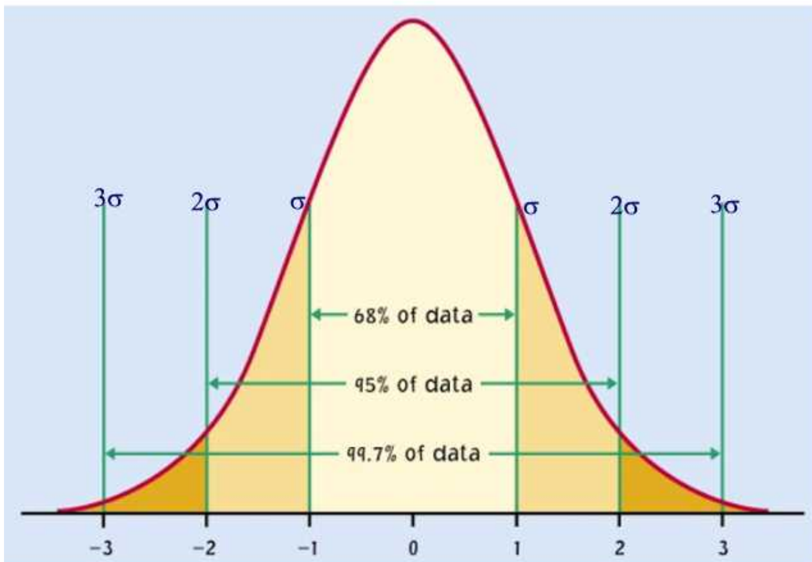


Figure 32.2

Example 32.12
When a standardized test was scored, there was a mean of 500 and a standard deviation of 100. Suppose that 10,000 students took the test and their scores had a bell-shaped distribution.

(a) How many scored between 400 and 600?
(b) How many scored between 300 and 700?
(c) How many scored between 200 and 800?

Solution.
(a) Since one standard deviation on either side of the mean is from 400 to 600, about 68% of the scores fall in this interval. Thus, $0.68 \times 10,000 = 6800$ students scored between 400 and 600.
(b) About 95% of 10,000 or 9500 students scored between 300 and 700.
(c) About 99.7% of 10,000 or 9970 students scored between 200 and 800■

By the discussion above we get information of the percentage of data within a certain number of standard deviations. However, if we want to find the location of a data value x from the mean we can use the so-called **z-score** given by the formula

$$z = \frac{x - \mu}{\sigma}$$

Now, if the z-score is given then the raw data can be found by solving the equation $z = \frac{x-\mu}{\sigma}$ for x to obtain

$$x = \mu + z\sigma.$$

Note that from this formula we see that the value of z tells us how many standard deviation the corresponding value of x lies above (if $z > 0$) or below (if $z < 0$) the mean of its distribution.

Example 32.13
Scores on intelligence tests (IQs) are normally distributed in children. IQs from the Wechsler intelligence tests are known to have means of 100 and standard deviations of 15. In almost all the states in the United States (Pennsylvania and Nebraska are exceptions) children can be labeled as mentally retarded if their IQ falls to 70 points or below. What is the maximum z score one could obtain on an intelligence test and still be considered to be mentally retarded?

Solution.
Applying the z-score formula we find $z = \frac{70-100}{15} = -2.$ ■

Example 32.14
In a certain city the mean price of a quart of milk is 63 cents and the standard deviation is 8 cents. The average price of a package of bacon is \$1.80 and the standard deviation is 15 cents. If we pay \$0.89 for a quart of milk and \$2.19 for a package of bacon at a 24-hour convenience store, which is relatively more expensive?

Solution.
To answer this, we compute z-scores for each:

$$z_{Milk} = \frac{0.89 - 0.63}{0.08} = 3.25$$

and

$$z_{Bacon} = \frac{2.19 - 1.80}{0.15} = 2.60.$$

Our z-scores show us that we are overpaying quite a bit more for the milk than we are for the bacon.■

Example 32.15
Graduate Record Examination (GRE) scores have means equal to 500 and standard deviations of 100. If a person receives a z-score on the GRE of 1.45, what would their raw score be?

Solution.
Using the formula $x = \mu + z\sigma$ we find $x = 500 + 1.45(100) = 645$.■

Practice Problems

Problem 32.22
On a final examination in Statistics, the mean was 72 and the standard deviation was 15. Assuming normal distribution, determine the z-score of students receiving the grades (a) 60, (b) 93, and (c) 72.

Problem 32.23
Referring to the previous exercise, find the grades corresponding to the z-score $z = 1.6$.

Problem 32.24
If $z_1 = 0.8$, $z_2 = -0.4$ and the corresponding x-values are $x_1 = 88$ and $x_2 = 64$ then find the mean and the standard deviation, assuming we have a normal distribution.

Problem 32.25
A student has computed that it takes an average (mean) of 17 minutes with a standard deviation of 3 minutes to drive from home, park the car, and walk to an early morning class. Assuming normal distribution,

(a) One day it took the student 21 minutes to get to class. How many standard deviations from the average is that?
(b) Another day it took only 12 minutes for the student to get to class. What is this measurement in standard units?
(c) Another day it took him 17 minutes to be in class. What is the z-score?

Problem 32.26
Mr. Eyha's z-score on a college exam is 1.3. If the x-scores have a mean of 480 and a standard deviation of 70 points, what is his x-score?

Problem 32.27
(a) If $\mu = 80, \sigma = 10$, what is the z-score for a person with a score of 92?
(b) If $\mu = 65, \sigma = 12$, what is the raw score for a z-score of -1.5?

Problem 32.28
Sketch a normal curve. Mark the axis corresponding to the parameter μ and the axis corresponding to $\mu + \sigma$ and $\mu - \sigma$.

Problem 32.29
For the population of Canadian high school students, suppose that the number of hours of TV watched per week is normally distributed with a mean of 20 hours and a standard deviation of 4 hours. Approximately, what percentage of high school students watch

(a) between 16 and 24 hours per week?
(b) between 12 and 28 hours per week?
(c) between 8 and 32 hours per week?

Problem 32.30
The length of human pregnancies from conception to birth varies according to a distribution that is approximately normal with mean 266 days and standard deviation 16 days. Use the emperical rule to answer the following questions.

(a) Between what values do the lengths of the middle 95% of all pregnancies fall?
(b) How short are the shortest 2.5% of all pregnancies?

33 Probability: Some Basic Terms

In this and the coming sections we discuss the fundamental concepts of probability at a level at which no previous exposure to the topic is assumed.
Probability has been used in many applications ranging from medicine to business and so the study of probability is considered an essential component of any mathematics curriculum.
So what is probability? Before answering this question we start with some basic definitions.

An **experiment** is any situation whose outcome cannot be predicted with certainty. Examples of an experiment include rolling a die, flipping a coin, and choosing a card from a deck of playing cards.
By an **outcome** or **simple event** we mean any result of the experiment. For example, the experiment of rolling a die yields six outcomes, namely, the outcomes 1,2,3,4,5, and 6.
The **sample space** S of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $S = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die.
An **event** is a subset of the sample space. For example, the event of rolling an odd number with a die consists of three simple events $\{1, 3, 5\}$.

Example 33.1
Consider the random experiment of tossing a coin three times.

(a) Find the sample space of this experiment.
(b) Find the outcomes of the event of obtaining more than one head.

Solution.
(a) The sample space is composed of eight simple events:

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

(b) The event of obtaining more than one head is the set $\{THH, HTH, HHT, HHH\}$.■

Example 33.2
Consider the random experiment of rolling a die.

(a) Find the sample space of this experiment.
(b) Find the event of rolling the die an even number.

Solution.
(a) The sample space is composed of six simple events:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

(b) The event of rolling the die an even number is the set $\{2, 4, 6\}$.■

Example 33.3
An experiment consists of the following two stages: (1) first a fair die is rolled and the number of dots recorded, (2) if the number of dots appearing is even, then a fair coin is tossed and its face recorded, and if the number of dots appearing is odd, then the die is tossed again, and the number of dots recorded. Find the sample space of this experiment.

Solution.
The sample space for this experiment is the set of 24 ordered pairs

$$\begin{gathered}\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,H),(2,T),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,H),(4,T),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,H),(6,T)\}\blacksquare.\end{gathered}$$

Probability is the measure of occurrence of an event. If the event is impossible to occur then its probability is 0. If the occurrence is certain then the probability is 1. The closer to 1 the probability is, the more likely the event is. The probability of occurrence of an event E (called its **success**) will be denoted by $P(E)$. If an event has no outcomes, that is as a subset of S if $E = \emptyset$ then $P(\emptyset) = 0$. On the other hand, if $E = S$ then $P(S) = 1$.

Example 33.4
A hand of 5 cards is dealt from a deck. Let E be the event that the hand contains 5 aces. List the elements of E.

Solution.
Recall that a standard deck of 52 playing cards can be described as follows:

hearts (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
clubs (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
diamonds (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
spades (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King

Cards labeled Jack, Queen, or King are called **face cards.**
Since there are only 4 aces in the deck, event E cannot occur. Hence E is an impossible event and $E = \emptyset$ so that $P(E) = 0$ ■

When the outcome of an experiment is just as likely as another, as in the example of tossing a coin, the outcomes are said to be **equally likely**. Various probability concepts exist nowadays. The **classical** probability concept applies *only* when all possible outcomes are equally likely, in which case we use the formula

$$P(E) = \frac{\text{number of outcomes favorable to event}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)},$$

where $n(E)$ denotes the number of elements in E.
Since for any event E we have $\emptyset \subseteq E \subseteq S$ then $0 \leq n(E) \leq n(S)$ so that $0 \leq \frac{n(E)}{n(S)} \leq 1$. It follows that $0 \leq P(E) \leq 1$.

Example 33.5
Which of the following numbers cannot be the probability of some event?
(a) 0.71 (b)-0.5 (c) 150% (d) $\frac{4}{3}$.

Solution.
Only (a) can represent the probability of an event. The reason that -0.5 is not because it is a negative number. As for 150% this is a number greater than 1 and so cannot be a probability of an event. The same is true for $\frac{4}{3}$.■

Example 33.6
What is the probability of drawing an ace from a well-shuffled deck of 52 playing cards?

Solution.
Since there are four aces in a deck of 52 playing cards then the probabiltiy of getting an ace is $\frac{4}{52} = \frac{1}{13}$.■

Example 33.7
What is the probability of rolling a 3 or a 4 with a fair die?

Solution.
Since the event of having a 3 or a 4 has two simple events $\{3, 4\}$ then the

probability of rolling a 3 or a 4 is $\frac{2}{6} = \frac{1}{3}$.■

It is important to keep in mind that the above definition of probability applies only to a sample space that has equally likely outcomes. Applying the definition to a space with outcomes that are not equally likely leads to incorrect conclusions. For example, the sample space for spinning the spinner in Figure 33.1 is given by S={Red,Blue}, but the outcome Blue is more likely to occur than is the outcome Red. Indeed, $P(Blue) = \frac{3}{4}$ whereas $P(Red) = \frac{1}{4}$.

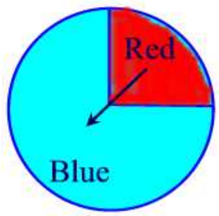


Figure 33.1

A widely used probability concept is the **experimental** probability which uses the relative frequency of an event and is given by the formula:

$$P(E) = \text{Relative frequency} = \frac{f}{n},$$

where f is the frequency of the event and n is the size of the sample space.

Example 33.8
Personality types are broadly defined according to four main preferences. Do married couples choose similar or different personality types in their mates? The following data give an indication:

Similarities and Differences in a Random Sample of 545 Married Couples

Number of Similar Preferences	Number of Married Couples
All four	108
Three	59
Two	146
One	141
None	91

Suppose that a married couple is selected at random. Use the data to estimate the probability (to two decimal places) that they will have no personality preferences in common.

Solution.
The probability they will have no personality preferences in common is $P(0) = \frac{91}{545} = 0.17$■

Next, we define the probability of nonoccurrence of an event E (called its **failure**) to be the number $P(E^c)$. Not surprisingly, the probabilities of an event E and its complement E^c are related. The probability of the event E^c is easily found from the identity

$$\frac{\text{number of outcomes in A}}{\text{total number of outcomes}} + \frac{\text{number of outcomes not in A}}{\text{total number of outcomes}} = 1,$$

so that $P(E) + P(E^c) = 1$.

Example 33.9
The probability that a college student without a flu shot will get the flu is 0.45. What is the probability that a college student without the flu shot will not get the flu?

Solution.
Let E denote the event with outcomes those students without a flu shot. Then $P(E) = 0.45$. The probability that a student without the flu shot will not get the flu is then $P(E^c) = 1 - P(E) = 1 - 0.45 = 0.55$.■

Practice Problems

Problem 33.1
An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.

Problem 33.2
Three coins are thrown. List the outcomes which belong to each of the following events.

(a) exactly two tails (b) at least two tails (c) at most two tails.

Problem 33.3
For each of the following events A, B, C, list and count the number of outcomes it contains and hence calculate the probability of A, B or C occurring.

(a) A = "throwing 3 or higher with one die",
(b) B = "throwing exactly two heads with three coins",
(c) C = "throwing a total score of 14 with two dice".

Problem 33.4
An experiment consists of throwing two four-faced dice.

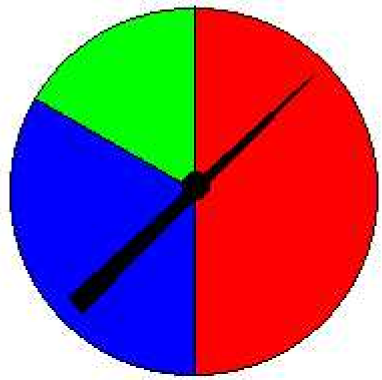
(a) Write down the sample space of this experiment.
(b) If E is the event total score is at least 4 list the outcomes belonging to E^c.
(c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?
(d) What is the probability that a double: (i.e. $\{(1,1),(2,2),(3,3),(4,4)\}$) will not be thrown?
(e) What is the probability that a double is not thrown nor is the score greater than 6?

Problem 33.5
A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. One article is chosen at random. Find the probability that:

(a) it has no defects,
(b) it has no major defects,
(c) it is either good or has major defects.

Problem 33.6
Consider the experiment of spinning the pointer on the game spinner pictured below. There are three possible outcomes, that is, when the pointer stops it must point to one of the three colors. (We rule out the possibility of landing on the border between two colors.)

(a) What is the probability that the spinner is pointing to the red area?
(b) What is the probability that the spinner is pointing to the blue area?
(c) What is the probability that the spinner is pointing to the green area?



Problem 33.7
Consider the experiment of flipping a coin three times. If we denote a head by H and a tail by T, we can list the 8 possible ordered outcomes as (H,H,H), (H,H,T) each of which occurs with probability of 1/8. Finish listing the remaining members of the sample space. Calculate the probability of the following events:

(a) All three flips are heads.
(b) Exactly two flips are heads.
(c) The first flip is tail.
(d) At least one flip is head.

Problem 33.8
Suppose an experimet consists of drawing one slip of paper from a jar containing 12 slips of paper, each with a different month of the year written on it. Find each of the following:

(a) The sample space S of the experiment.
(b) The event A consisting of the outcomes having a month begining with J.
(c) The event B consisting of outcomes having the name of a month that has exactly four letters.
(d) The event C consisting of outcomes having a month that begins with M or N.

Problem 33.9
Let $S = \{1, 2, 3, \cdots, 25\}$. If a number is chosen at **random**, that is, with the same chance of being drawn as all other numbers in the set, calculate each

of the following probabilities:

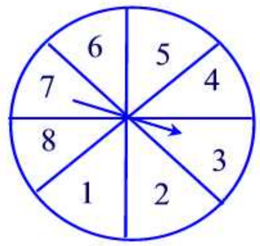
(a) The even A that an even number is drawn.
(b) The event B that a number less than 10 and greater than 20 is drawn.
(c) The event C that a number less than 26 is drawn.
(d) The event D that a prime number is drawn.
(e) The event E that a number both even and prime is drawn.

Problem 33.10
Consider the experiment of drawing a single card from a standard deck of cards and determine which of the following are sample spaces with equally likely outcomes:
(a) {face card, not face card}
(b) {club, diamond, heart, spade}
(c) {black, red}
(d) {king, queen, jack, ace, even card, odd card}

Problem 33.11
An experiment consists of selecting the last digit of a telephone number. Assume that each of the 10 digits is equally likely to appear as a last digit. List each of the following:
(a) The sample space
(b) The event consisting of outcomes that the digit is less than 5
(c) The event consisting of outcomes that the digit is odd
(d) The event consisting of outcomes that the digit is not 2
(e) Find the probability of each of the events in (b) - (d)

Problem 33.12
Each letter of the alphabet is written on a seperate piece of paper and placed in a box and then one piece is drawn at random.
(a) What is the probability that the selected piece of paper has a vowel written on it?
(b) What is the probability that it has a consonant written on it?

Problem 33.13
The following spinner is spun:



Find the probabilities of obtaining each of the following:
(a) P(factor of 35)
(b) P(multiple of 3)
(c) P(even number)
(d) P(11)
(e) P(composite number)
(f) P(neither prime nor composite)

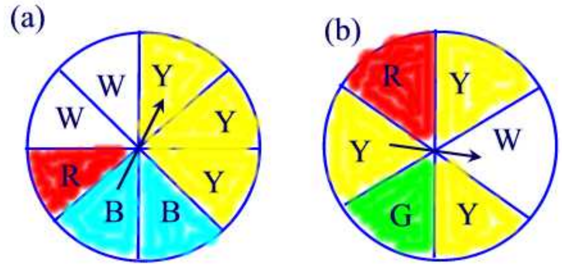
Problem 33.14
An experiment consists of tossing four coins. List each of the following.

(a) The sample space
(b) The event of a head on the first coin
(c) The event of three heads

Problem 33.15
Identify which of the following events are certain, impossible, or possible.
(a) You throw a 2 on a die
(b) A student in this class is less than 2 years old
(c) Next week has only 5 days

Problem 33.16
Two dice are thrown. If each face is equally likely to turn up, find the following probabilities.
(a) The sum is even
(b) The sum is not 10
(c) The sum is a prime
(d) The sum is less than 9
(e) The sum is not less than 9

Problem 33.17
What is the probability of getting yellow on each of the following spinners?



Problem 33.18
A department store's records show that 782 of 920 women who entered the store on a Saturday afternoon made at least one purchase. Estimate the probability that a woman who enters the store on a Saturday afternoon will make at least one purchase.

Problem 33.19
Suppose that a set of 10 rock samples includes 3 that contain gold nuggets. If you were to pick up a sample at random, what is the probability that it includes a gold nugget?

Problem 33.20
When do creative people get their best ideas? A magazine did a survey of 414 inventors (who hold U.S. patents) and obtained the following information:

Time of Day When Best Ideas Occur

Time	Number of Inventors
6 A.M. - 12 noon	46
12 noon - 6 P.M.	188
6 P.M. - 12 midnight	63
12 midnight - 6 A.M.	117

Assuming that the time interval includes the left limit and all the times up to but not including the right limit, estimate the probability (to two decimal places) that an inventor has a best idea during the time interval 6 A.M. - 12 noon.

Probability of Union and Intersection of Two Events
The **union** of two events A and B is the event $A \cup B$ whose outcomes are either in A or in B. The **intersection** of two events A and B is the event $A \cap B$ whose outcomes are outcomes of both events A and B. Two events A and B are said to be **mutually exclusive** if they have no outcomes in common. In this case $A \cap B = \emptyset$. Thus, $P(A \cap B) = P(\emptyset) = 0$.

Example 33.10
Consider the sample space of rolling a die. Let A be the event of rolling an even number, B the event of rolling an odd number, and C the event of rolling a 2. Find

(a) $A \cup B, A \cup C$, and $B \cup C$.
(b) $A \cap B, A \cap C$, and $B \cap C$.
(c) Which events are mutually exclusive?

Solution.
(a) We have

$$\begin{array}{rcl} A \cup B & = & \{1,2,3,4,5,6\} \\ A \cup C & = & \{2,4,6\} \\ B \cup C & = & \{1,2,3,5\} \end{array}$$

(b)

$$\begin{array}{rcl} A \cap B & = & \emptyset \\ A \cap C & = & \{2\} \\ B \cap C & = & \emptyset \end{array}$$

(c) A and B are mutually exclusive as well as B and C.$\blacksquare$

Example 33.11
Let A be the event of drawing a King from a well-shuffled standard deck of playing cards and B the event of drawing a "ten" card. Are A and B mutually exclusive?

Solution.
Since $A = \{\text{king of diamonds, king of hearts, king of clubs, king of spades}\}$ and $B = \{\text{ten of diamonds, ten of hearts, ten of clubs, ten of spades}\}$ then A and B are mutually exclusive since there are no cards common to both events. $\blacksquare$

The next result provides a relationship between the probabilities of the events A, B, and $A \cup B$ when A and B are mutually exclusive.

Theorem 33.1
If A and B are mutually exclusive events of a sample space S than $P(A \cup B) = P(A) + P(B)$.

Proof.
Since A and B are mutually exclusive then A and B have no elements in common so that $n(A \cup B) = n(A) + n(B)$. Thus,

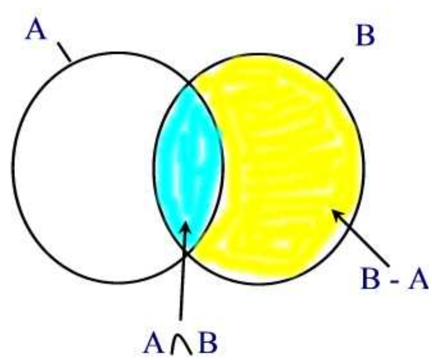
$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = P(A) + P(B). \blacksquare$$

Now, for any events A and B, not necessarily exclusive, the probability of $A \cup B$ is given by the addition rule.

Theorem 33.2
Let A and B be two events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof.
Let $B - A$ denote the event whose outcomes are the outcomes in B that are not in A. Then using the Venn diagram below we see that $B = (A \cap B) \cup (B - A)$ and $A \cup B = A \cup (B - A)$.



Since $(A \cap B)$ and $(B - A)$ are disjoint then

$$P(B) = P(A \cap B) + P(B - A).$$

Thus,

$$P(B - A) = P(B) - P(A \cap B).$$

Similarly, since A and $B - A$ are disjoint then

$$P(A \cup B) = P(A) + P(B - A) = P(A) + P(B) - P(A \cap B). \blacksquare$$

Example 33.12
Let $P(A) = 0.9$ and $P(B) = 0.6$. Find the minimum possible value for $P(A \cap B)$.

Solution.
Since $P(A) + P(B) = 1.5$ and $0 \leq P(A \cup B) \leq 1$ then by the previous theorem

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq 1.5 - 1 = 0.5.$$

So the minimum value of $P(A \cap B)$ is $0.5 \blacksquare$

Example 33.13
Suppose there's 40% chance of colder weather, 10% chance of rain and colder weather, 80% chance of rain or colder weather. Find the chance of rain.

Solution.
By the addition rule we have

$$P(R) = P(R \cup C) - P(C) + P(R \cap C) = 0.8 - 0.4 + 0.1 = 0.5. \blacksquare$$

Practice Problem

Problem 33.21
Which of the following are mutually exclusive? Explain your answers.

(a) A driver getting a ticket for speeding and a ticket for going through a red light.
(b) Being foreign-born and being President of the United States.

Problem 33.22
If A and B are the events that a consumer testing service will rate a given stereo system very good or good, $P(A) = 0.22$, $P(B) = 0.35$. Find
(a) $P(A^c)$;
(b) $P(A \cup B)$;
(c) $P(A \cap B)$.

Problem 33.23
If the probabilities are 0.20, 0.15, and 0.03 that a student will get a failing grade in Statistics, in English, or in both, what is the probability that the student will get a failing grade in at least one of these subjects?

Problem 33.24
If A is the event "drawing an ace" from a deck of cards and B is the event "drawing a spade". Are A and B mutually exclusive? Find $P(A \cup B)$.

Problem 33.25
A bag contains 18 coloured marbles: 4 are coloured red, 8 are coloured yellow and 6 are coloured green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

Problem 33.26
Show that for any events A and B, $P(A \cap B) \geq P(A) + P(B) - 1$.

Problem 33.27
A golf bag contains 2 red tees, 4 blue tees, and 5 white tees.

(a) What is the probability of the event R that a tee drawn at random is red?
(b) What is the probability of the event "not R" that is, that a tee drawn at random is not red?
(c) What is the probability of the event that a tee drawn at random is either red or blue?

Problem 33.28
A fair pair of dice is rolled. Let E be the event of rolling a sum that is an even number and P the event of rolling a sum that is a prime number. Find the probability of rolling a sum that is even or prime?

Problem 33.29
If events A nd B are from the same sample space, and if P(A)=0.8 and P(B)=0.9, can events A and B be mutually exclusive?

34 Probability and Counting Techniques

If you recall that the classical probability of an event $E \subseteq S$ is given by

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ and $n(S)$ denote the number of elements of E and S respectively. Thus, finding $P(E)$ requires counting the elements of the sample space S. Sometimes the sample space is so large that shortcuts are needed to count all the possibilities.
All the examples discussed thus far have been experiments consisting of one action such as tossing three coins or rolling two dice. We now want to consider experiments that consist of doing two or more actions in succession. For example, consider the experiment of drawing two balls in succession and with replacement from a box containing one red ball (R), one white ball (W), and one green ball (G). The outcomes of this experiment, i.e. the elements of the sample space can be found in two different ways by using

Organized Table or an Orderly List
An organized table of our experiment looks like

	R	W	G
R	RR	RW	RG
W	WR	WW	WG
G	GR	GW	GG

Thus, there are nine equally likely outcomes so that

$$S = \{RR, RW, RG, WR, WW, WG, GR, GW, GG\}$$

Tree Diagrams
An alternative way to generate the sample space is to use a tree diagram as shown in Figure 34.1.

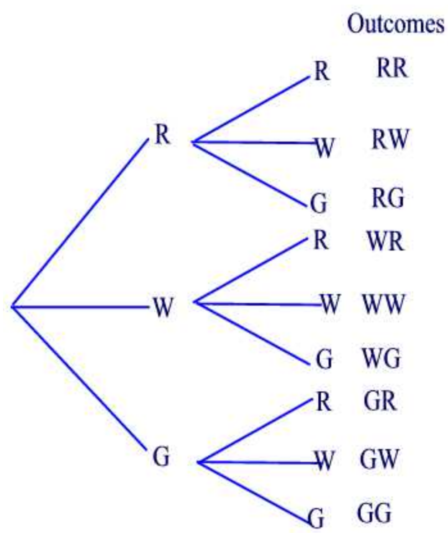


Figure 34.1

Example 34.1
Show the sample space for tossing one penny and rolling one die. (H = heads, T = tails)

Solution.
According to Figure 34.2, the sample space is

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

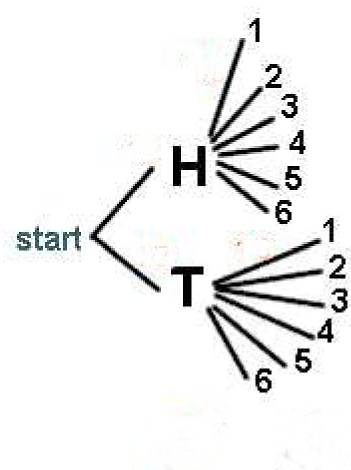


Figure 34.2

Fundamental Principle of Counting
If there are many stages to an experiment and several possibilities at each stage, the tree diagram associated with the experiment would become too large to be manageable. For such problems the counting of the outcomes is simplified by means of algebraic formulas. The commonly used formula is the **multiplication rule of counting** which states:
" If a choice consists of k steps, of which the first can be made in n_1 ways, for each of these the second can be made in n_2 ways,... and for each of these the kth can be made in n_k ways, then the whole choice can be made in $n_1 \cdot n_2 \cdot ... n_k$ ways."

Example 34.2
How many license-plates with 3 letters followed by 3 digits exist?

Solution.
A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third letter, (4) choose the first digit, (5) choose the second digit, and (6) choose the third digit. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ ways.$\blacksquare$

Example 34.3
How many numbers in the range 1000 - 9999 have no repeated digits?

Solution.
A 4-step process: (1) Choose first digit, (2) choose second digit, (3) choose third digit, (4) choose fourth digit. Every step can be done in a number of ways that does not depend on previous choices, and each number can be specified in this manner. So there are $9 \cdot 9 \cdot 8 \cdot 7 = 4,536$ ways.$\blacksquare$

Example 34.4
How many license-plates with 3 letters followed by 3 digits exist if exactly one of the digits is 1?

Solution.
In this case, we must pick a place for the 1 digit, and then the remaining digit places must be populated from the digits $\{0, 2, \cdots 9\}$. A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third

letter, (4) choose which of three positions the 1 goes, (5) choose the first of the other digits, and (6) choose the second of the other digits. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 3 \cdot 9 \cdot 9 = 4,270,968$ ways.■

Practice Problems

Problem 34.1
If each of the 10 digits is chosen at random, how many ways can you choose the following numbers?

(a) A two-digit code number, repeated digits permitted.
(b) A three-digit identification card number, for which the first digit cannot be a 0.
(c) A four-digit bicycle lock number, where no digit can be used twice.
(d) A five-digit zip code number, with the first digit not zero.

Problem 34.2
(a) If eight horses are entered in a race and three finishing places are considered, how many finishing orders can they finish?
(b) If the top three horses are Lucky one, Lucky Two, and Lucky Three, in how many possible orders can they finish?

Problem 34.3
You are taking 3 shirts(red, blue, yellow) and 2 pairs of pants (tan, gray) on a trip. How many different choices of outfits do you have?

Problem 34.4
The state of Maryland has automobile license plates consisting of 3 letters followed by three digits. How many possible license plates are there?

Problem 34.5
A club has 10 members. In how many ways can the club choose a president and vice-president if everyone is eligible?

Problem 34.6
A lottery allows you to select a two-digit number. Each digit may be either 1,2 or 3. Use a tree diagram to show the sample space and tell how many different numbers can be selected.

Problem 34.7
In a medical study, patients are classified according to whether they have blood type A, B, AB, or O, and also according to whether their blood pressure is low, normal, or high. Use a tree diagram to represent the various outcomes that can occur.

Problem 34.8
If a travel agency offers special weekend trips to 12 different cities, by air, rail, or bus, in how many different ways can such a trip be arranged?

Problem 34.9
If twenty paintings are entered in art show, in how many different ways can the judges award a first prize and a second prize?

Problem 34.10
In how many ways can the 52 members of a labor union choose a president, a vice-president, a secretary, and a treasurer?

Problem 34.11
Find the number of ways in which four of ten new movies can be ranked first, second, third, and fourth according to their attendance figures for the first six months.

Problem 34.12
To fill a number of vacancies, the personnel manager of a company has to choose three secretaries from among ten applicants and two bookkeepers from among five applicants. In how many different ways can the personnel manager fill the five vacancies?

Finding Probabilities Using the Fundamental Principle of Counting
The Fundamental Principle of Counting can be used to compute probabilities as shown in the following example.

Example 34.5
A quizz has 5 multiple-choice questions. Each question has 4 answer choices, of which 1 is correct answer and the other 3 are incorrect. Suppose that you guess all the answers.

(a) How many ways are there to answer the 5 questions?
(b) What is the probability of getting all 5 questions right?
(c) What is the probability of getting exactly 4 questions right and 1 wrong?
(d) What is the probability of doing well (getting at leat 4 right)?

Solution.
(a) We can look at this question as a decision consisting of five steps. There are 4 ways to do each step so that by the Fundamental Principle of Counting there are

$$(4)(4)(4)(4)(4) = 1024 \text{ ways}$$

(b) There is only one way to answer each question correctly. Using the Fundamental Principle of Counting there is $(1)(1)(1)(1)(1) = 1$ way to answer all 5 questions correctly out of 1024 possibilities. Hence,

$$\text{P(all 5 right)} = \tfrac{1}{1024}$$

(c) The following table lists all possible responses that involve at least 4 right answers, R stands for right and W stands for a wrong answer

Five Responses	Number of ways to fill out the test
WRRRR	$(3)(1)(1)(1)(1) = 3$
RWRRR	$(1)(3)(1)(1)(1) = 3$
RRWRR	$(1)(1)(3)(1)(1) = 3$
RRRWR	$(1)(1)(1)(3)(1) = 3$
RRRRW	$(1)(1)(1)(1)(3) = 3$

So there are 15 ways out of the 1024 possible ways that result in 4 right answers and 1 wrong answer so that

$$\text{P(4 right,1 wrong)} = \tfrac{15}{1024} \approx 1.5\%$$

(d) "At least 4" means you can get either 4 right and 1 wrong or all 5 right. Thus,

$$\text{P(at least 4 right)} = \text{P(4 right, 1 wrong)} + \text{P(5 right)} = \\ \tfrac{15}{1024} + \tfrac{1}{1024} = \tfrac{16}{1024} \approx 0.016 \blacksquare$$

Probability Trees
Probability trees can be used to compute the probabilities of combined outcomes in a sequence of experiments.

Example 34.6
Construct the probability tree of the experiment of flipping a fair coin twice.

Solution.
The probability tree is shown in Figure 34.3■

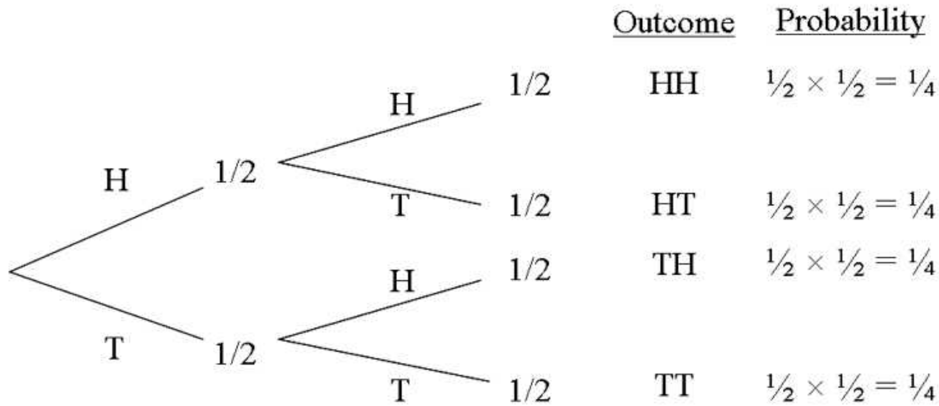


Figure 34.3

The probabilities shown in Figure 34.3 are obtained by following the paths leading to each of the four outcomes and multiplying the probabilities along the paths. This procedure is an instance of the following general property

Multiplication Rule for Probabilities for Tree Diagrams
For all multistage experiments, the probability of the outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Example 34.7
Suppose that out of 500 computer chips there are 9 defective. Construct the probability tree of the experiment of sampling two of them without replacement.

Solution.
The probability tree is shown in Figure 34.4■

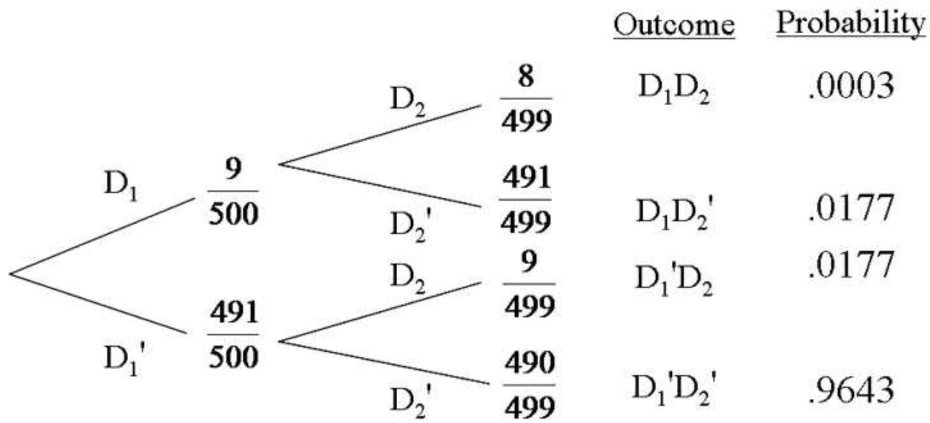


Figure 34.4

Practice Problems

Problem 34.13
A box contains three red balls and two blue balls. Two balls are to be drawn without replacement. Use a tree diagram to represent the various outcomes that can occur. What is the probability of each outcome?

Problem 34.14
Repeat the previous exercise but this time replace the first ball before drawing the second.

Problem 34.15
If a new-car buyer has the choice of four body styles, three engines, and ten colors, in how many different ways can s/he order one of these cars?

Problem 34.16
A jar contains three red gumballs and two green gumballs. An experiment consists of drawing gumballs one at a time from the jar, without replacement, until a red one is obtained. Find the probability of the following events.

A: Only one draw is needed.
B: Exactly two draws are needed.
C: Exactly three draws are needed.

Problem 34.17
Consider a jar with three black marbles and one red marble. For the experiment of drawing two marbles with replacement, what is the probability of drawing a black marble and then a red marble in that order?

Problem 34.18
A jar contains three marbles, two black and one red. Two marbles are drawn with replacement. What is the probability that both marbles are black? Assume that the marbles are equally likely to be drawn.

Problem 34.19
A jar contains four marbles-one red, one green, one yellow, and one white. If two marbles are drawn without replacement from the jar, what is the probability of getting a red marble and a white marble?

Problem 34.20
A jar contains 3 white balls and 2 red balls. A ball is drawn at random from the box and not replaced. Then a second ball is drawn from the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Problem 34.21
Suppose that a ball is drawn from the box in the previous problem, its color recorded, and then it is put back in the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Problem 34.22
Suppose there are 19 balls in an urn. They are identical except in color. 16 of the balls are black and 3 are purple. You are instructed to draw out one ball, note its color, and set it aside. Then you are to draw out another ball and note its color. What is the probability of drawing out a black on the first draw and a purple on the second?

Binary Experiments
Binary experiments are experiments with exactly two outcomes such as coin-tossing. Our first question is to find the total number of outcomes when a coin is tossed n times which is equivalent to saying that n coins are tossed. So what we have here is a decision consisting of n steps each step has two outcomes (head/tail) so by the Fundamental Principle of Counting there are

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n\ factors} = 2^n \text{ outcomes}$$

Next, let's list the outcomes. For simplicity, assume $n = 3$. In this case, the tree diagram of Figure 34.5 lists all the outcomes of tossing one, two, and three coins.

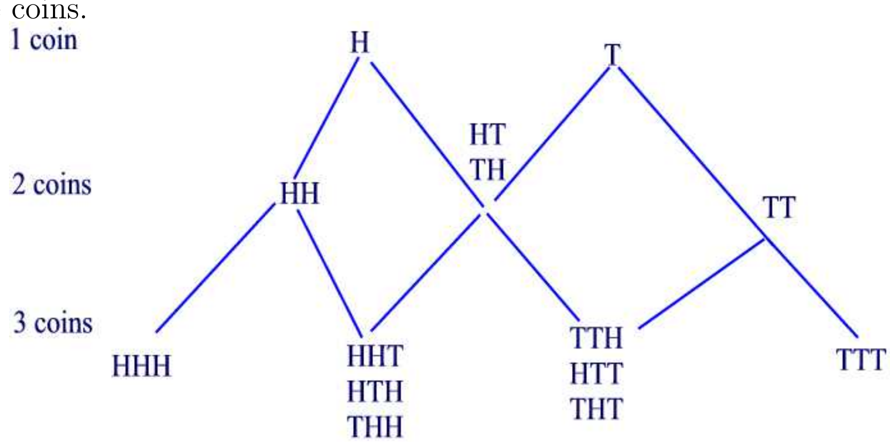


Figure 34.5

From Figure 34.5 we can create a tree diagram that counts the coin outcomes with a given number of heads as shown in Figure 34.6.

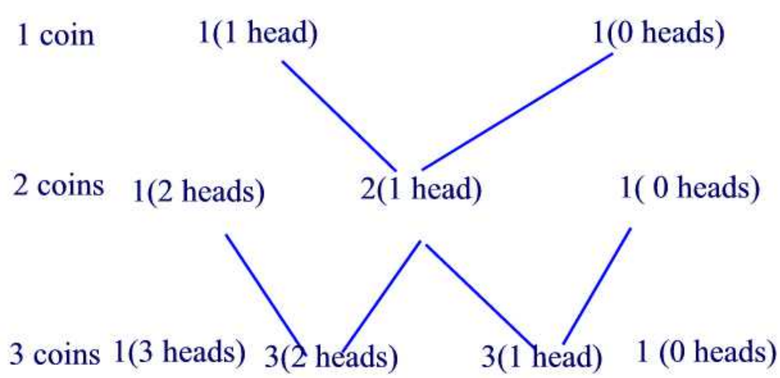


Figure 34.6

The pattern of Figure 34.6 holds for any number of coins and thus leads to the following diagram known as **Pascal's triangle**.

	1
1 coin	1 1
2coins	1 2 1
3coins	1 3 3 1
4 coins	1 4 6 4 1
5 coins	1 5 10 10 5 1
6 coins	1 6 15 20 15 6 1

Example 34.8
Six fair coins are tossed.

(a) Find the probability of getting exactly 3 heads.
(b) Find the probability of getting at least four heads.

Solution.
(a) The 6-coins row of Pascal's triangle may be interpreted as follows

$$1(6H)\ 6(5H)\ 15(4H)\ 20(3H)\ 15(2H)\ 6(1H)\ 1(0H)$$

Thus, there are 20 ways of getting exactly three heads, and the probability of 3 heads is $\frac{20}{64} = \frac{5}{16}$
(b) The first numbers $-1, 6,$ and $15-$ represent the number of outcomes for which there are at least 4 heads. Thus, the probability of getting at least four heads is

$$\frac{1+6+15}{64} = \frac{22}{64} = \frac{11}{32} \blacksquare$$

Practice Problems

Problem 34.23
The row of Pascal's triangle that starts 1,4,... would be useful in finding probabilities for an experiment of tossing four coins.
(a) Interpret the meaning of each number.
(b) Find the probability of exactly one head and three tails.
(c) Find the probability of at least one tail turning up.

Problem 34.24
Four coins are tossed.

(a) Draw a tree diagram to represent the arrangements of heads (H) and

tails (T).
(b) How many outcomes involve all heads? three heads, one tail? two heads, two tails? one head, three tails? no heads?
(c) How do these results relate to Pascal's triangle?

Problem 34.25
A true-false problem has 6 questions.

(a) How many ways are there to answer the 6-question test?
(b) What is the probability of getting at least 5 right by guessing the answers at random?

Problem 34.26
(a) Write the 7th row of Pascal's triangle.
(b) What is the probability of getting at least four heads when tossing seven coins?

Problem 34.27
Assume the probability is $\frac{1}{2}$ that a child born is a boy. What is the probability that if a family is going to have four children, they will all be boys?

35 Permutations, Combinations and Probability

Thus far we have been able to list the elements of a sample space by drawing a tree diagram. For large sample spaces tree diagrams become very complex to construct. In this section we discuss counting techniques for finding the number of elements of a sample space or an event without having to list them.

Permutations
Consider the following problem: In how many ways can 8 horses finish in a race (assuming there are no ties)? We can look at this problem as a decision consisting of 8 steps. The first step is the possibility of a horse to finish first in the race, the second step the horse finishes second, ... , the 8th step the horse finishes 8th in the race. Thus, by the Fundamental Principle of counting there are

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320 \text{ ways}$$

This problem exhibits an example of an ordered arrangement, that is, the order the objects are arranged is important. Such ordered arrangement is called a **permutation**. Products such as $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written in a shorthand notation called factoriel. That is, $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$ (read "8 factoriel"). In general, we define **n factoriel** by

$$n! = \begin{cases} n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

where n is a whole number n.

Example 35.1
Evaluate the following expressions:

(a) $6!$ (b) $\frac{10!}{7!}$.

Solution.
(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
(b) $\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720\blacksquare$

Using factoriels we see that the number of permutations of n objects is $n!$.

Example 35.2
There are 6! permutations of the 6 letters of the word "square." In how many of them is r the second letter?

Solution.
Let r be the second letter. Then there are 5 ways to fill the first spot, 4 ways to fill the third, 3 to fill the fourth, and so on. There are 5! such permutations.■

Example 35.3
Five different books are on a shelf. In how many different ways could you arrange them?

Solution.
The five books can be arranged in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways. ■

Counting Permutations
We next consider the permutations of a set of objects taken from a larger set. Suppose we have n items. How many ordered arrangements of r items can we form from these n items? The number of permutations is denoted by $P(n, r)$. The n refers to the number of different items and the r refers to the number of them appearing in each arrangement. This is equivalent to finding how many different ordered arrangements of people we can get on r chairs if we have n people to choose from. We proceed as follows.
The first chair can be filled by any of the n people; the second by any of the remaining $(n-1)$ people and so on. The rth chair can be filled by $(n-r+1)$ people. Hence we easily see that

$$P(n, r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}.$$

Example 35.4
How many ways can gold, silver, and bronze medals be awarded for a race run by 8 people?

Solution.
Using the permuation formula we find $P(8, 3) = \frac{8!}{(8-3)!} = 336$ ways.■

Example 35.5
How many five-digit zip codes can be made where all digits are unique? The possible digits are the numbers 0 through 9.

Solution.
$P(10,5) = \frac{10!}{(10-5)!} = 30,240$ zip codes.■

Practice Problems

Problem 35.1
Compute each of the following expressions.

(a) $(2!)(3!)(4!)$
(b) $(4 \times 3)!$
(c) $4 \cdot 3!$
(d) $4! - 3!$
(e) $\frac{8!}{5!}$
(f) $\frac{8!}{0!}$

Problem 35.2
Compute each of the following.

(a) $P(7,2)$ (b) $P(8,8)$ (c) $P(25,2)$

Problem 35.3
Find m and n so that $P(m,n) = \frac{9!}{6!}$

Problem 35.4
How many four-letter code words can be formed using a standard 26-letter alphabet
(a) if repetition is allowed?
(b) if repetition is not allowed?

Problem 35.5
Certain automobile license plates consist of a sequence of three letters followed by three digits.

(a) If no repetitions of letters are permitted, how many possible license plates are there?
(b) If no letters and no digits are repeated, how many license plates are possible?

Problem 35.6
A combination lock has 40 numbers on it.

(a) How many different three-number combinations can be made?
(b) How many different combinations are there if the numbers must be all different?

Problem 35.7
(a) Miss Murphy wants to seat 12 of her students in a row for a class picture. How many different seating arrangements are there?
(b) Seven of Miss Murphy's students are girls and 5 are boys. In how many different ways can she seat the 7 girls together on the left, and then the 5 boys together on the right?

Problem 35.8
Using the digits 1, 3, 5, 7, and 9, with no repetitions of the digits, how many

(a) one-digit numbers can be made?
(b) two-digit numbers can be made?
(c) three-digit numbers can be made?
(d) four-digit numbers can be made?

Problem 35.9
There are five members of the Math Club. In how many ways can the positions of officers, a president and a treasurer, be chosen?

Problem 35.10
(a) A baseball team has nine players. Find the number of ways the manager can arrange the batting order.
(b) Find the number of ways of choosing three initials from the alphabet if none of the letters can be repeated.

Combinations
As mentioned above, in a permutation the order of the set of objects or people is taken into account. However, there are many problems in which we want to know the number of ways in which r objects can be selected from n distinct objects in arbitrary order. For example, when selecting a two-person committee from a club of 10 members the order in the committee is irrelevant. That is choosing Mr A and Ms B in a committee is the same as

choosing Ms B and Mr A. A **combination** is a group of items in which the order does not make a difference.

Counting Combinations
Let $C(n,r)$ denote the number of ways in which r objects can be selected from a set of n distinct objects. Since the number of groups of r elements out of n elements is $C(n,r)$ and each group can be arranged in $r!$ ways then $P(n,r) = r!C(n,r)$. It follows that

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Example 35.6
How many ways can two slices of pizza be chosen from a plate containing one slice each of pepperoni, sausage, mushroom, and cheese pizza.

Solution.
In choosing the slices of pizza, order is not important. This arrangement is a combination. Thus, we need to find $C(4,2) = \frac{4!}{2!(4-2)!} = 6$. So, there are six ways to choose two slices of pizza from the plate.■

Example 35.7
How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Solution.
There are $C(9,3) \cdot C(11,4) = \frac{9!}{3!(9-3)!} \cdot \frac{11!}{4!(11-4)!} = 27,720$ ways.■

Practice Problems

Problem 35.11
Compute each of the following: (a) C(7,2) (b) C(8,8) (c) C(25,2)

Problem 35.12
Find m and n so that $C(m,n) = 13$

Problem 35.13
The Library of Science Book Club offers three books from a list of 42. If you circle three choices from a list of 42 numbers on a postcard, how many possible choices are there?

Problem 35.14
At the beginning of the second quarter of a mathematics class for elementary school teachers, each of the class's 25 students shook hands with each of the other students exactly once. How many handshakes took place?

Problem 35.15
There are five members of the math club. In how many ways can the two-person Social Committee be chosen?

Problem 35.16
A consumer group plans to select 2 televisions from a shipment of 8 to check the picture quality. In how many ways can they choose 2 televisions?

Problem 35.17
The Chess Club has six members. In how many ways
(a) can all six members line up for a picture?
(b) can they choose a president and a secretary?
(c) can they choose three members to attend a regional tournament with no regard to order?

Problem 35.18
Find the smallest value m and n such that $C(m,n) = P(15,2)$

Problem 35.19
A school has 30 teachers. In how many ways can the school choose 3 people to attend a national meeting?

Problem 35.20
Which is usually greater the number of combinations of a set of objects or the number of permutations?

Problem 35.21
How many different 12-person juries can be chosen from a pool of 20 juries?

Finding Probabilities Using Combinations and Permutations
Combinations can be used in finding probabilities as illustrated in the next example.

Example 35.8
Given a class of 12 girls and 10 boys.

(a) In how many ways can a committee of five consisting of 3 girls and 2 boys be chosen?
(b) What is the probability that a committee of five, chosen at random from the class, consists of three girls and two boys?
(c) How many of the possible committees of five have no boys?(i.e. consists only of girls)
(d) What is the probability that a committee of five, chosen at random from the class, consists only of girls?

Solution.
(a) First note that the order of the children in the committee does not matter. From 12 girls we can choose $C(12,3)$ different groups of three girls. From the 10 boys we can choose $C(10,2)$ different groups. Thus, by the Fundamental Principle of Counting the total number of committee is

$$C(12,3)\cdot C(10,2) = \frac{12!}{3!9!}\cdot\frac{10!}{2!8!} = \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}\cdot\frac{10\cdot 9}{2\cdot 1} = 9900$$

(b) The total number of committees of 5 is $C(22,5) = 26,334$. Using part (a), we find the probability that a committee of five will consist of 3 girls and 2 boys to be

$$\frac{C(12,3)\cdot C(10,2)}{C(22,5)} = \frac{9900}{26,334} \approx 0.3759.$$

(c) The number of ways to choose 5 girls from the 12 girls in the class is

$$C(10,0)\cdot C(12,5) = C(12,5) = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 792$$

(d) The probability that a committee of five consists only of girls is

$$\frac{C(12,5)}{C(22,5)} = \frac{792}{26,334} \approx 0.03\blacksquare$$

Practice Problems

Problem 35.22
John and Beth are hoping to be selected from their class of 30 as president and vice-president of the Social Committee. If the three-person committee (president, vice-president, and secretary) is selected at random, what is the probability that John and Beth would be president and vice president?

Problem 35.23
There are 10 boys and 13 girls in Mr. Benson's fourth-grade class and 12 boys and 11 girls in Mr. Johnson fourth-grade class. A picnic committee of six people is selected at random from the total group of students in both classes.

(a) What is the probability that all the committee members are girls?
(b) What is the probability that the committee has three girls and three boys?

Problem 35.24
A school dance committee of 4 people is selected at random from a group of 6 ninth graders, 11 eighth graders, and 10 seventh graders.

(a) What is the probability that the committee has all seventh graders?
(b) What is the probability that the committee has no seventh graders?

Problem 35.25
In an effort to promote school spirit, Georgetown High School created ID numbers with just the letters G, H, and S. If each letter is used exactly three times,

(a) how many nine-letter ID numbers can be generated?
(b) what is the probability that a random ID number starts with GHS?

Problem 35.26
The license plates in the state of Utah consist of three letters followed by three single-digit numbers.

(a) If Edward's initials are EAM, what is the probability that his license plate will have his initials on it (in any order)?

(b) What is the probability that his license plate will have his initials in the correct order?

36 Odds, Expected Value, and Conditional Probability

What's the difference between probabilities and odds? To answer this question, let's consider a game that involves rolling a die. If one gets the face 1 then he wins the game, otherwise he loses. The probability of winning is $\frac{1}{6}$ whereas the probability of losing is $\frac{5}{6}$. The odds of winning is 1:5(read 1 to 5). This expression means that the probability of losing is five times the probability of winning. Thus, probabilities describe the frequency of a favorable result in relation to all possible outcomes whereas the "odds in favor" compare the favorable outcomes to the unfavorable outcomes. More formally,

$$\text{odds in favor} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}}$$

If E is the event of all favorable outcomes then its complementary, $\overline{E}$, is the event of unfavorable outcomes. Hence,

$$\text{odds in favor} = \frac{n(E)}{n(\overline{E})}$$

Also, we define the **odds against** an event as

$$\text{odds against} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{n(\overline{E})}{n(E)}$$

Any probability can be converted to odds, and any odds can be converted to a probability.

Converting Odds to Probability

Suppose that the odds for an event E is a:b. Thus, $n(E) = ak$ and $n(\overline{E}) = bk$ where k is a positive integer. Since E and $\overline{E}$ are complementary then

$n(S) = n(E) + n(\overline{E})$. Therefore,

$$\begin{array}{rcl} P(E) & = & \frac{n(E)}{n(S)} \\ & = & \frac{n(E)}{n(E)+n(\overline{E})} \\ & = & \frac{ak}{ak+bk} = \frac{a}{a+b} \\ P(\overline{E}) & = & \frac{n(\overline{E})}{n(S)} \\ & = & \frac{n(\overline{E})}{n(E)+n(\overline{E})} \\ & = & \frac{bk}{ak+bk} = \frac{b}{a+b} \end{array}$$

Example 36.1
If the odds in favor of an event E is 5 to 4, compute $P(E)$ and $P(\overline{E})$.

Solution.
We have

$$P(E) = \frac{5}{5+4} = \frac{5}{9}$$

and

$$P(\overline{E}) = \frac{4}{5+4} = \frac{4}{9} \blacksquare$$

Converting Probability to Odds
Given $P(E)$, we want to find the odds in favor of E and the odds against E. The odds in favor of E are

$$\begin{array}{rcl} \frac{n(E)}{n(\overline{E})} & = & \frac{n(E)}{n(S)} \cdot \frac{n(S)}{n(\overline{E})} \\ & = & \frac{P(E)}{P(\overline{E})} \\ & = & \frac{P(E)}{1-P(E)} \end{array}$$

and the odds against E are

$$\frac{n(\overline{E})}{n(E)} = \frac{1-P(E)}{P(E)}$$

Example 36.2
For each of the following, find the odds in favor of the event's occurring:

(a) Rolling a number less than 5 on a die.
(b) Tossing heads on a fair coin.
(c) Drawing an ace from an ordinary 52-card deck.

Solution.
(a) The probability of rolling a number less than 5 is $\frac{4}{6}$ and that of rolling 5 or 6 is $\frac{2}{6}$. Thus, the odds in favor of rolling a number less than 5 is $\frac{4}{6} \div \frac{2}{6} = \frac{2}{1}$ or 2:1
(b) Since $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$ then the odds in favor of getting heads is $\left(\frac{1}{2}\right) \div \left(\frac{1}{2}\right)$ or 1:1
(c) We have P(ace) $=\frac{4}{52}$ and P(not an ace) $=\frac{48}{52}$ so that the odds in favor of drawing an ace is $\left(\frac{4}{52}\right) \div \left(\frac{48}{52}\right) = \frac{1}{12}$ or 1:12$\blacksquare$

Remark 36.1
A probability such as $P(E) = \frac{5}{6}$ is just a ratio. The exact number of favorable outcomes and the exact total of all outcomes are not necessarily known.

Practice Problems

Problem 36.1
If the probability of a boy's being born is $\frac{1}{2}$, and a family plans to have four children, what are the odds against having all boys?

Problem 36.2
If the odds against Deborah's winning first prize in a chess tournament are 3 to 5, what is the probability that she will win first prize?

Problem 36.3
What are the odds in favor of getting at least two heads if a fair coin is tossed three times?

Problem 36.4
If the probability of rain for the day is 60%, what are the odds against its raining?

Problem 36.5
On a tote board at a race track, the odds for Gameylegs are listed as 26:1. Tote boards list the odds that the horse will lose the race. If this is the case, what is the probability of Gameylegs's winning the race?

Problem 36.6
If a die is tossed, what are the odds in favor of the following events?
(a) Getting a 4
(b) Getting a prime
(c) Getting a number greater than 0
(d) Getting a number greater than 6.

Problem 36.7
Find the odds against E if $P(E) = \frac{3}{4}$.

Problem 36.8
Find P(E) in each case.

(a) The odds in favor of E are 3:4
(b) The odds against E are 7:3

Expected Value
A cube has three red faces, two green faces, and one blue face. A game consists of rolling the cube twice. You pay \$ 2 to play. If both faces are the same color, you are paid \$ 5(that is you win \$3). If not, you lose the \$2 it costs to play. Will you win money in the long run? Let W denote the event that you win. Then $W = \{RR, GG, BB\}$ and

$$P(W) = P(RR) + P(GG) + P(BB) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6} = \frac{7}{18} \approx 39\%.$$

Thus, $P(L) = \frac{11}{18} = 61\%$. Hence, if you play the game 18 times you expect to win 7 times and lose 11 times on average. So your winnings in dollars will be $3 \times 7 - 2 \times 11 = -1$. That is, you can expect to lose \$1 if you play the game 18 times. On the average, you will lose $\frac{1}{18}$ per game (about 6 cents). This can be found also using the equation

$$3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}$$

We call this number the expected value. More formally, let the outcomes of an experiment be a sequence of real numbers $n_1, n_2, \cdots, n_k$, and suppose that the outcomes occur with respective probabilities $p_1, p_2, \cdots, p_k$. Then the **expected value** of the experiment is

$$E = n_1p_1 + n_2p_2 + \cdots + n_kp_k.$$

Example 36.3
Suppose that an insurance company has broken down yearly automobile claims for drivers from age 16 through 21 as shown in the following table.

Amount of claim	Probability
0	0.80
2,000	0.10
4,000	0.05
6,000	0.03
8,000	0.01
10,000	0.01

How much should the company charge as its average premium in order to break even on costs for claims?

Solution.
Finding the expected value

$$E = 0(0.80)+2,000(0.10)+4,000(0.05)+6,000(0.03)+8,000(0.01)+10,000(0.01) = 760$$

Since average claim value is \$760, the average automobile insurance premium should be set at \$760 per year for the insurance company to break even■

Example 36.4
An American roulette wheel has 38 compartments around its rim. Two of these are colored green and are numbered 0 and 00. The remaining compartments are numbered 1 through 36 and are alternately colored black and red. When the wheel is spun in one direction, a small ivory ball is rolled in the opposite direction around the rim. When the wheel and the ball slow down, the ball eventually falls in any one compartments with equal likelyhood if the wheel is fair. One way to play is to bet on whether the ball will fall in a red slot or a black slot. If you bet on red for example, you win the amount of the bet if the ball lands in a red slot; otherwise you lose. What is the expected win if you consistently bet \$5 on red?

Solution.
The probability of winning is $\frac{18}{38}$ and that of losing is $\frac{20}{38}$. Your expected win is

$$\frac{18}{38} \times 5 - \frac{20}{38} \times 5 \approx -0.26$$

On average you should expect to lose 26 cents per play■

Practice Problems

Problem 36.9
Compute the expected value of the score when rolling two dice.

Problem 36.10
A game consists of rolling two dice. You win the amounts shown for rolling the score shown.

Score	2	3	4	5	6	7	8	9	10	11	12
$ won	4	6	8	10	20	40	20	10	8	6	4

Compute the expected value of the game.

Problem 36.11
Consider the spinner in Figure 36.1, with the payoff in each sector of the circle. Should the owner of this spinner expect to make money over an extended period of time if the charge is $2.00 per spin?

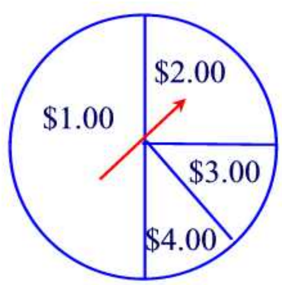


Figure 36.1

Problem 36.12
You play a game in which two dice are rolled. If a sum of 7 appears, you win $10; otherwise, you lose $2.00. If you intend to play this game for a long time, should you expect to make money, lose money, or come out about even? Explain.

Problem 36.13
Suppose it costs \$8 to roll a pair of dice. You get paid the sum of the numbers in dollars that appear on the dice. What is the expected value of this game?

Problem 36.14
An insurance company will insure your dorm room against theft for a semester. Suppose the value of your possessions is \$800. The probability of your being robbed of \$400 worth of goods during a semester is $\frac{1}{100}$, and the probability of your being robbed of \$800 worth of goods is $\frac{1}{400}$. Assume that these are the only possible kinds of robberies. How much should the insurance company charge people like you to cover the money they pay out and to make an additional \$20 profit per person on the average?

Problem 36.15
Consider a lottery game in which 7 out of 10 people lose, 1 out of 10 wins \$50, and 2 out of 10 wins \$35. If you played 10 times, about how much would you expect to win?

Problem 36.16
Suppose a lottery game allows you to select a 2-digit number. Each digit may be either 1, 2, 3, 4, or 5. If you pick the winning number, you win \$10. Otherwise, you win nothing. What is the expected payoff?

Conditional Probability and Independent Events
When the sample space of an experiment is affected by additional information, the new sample space is reduced in size. For example, suppose we toss a fair coin three times and consider the following events:

A : getting a tail on the first toss
B : getting a tail on all three tosses

Since

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

then $P(A) = \frac{4}{8} = \frac{1}{2}$ and $P(B) = \frac{1}{8}$. What if we were told that event A has occurred (that is, a tail occurred on the first toss), and we are now asked to find $P(B)$. The sample space is now reduced to $\{THH, THT, TTH, TTT\}$. The probability that all three are tails given that the first toss is a tail is $\frac{1}{4}$.

The notation we use for this situation is $P(B|A)$, read "the probability of B given A," and we write $P(B|A) = \frac{1}{4}$. Notice that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

This is true in general, and we have the following:

Given two events A and B belonging to the same sample S. The **conditional probability** $P(B|A)$ denotes the probability that event B will occur given that event A has occurred. It is given by the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Example 36.5
Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{aligned} A &= \{\text{Observing an even number of dots on the upper face of the die}\}, \\ B &= \{\text{Observing a number of dots less than or equal to 3 on the upper face of the die}\}. \end{aligned}$$

Find the probability of the event A, given the event B.

Solution.
Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$ then $A \cap B = \{2\}$ and therefore $P(A|B) = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$.■

If $P(B|A) = P(B)$, i.e., the occurrence of the event A does not affect the probability of the event B, then we say that the two events A and B are **independent**. In this case the above formula gives

$$P(A \cap B) = P(A) \cdot P(B).$$

This formula is known as the "multiplication rule of probabilities". If two events are not independent, we say that they are **dependent**. In this case, $P(B|A) \neq P(B)$ or equivalently $P(A \cap B) \neq P(A) \cdot P(B)$.

Example 36.6
Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{aligned} A &= \{\text{Observing an even number of dots on the upper face of the die}\} &, \\ B &= \{\text{Observing a number of dots less than or equal to 4 on the upper face of the die}\}. \end{aligned}$$

Are A and B independent?

Solution.
Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ then $A \cap B = \{2, 4\}$ and therefore $P(A|B) = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2} = P(A)$. Thus, A and B are independent.■

Practice Problems

Problem 36.17
Suppose that A is the event of rolling a sum of 7 with two fair dice. Make up an event B so that
(a) A and B are independent.
(b) A and B are dependent.

Problem 36.18
When tossing three fair coins, what is the probability of getting two tails given that the first coin came up heads?

Problem 36.19
Suppose a 20-sided die has the following numerals on its face:1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 7, ,8 9, 10, 11, 12, 13, 14, 15, 16. The die is rolled once and the number on the top face is recorded. Let A be the event the number is prime, and B be the event the number is odd. Find $P(A|B)$ and $P(B|A)$.

Problem 36.20
What is the probability of rolling a 6 on a fair die if you know that the roll is an even number?

Problem 36.21
A red die and a green die are rolled. What is the probability of obtaining an even number on the red die and a multiple of 3 on the green die?

Problem 36.22
Two coins are tossed. What is the probability of obtaining a head on the first coin and a tail on the second coin?

Problem 36.23
Consider two boxes: Box 1 contains 2 white and 2 black balls, and box 2 contains 2 white balls and three black balls. What is the probability of drawing a black ball from each box?

Problem 36.24
A container holds three red balls and five blue balls. One ball is drawn and discarded. Then a second ball is drawn.
(a) What is the probability that the second ball drawn is red if you drew a red ball the first time?
(b) What is the probability of drawing a blue ball second if the first ball was red?
(c) What is the probability of drawing a blue ball second if the first ball was blue?

Problem 36.25
Consider the following events.

A: rain tomorrow
B: You carry an umbrella
C: coin flipped tomorrow lands on heads

Which of two events are dependent and which are independent?

Problem 36.26
You roll a regular red die and a regular green die. Consider the following events.

A: a 4 on the red die
B: a 3 on the green die
C: a sum of 9 on the two dice

Tell whether each pair of events is independent or dependent.
(a) A and B (b) B and C

37 Basic Geometric Shapes and Figures

In this section we discuss basic geometric shapes and figures such as points, lines, line segments, planes, angles, triangles, and quadrilaterals.

The three pillars of geometry are points, lines, and planes: A **point** is an undefined term used to describe for example a location on a map. A point has no length, width, or thickness and we often use a dot to represent it. Points are usually labeled with uppercase letters.
Line is another basic term of geometry. Like a point, a line is an undefined term used to describe a tightly stretched thread or a laser beam. We can say that a line is a straight arrangement of points. A line has no thickness but its length goes on forever in two directions as shown in Figure 37.1. The arrows represent the fact that the line extends in both directions forever. A line is often named by a lowercase letter such as the line k in Figure 37.1.

Figure 37.1

The subset of the line k consisting of all points between A and B together with A and B forms a **line segment** denoted by $\overline{AB}$. We call A the **left endpoint** and B the **right endpoint.** The distance between the endpoints is known as the **length** of the line segment and will be denoted by AB. Two line segments with the same length are said to be **congruent**.
Any three or more points that belong to the same line are called **collinear**. See Figure 37.2

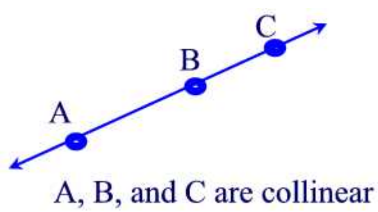


Figure 37.2

Three noncollinear points (also knwon as **coplanar points**) determine a **plane**, which is yet another undefined term used to describe a flat space such as a tabletop.

Subsets of a plane are called **plane shapes** or **planes figures**. We have already discussed a geometric figure, namely, a line. Another important example of a geometric figure is the concept of an angle.
By an **angle** we mean the opening between two line segments that have a common endpoint, known as the **vertex**, as shown in Figure 37.3(a). The line segments are called the **sides** of the angle.
If one of the line segments of an angle is horizontal and the other is vertical then we call the angle a **right angle**. See Figure 37.3(b). Note that the sides of an angle partition the plane into two regions, the **interior** and the **exterior** of the angle as shown in Figure 37.3(c). Two angles with the same opening are said to be **congruent**.

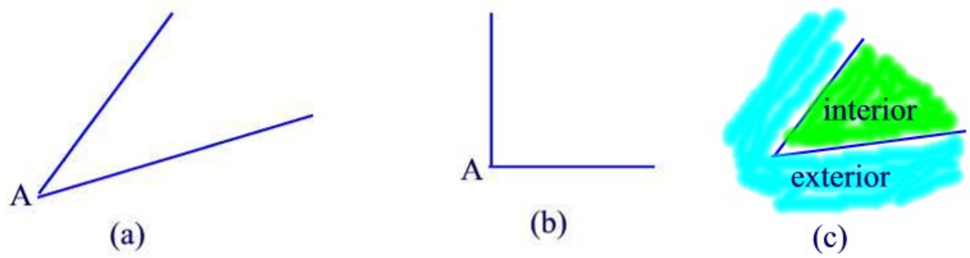


Figure 37.3

The Early Stages of Learning Geometry
The first stage of a child's learning geometry consists on **recognizing** geometric shapes by their appearances without paying attention to their component parts (such as the sides and the angles). For example, a rectangle may be recognized because it "looks like a door," not because it has four straight sides and four right angles. The second stage, known as **description**, students are able to describe the component parts and properties of a shape, such as how many sides it has and whether it has some congruent sides or angles. At the third stage, students become aware of **relationships** between different shapes such as a rhombus is a quadrilateral with four congruent sides and a parallelogram is a quadrilateral with parallel opposite sides, etc.

Triangles
A **triangle** is a closed figure composed exactly of three line segments called the **sides**. The points of intersection of any two line segments is called a **vertex**. Thus, a triangle has three vertices. Aslo, a triangle has three interior angles. See Figure 37.4(a).
Triangles may be classified according to their angles and sides. If exactly one

of the angle is a right angle then the triangle is called a **right triangle**. See Figure 37.4(b). A triangle with three congruent sides is called an **equilateral triangle**. See Figure 37.4(c). A triangle with two or more congruent sides is called an **isosceles** triangle. A triangle with no congruent sides is called a **scalene** triangle.

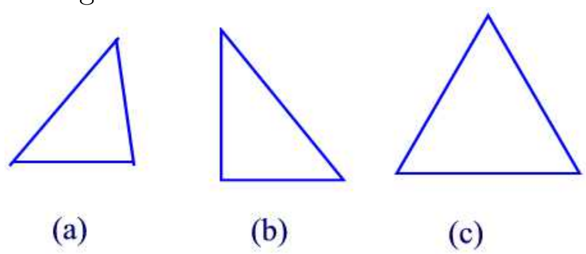


Figure 37.4

Quadrilaterals

By a **quadrilateral** we mean a closed figure with exactly four line segments (or sides). Quadrilaterals are classified as follows:

• A **trapezoid** is a quadrilateral that has exactly one pair of parallel sides. Model: the middle part of a bike frame.

• An **isosceles trapezoid** is a quadrilateral with exactly two parallel sides and the remaining two sides are congruent. Model: A water glass silhouette.

• A **parallelogram** is a quadrilateral in which each pair of opposite sides is parallel.

• A **rhombus** is a parallelogram that has four congruent sides. Model: diamond.

• A **kite** is a quadrilateral with two nonoverlapping pairs of adjacent sides that are the same length. Model: a kite.

• A **rectangle** is a parallelogram that has four right angles. Model: a door.

• A **square** is a rectangle that has four congruent sides. Model: Floor tile.

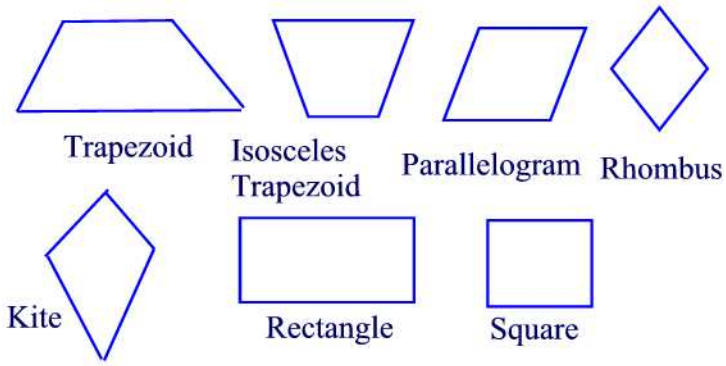


Figure 37.5

Practice Problems

Problem 37.1
Find three objects in your classroom with surfaces that suggests common geometric figures.

Problem 37.2
A fifth grader says a square is not a rectangle because a square has four congruent sides and rectangles don't have that. A second fifth grader says a square is a type of rectangle because it is a parallelogram and it has four right angles.

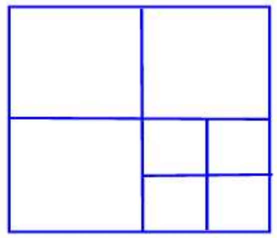
(a) Which child is right?
(b) How can you use the definitions to help the other child understand?

Problem 37.3
Suppose P={parallelograms}, Rh={rhombus}, S={squares}, Re={rectangles}, T={trapezoids}, and Q={quadrilaterals}. Find
(a) $Rh \cap Re$ (b) $T \cap P$

Problem 37.4
Organize the sets P, Rh, S, Re, T, and Q using Venn diagram.

Problem 37.5
(a) True or false? No scalene triangle is isosceles.
(b) What shape is the diamond in a deck of cards?

Problem 37.6
How many squares are in the following design?



Problem 37.7
Tell whether each of the following shapes must, can, or cannot have at least one right angle.

(a) Rhombus
(b) Square
(c) Trapezoid
(d) Rectangle
(e) Parallelogram

Problem 37.8
In which of the following shapes are both pairs of opposite sides parallel?

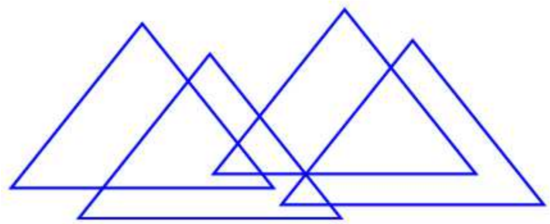
(a) Rhombus
(b) Square
(c) Trapezoid
(d) Rectangle
(e) Parallelogram

Problem 37.9
A square is also which of the following?
(a) Quadrilateral
(b) Parallelogram
(c) Rhombus
(d) Rectangle

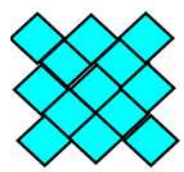
Problem 37.10
Fill in the blank with "All", "Some", or "No"

(a) _____ rectangles are squares.
(b) _____ parallelograms are trapezoids.
(c) _____ rhombuses are quadrilaterals.

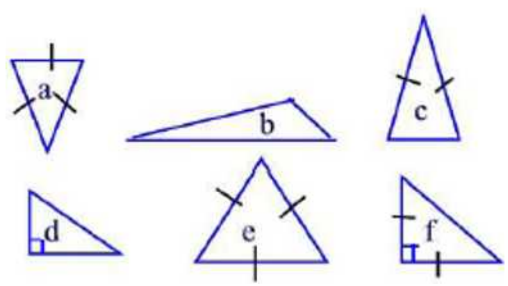
Problem 37.11
How many triangles are in the following design?



Problem 37.12
How many squares are found in the following figure?



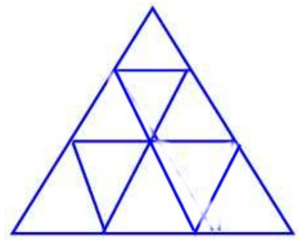
Problem 37.13
Given are a variety of triangles. Sides with the same length are indicated. Right angles are indicated.



(a) Name the triangles that are scalene.
(b) Name the triangles that are isosceles.
(c) Name the triangles that are equilateral.
(d) Name the triangles that contain a right angle.

Problem 37.14
(a) How many triangles are in the figure?

(b) How many parallelograms are in the figure?
(c) How many trapezoids are in the figure?



Problem 37.15
If possible, sketch two parallelograms that intersect at exactly
(a) one point
(b) two points
(c) three points
(d) four points.

Problem 37.16
If possible, draw a triangle and a quadrilateral that intersect at exactly
(a) one point
(b) two points
(c) three points.

Problem 37.17
Suppose P={parallelograms}, S={squares}, T={trapezoids}, and Q={quadrilaterals}.
Find

(a) $P \cap S$
(b) $P \cup Q$

Problem 37.18
A fifth grader does not think that a rectangle is a type of parallelogram. Tell why it is.

Problem 37.19
Tell whether each definition has sufficient information. If it is not sufficient, tell what information is missing.

(a) A rhombus is a quadrilateral with both pairs of opposite sides parallel.
(b) A square is a quadrilateral with four congruent sides.
(c) A rhombus is a quadrilateral that has four congruent sides.

Problem 37.20
Name properties that a square, parallelogram, and rhombus have in common.

Problem 37.21
How many different line segments are contained in the following portion of a line?

38 Properties of Geometric Shapes

In this section we study the properties of geometric shapes that belong to the same plane such as lines and angles.

Lines
Two lines in a plane that do not have a point in common are called **parallel**. See Figure 38.1 (a). If the two lines are l and m then we write $l||m$. When two or more lines have exactly one point in common then we say that they are **concurrent**. See Figure 38.1(b).
The **midpoint** of a line segment is a point that is **equidistant** from A and B, that is a point that has the same distance from A and B. See Figure 38.1(c). A **ray** is a subset of a line that contains the endpoint and all points on the line on one side of the point. See Figure 38.1(d).

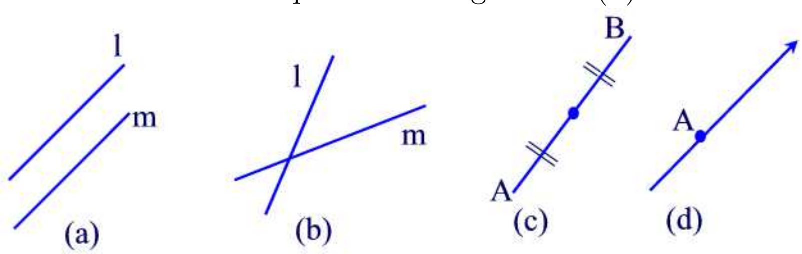


Figure 38.1

Next, we list some of the important properties of lines.

Properties
(1) Through any distinct two points there is exactly one line going through them.
(2) If two points are in a plane then the line containing them lies in that plane.
(3) There is exactly one plane containing any three noncollinear distinct points.
(4) A line and a point not on the line determine a plane.
(5) Two parallel lines determine a plane.
(6) Two concurrent lines determine a plane.

Angles: Measurements and Types
We have seen that an angle is the opening between two line segments with

one common endpoint. An **angle** can be also defined as the opening between two rays with the same endpoint called the **vertex**. We call the rays the **sides** of the angle. If A is the vertex, B is a point on one side and C is a point on the other side then we write $\angle BAC$. See Figure 38.2. Note that the middle letter in the notation of an angle is always the vertex. Sometimes only the vertex A is given. In this case we use the notation $\angle A$. Also, angles can be labeled using numbers. In this case we use the notation $\angle 1$ to denote the angle labeled 1.

An angle is measured according to the amount of "opening" between its sides. A unit of angle measure is the **degree**. A complete rotation about a point has a measure of 360 degrees, written $360°$. Thus, one degree is $\frac{1}{360}$ of a complete rotation. We will write $m(\angle BAC)$ for the measure of the angle at vertex A.
Angles are measured using a **protractor**. To measure an angle $\angle BAC$, place the center of the protractor at the vertex A while lining up the side AB with the base of the protractor. The other side of the angle will pass through a number on the protractor representing $m(\angle BAC)$ in degrees. See Figure 38.2.

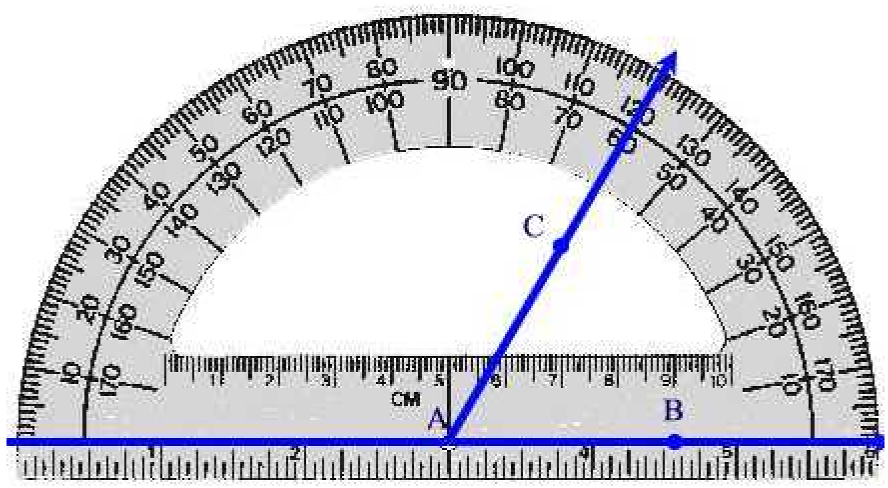


Figure 38.2

Next, we categorize angles according to their measures. If $m(\angle BAC) = 90°$ then we will say that the angle is a **right** angle. See Figure 38.3(a). Two lines that intersect at a right angle are called **perpendicular**. If

$m(\angle BAC) = 180°$ we say that the angle is **straight**. See Figure 38.3(b). If $0° < m(\angle BAC) < 90°$ then the angle is called **acute**. See Figure 38.3(c). If $90° < m(\angle BAC) < 180°$ then the angle is called **obtuse**. See Figure 38.3(d). If $m(\angle BAC) > 180°$ then the angle is called **reflex**. See Figure 38.3(e).

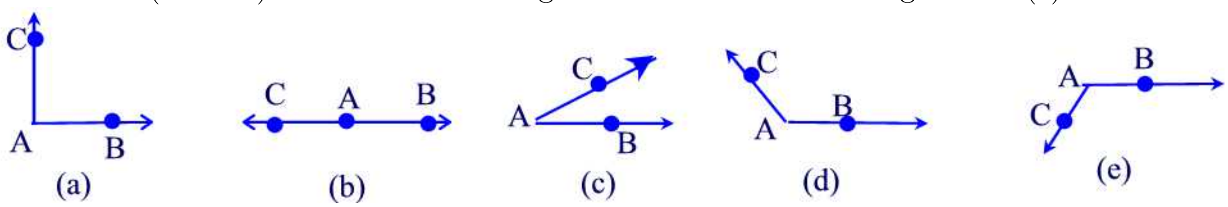


Figure 38.3

Now, recall that a triangle has three interior angles. If one of the angles is a right angle then we call the triangle a **right** triangle. If one of the angle is obtuse then we call the triangle an **obtuse triangle**. If all three angles are acute angles then the triangle is called an **acute** triangle.

Pairs of Angles and the Vertical Angles Theorem

Two angles that have a common side and nonoverlapping interiors are called **adjacent** angles. See Figure 38.4(a). If the sum of two adjacent angles is $90°$ then the two angles are called **complementary**. See Figure 38.4(b). Two adjacent angles whose sum is $180°$ are called **supplementary** angles. See Figure 38.4(c). A pair of nonadjacent angles formed by two intersecting lines are called **vertical angles**. For example, $m(\angle 1)$ and $m(\angle 2)$ are vertical angles as well as $m(\angle 3)$ and $m(\angle 4)$. See Figure 38.4(d)

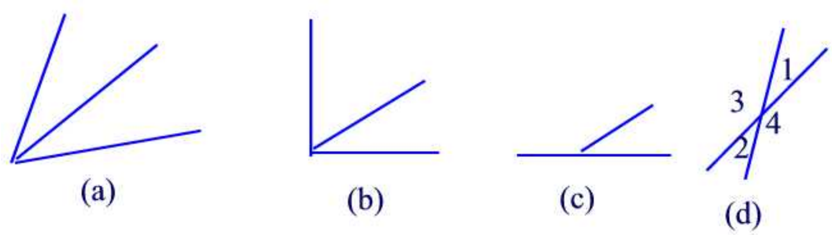


Figure 38.4

Vertical angles are congruent as shown in the following theorem.

Theorem 38.1 *(Vertical Angles Theorem)*
Vertical angles have the same measure.

Proof.
The angles $\angle 1$ and $\angle 3$ are supplementary so that $m(\angle 1) + m(\angle 3) = 180°$. Similarly, the angles $\angle 2$ and $\angle 3$ are supplementary so that $m(\angle 2)+m(\angle 3) = 180°$. Thus, $m(\angle 1)+m(\angle 3) = m(\angle 2)+m(\angle 3)$. Subtracting $m(\angle 3)$ from both sides of the equation gives $m(\angle 1) = m(\angle 2)$. A similar argument shows that $m(\angle 3) = m(\angle 4)$■

Angles Associated with Parallel Lines
If two lines k and l are parallel then a line m as shown in Figure 38.5 is called a **transversal line**. In this case, there are 8 angles determined by these three lines numbered 1 through 8.

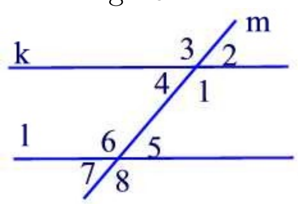


Figure 38.5

Angles $\angle 1$ and $\angle 6$ as well as angles $\angle 4$ and $\angle 5$ are called **alternate interior angles**. The following theorem shows that any pair of alternate interior angles are congruent.

Theorem 38.2
Any pair of alternate interior angles have the same measure.

Proof.
We will show that $m(\angle 4) = m(\angle 5)$. The other equality is similar. According to Figure 38.6 we have the following equalities: $m(\angle c) + m(\angle 5) = 90°$ and $m(\angle 4)+m(\angle c) = 90°$. Thus, $m(\angle c)+m(\angle 5) = m(\angle 4)+m(\angle c)$. Subtracting $m(\angle c)$ from both sides we obtain $m(\angle 4) = m(\angle 5)$■

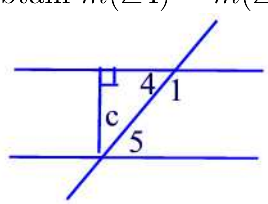


Figure 38.6

Now, going back to Figure 38.5, the angles $\angle 1$ and $\angle 8$; $\angle 2$ and $\angle 5$; $\angle 3$ and $\angle 6$; $\angle 4$ and $\angle 7$ are called **corresponding angles**. Every pair of corresponding angles are congruent as shown in the next theorem.

Theorem 38.3
Corresponding angles have the same measure.

Proof.
We will just show that $m(\angle 2) = m(\angle 5)$. By the vertical angles theorem we have $m(\angle 2) = m(\angle 4)$. By the alternate interior angles theorem we have $m(\angle 4) = m(\angle 5)$. From these two equalities we conclude that $m(\angle 2) = m(\angle 5)$ ■

Sum of the Measures of the Interior Angles of a Triangle
As an application to the results of the previous paragraph, we show that the sum of the measures of the interior angles of a triangle is $180°$. To see this, let's consider a triangle with vertices A, B, and C and interior angles $\angle 1, \angle 2$, and $\angle 3$ as shown in Figure 38.7(a). At A draw a line parallel to the line segment $\overline{BC}$. Then using the corresponding angles theorem we can draw Figure 38.7(b) which shows that

$$m(\angle 1) + m(\angle 2) + m(\angle 3) = 180°$$

as required.

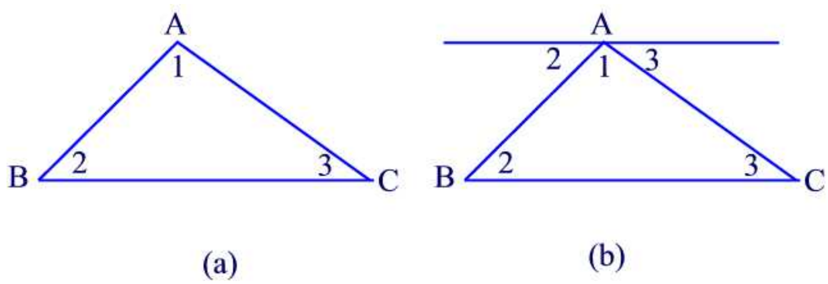


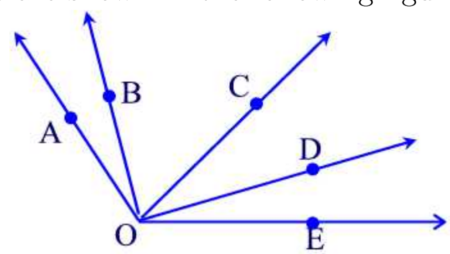
Figure 38.7

Practice Problems

Problem 38.1
Using Figure 38.5 show that $m(\angle 1) + m(\angle 5) = 180°$.

Problem 38.2

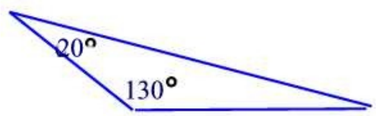
(a) How many angles are shown in the following figure?



(b) How many are obtuse?
(c) How many are acute?

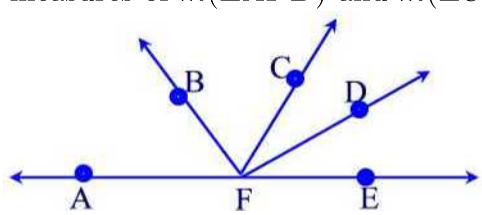
Problem 38.3

Find the missing angle in the following triangle.



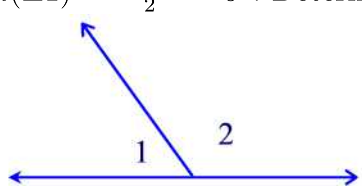
Problem 38.4

In the figure below, $m(\angle BFC) = 55°, m(\angle AFD) = 150°$, $m(\angle BFE) = 120°$. Determine the measures of $m(\angle AFB)$ and $m(\angle CFD)$.



Problem 38.5

In the following figure $m(\angle 1) = \frac{m(\angle 2)}{2} - 9°$. Determine $m(\angle 1)$ and $m(\angle 2)$.

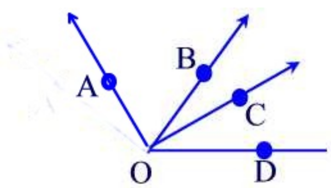


Problem 38.6
Angles 3 and 8; 2 and 7 in Figure 38.5 are called **alternate exterior angles**. Show that $m(\angle 3) = m(\angle 8)$.

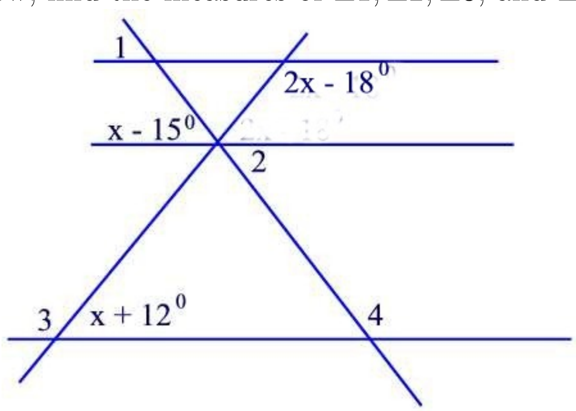
Problem 38.7
Following are the measures of $\angle A, \angle B, \angle C$. Can a triangle ΔABC be made that has the given angles? Explain.

(a) $m(\angle A) = 36°, m(\angle B) = 78°, m(\angle C) = 66°$.
(b) $m(\angle A) = 124°, m(\angle B) = 56°, m(\angle C) = 20°$.
(c) $m(\angle A) = 90°, m(\angle B) = 74°, m(\angle C) = 18°$.

Problem 38.8
In the following figure $\overline{AO}$ is perpendicular to $\overline{CO}$. If $m(\angle AOD) = 165°$ and $m(\angle BOD) = 82°$, determine the measures of $\angle AOB$ and $\angle BOC$.



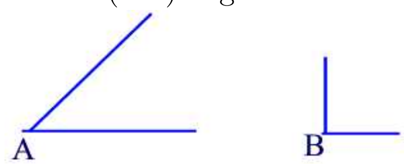
Problem 38.9
In the figure below, find the measures of $\angle 1, \angle 2, \angle 3$, and $\angle 4$.



Problem 38.10
(a) How is a line segment different from a line?

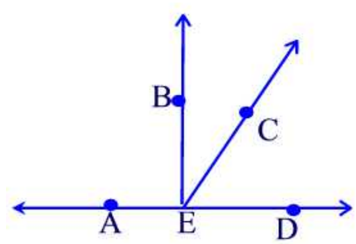
(b) What is the vertex of the angle $\angle PAT$?
(c) How is $\overline{AB}$ different from AB?

Problem 38.11
A fourth grader thinks that $m(\angle A)$ is greater than $m(\angle B)$.



(a) Why might the child think this?
(b) How could you put the angles together to show that $m(\angle A) < m(\angle B)$?

Problem 38.12
In the figure below



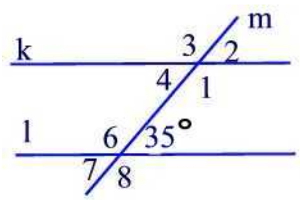
(a) name two supplementary angles
(b) name two complementary angles.

Problem 38.13
An angle measures $20°$. What is the measure of
(a) its supplement? (b) its complement?

Problem 38.14
True or false? If false give an example.

(a) All right angles are congruent.
(b) Two complementary angles are congruent.
(c) Two supplementary angles are congruent.

Problem 38.15
Find the measures of the angles in the following figure.



Problem 38.16
How many degrees does the minute hand of a clock turn through

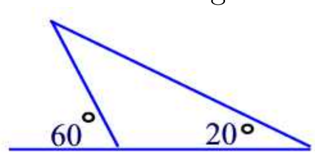
(a) in sixty minutes?
(b) in five minutes?
(c) in one minute?

Problem 38.17
How many degrees does the hour hand of a clock turn through

(a) in sixty minutes?
(b) in five minutes?

Problem 38.18
Find the angle formed by the minute and hour hands of a clock at these times.
(a) 3:00 (b) 6:00 (c) 4:30 (d) 10:20

Problem 38.19
Determine the measures of the interior angles.



Problem 38.20
(a) Can a triangle have two obtuse angles? Why?
(b) Can a triangle have two right angles? Why?
(c) Can a triangle have two acute angles?

Problem 38.21

A hiker started heading due north, then turned to the right $38°$, then turned to the left $57°$, and next turned right $9°$. To resume heading north, what turn must be made?

39 Symmetry of Plane Figures

In this section, we are interested in the symmetric properties of plane figures. By a **symmetry** of a plane figure we mean a motion of the plane that moves the figure so that it falls back on itself.
The two types of symmetry that we discuss are

Reflection Symmetry:
A plane figure is symmetric about a line if it is its own image when flipped across the line. We call the reflection line the **line of symmetry**. In other words, a figure has a line symmetry if it can be folded along the line so that one half of the figure matches the other half.
Reflection symmetry is also known as **mirror symmetry**, since the line of symmetry acts like a double-sided mirror. Points on each side are reflected to the opposite side.
Many plane figures have several line symmetries. Figure 39.1 shows some of the plane figures with their line symmetries shown dashed.

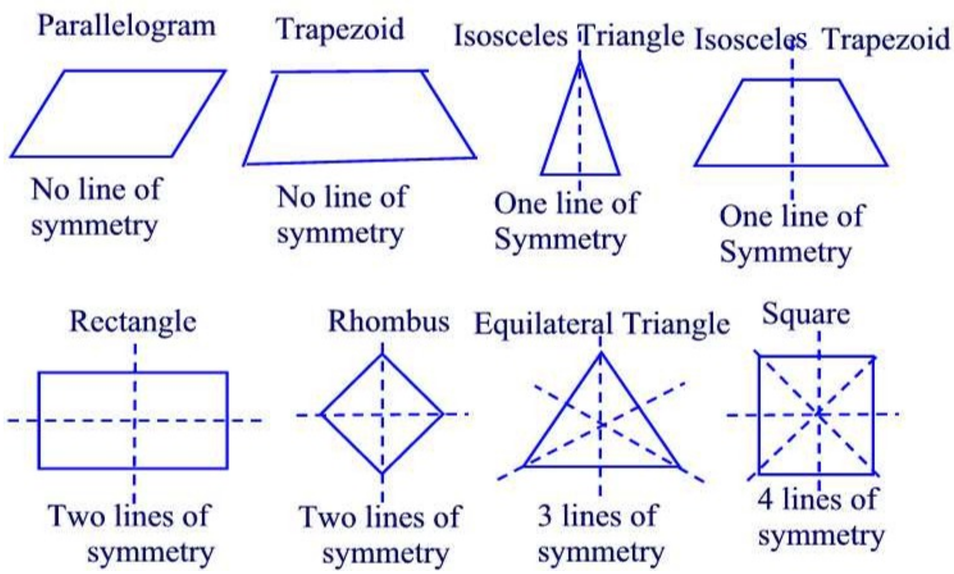


Figure 39.1

Remark 39.1
Using reflection symmetry we can establish properties for some plane figures. For example, since an isosceles triangle has one line of symmetry then the

base angles, i.e. angles opposed the congruent sides, are congruent. A similar property holds for isosceles trapezoids.

Rotational Symmetry:
A plane figure has **rotational symmetry** if and only if it can be rotated more than $0°$ and less than or equal to $360°$ about a fixed point called the **center of rotation** so that its image coincides with its original position. Figure 39.2 shows the four different rotations of a square.

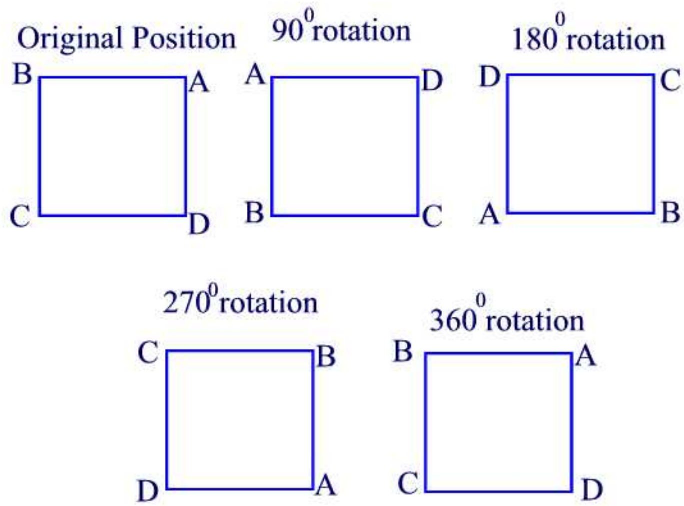


Figure 39.2

The following table list the number of turns, including a full turn, of some plane figures.

Figure	Rotation
Rectangle	2 rotations($180°$ and $360°$)
Square	4 rotations($90°, 180°, 270°, 360°$)
Rhombus	2 rotations($180°$ and $360°$)
Parallelogram	2 rotations($180°$ and $360°$)
Trapezoid	1 rotation($360°$)
Equilateral Triangle	3 rotations($120°, 240°$, and $360°$)

Example 39.1
Show that the opposite sides of a parallelogram are congruent.

Solution.
Rotate the parallelogram 180° about its center as shown in Figure 39.3.

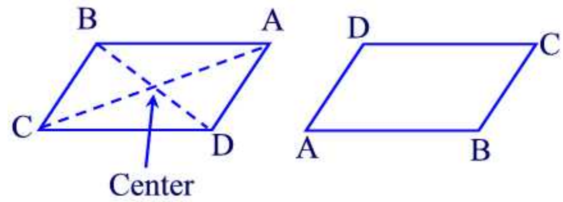
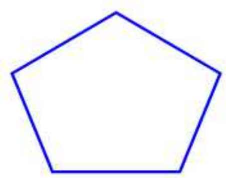


Figure 39.3

It follows that $\overline{AB} = \overline{DC}$ and $\overline{BC} = \overline{AD}$■

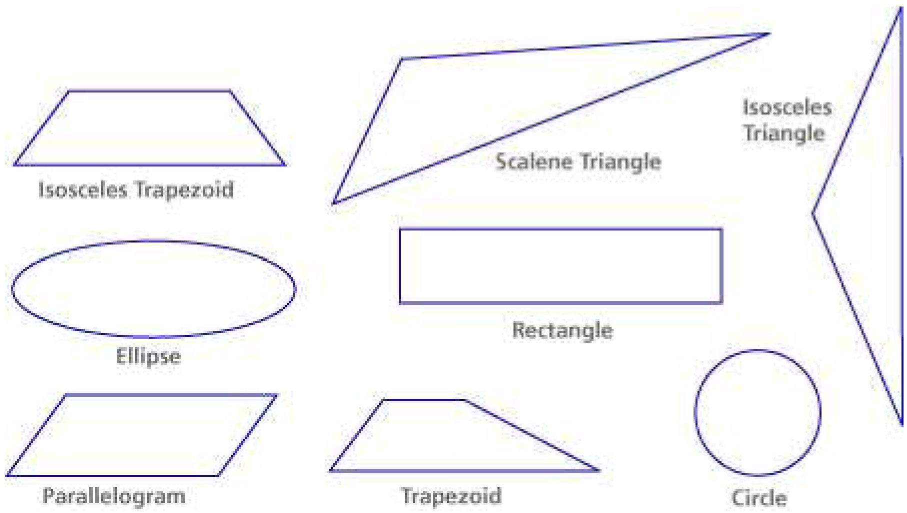
Practice Problems

Problem 39.1
Draw all lines of symmetry for the figure below.

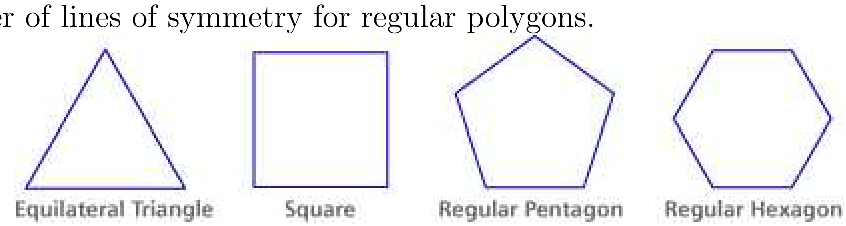


Problem 39.2
(a) Find the vertical line(s) of symmetry of the letters A, U, V, T, Y.
(b) Find the horizontal line(s) of symmetry of the letters D, E, C, B.
(c) Find the vertical and horizontal line(s) of symmetry of the letters H, I, O, X.

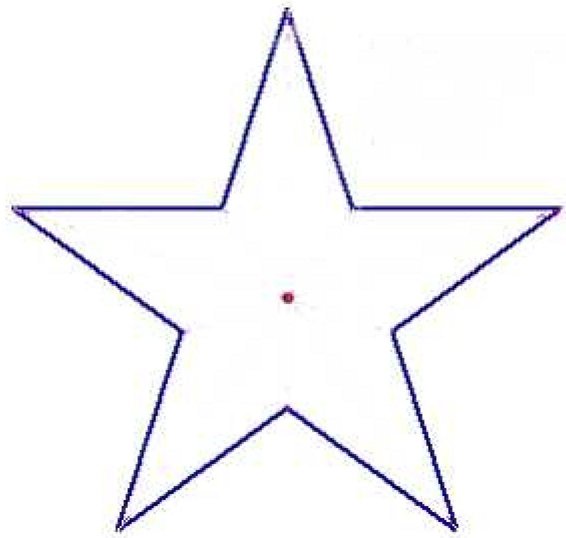
Problem 39.3
For each figure, find all the lines of symmetry you can.



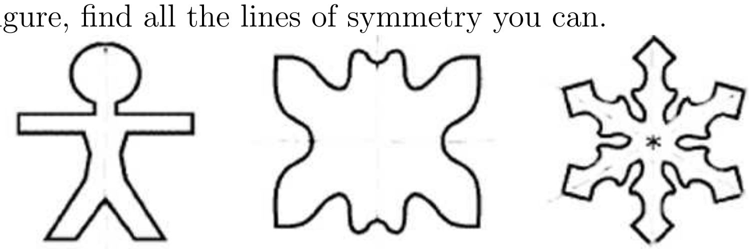
Problem 39.4
A regular polygon is a closed figure with all sides congruent. Find all the lines of symmetry for these regular polygons. Generalize a rule about the number of lines of symmetry for regular polygons.



Problem 39.5
Find the number of rotations of the following geometric shape.



Problem 39.6
For each figure, find all the lines of symmetry you can.



Diagonals of Quadrilaterals
A **diagonal** of a quadrilateral is a line segement formed by connecting non-adjacent vertices (i.e., not on the same side). Each quadrilateral discussed in Section 37 has two diagonals.

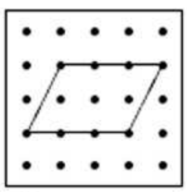
Example 39.2
For each geoboard quadrilateral below draw in the two diagonals. Enter the data requested below each figure. The meaning of each entry is given here.

Type: Which of the seven types of quadrilaterals best describes the figure: square, rhombus, rectangle, parallelogram, kite, trapezoid or isosceles trapezoid.
Cong $\cong$: Are the diagonals congruent? Yes or No.

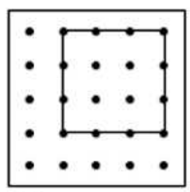
Perp $\perp$: Are the diagonals perpendicular? Yes or No.
Bisect: Do the diagonals bisect each other? Yes or No.
Bisect $\angle$: Do the diagonals bisect the corner angles? Yes or No.

Instructions: Complete this table by entering yes or no in each box.

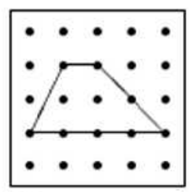
Quads	Diags. $\cong$	Diags $\perp$	Diags Bisect	Diags Bisect $\angle$
Square				
Rhombus				
Rectangle				
Parallelogram				
Kite				
Trapezoid				
Iso. Trap.				



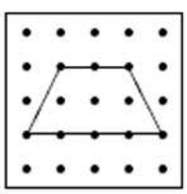
Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



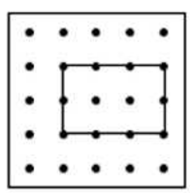
Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



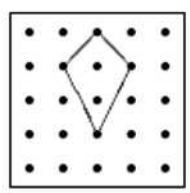
Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



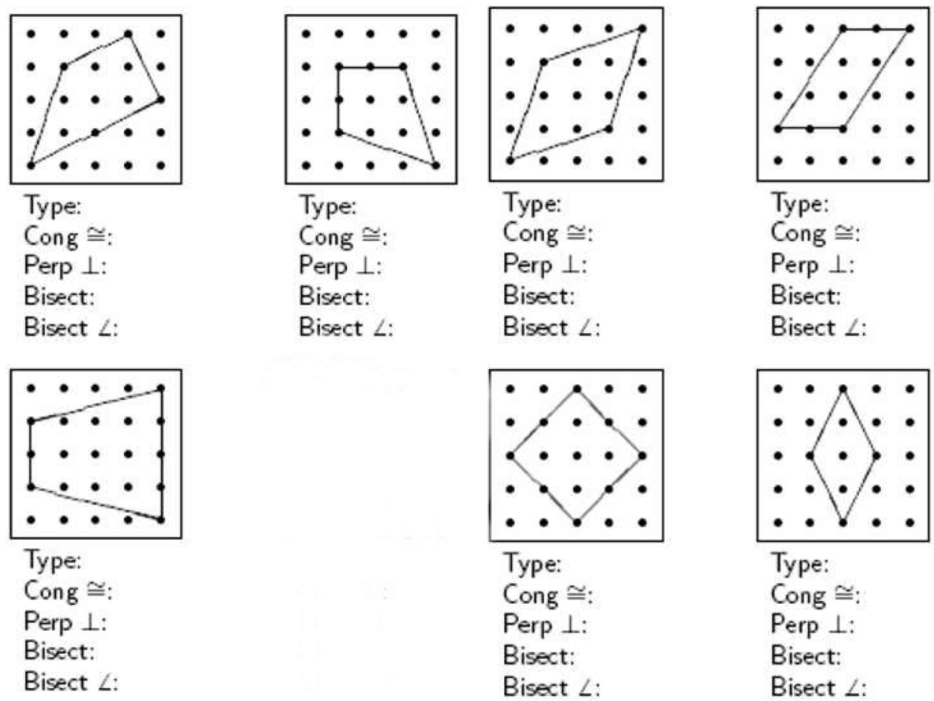
Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



Type:
Cong $\cong$:
Perp $\perp$:
Bisect:
Bisect $\angle$:



Solution.

Quads	Diags. ≅	Diags ⊥	Diags Bisect	Diags Bisect ∠
Square	yes	yes	yes	yes
Rhombus	yes	yes	yes	yes
Rectangle	yes	No	yes	No
Parallelogram	yes	No	yes	No
Kite	No	yes	No	No
Trapezoid	No	No	No	No
Iso. Trap.	yes	No	No	No

Circles

So far we have been studying plane figures with boundaries consisting of line segments. Circles are plane figures with curved boundary. By definition, a **circle** is the set of all points in the plane that are equidistant from a fixed point called the **center** of the circle. The distance from the center to a point on the circle is called the **radius**. Any line segment crossing the center and whose endpoints are on the circle is called a **diameter** of the circle. It is clear that the length of a diameter is twice the length of the radius of the circle. See Figure 39.4.

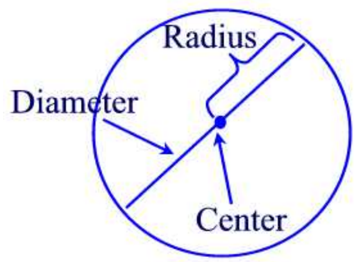


Figure 39.4

A useful device for drawing circles is the **compass** as shown in Figure 39.5.

Figure 39.5

The steps for drawing a circle with a compass are as follows:

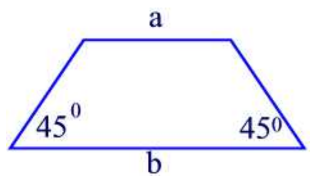
1. Insert a sharp pencil into the holder on the compass.
2. Open the compass to the radius desired for your circle.
3. Place the compass point on a piece of paper where you would like the center point of the circle to be.
4. Place the point of the pencil on the paper.
5. Rotate the compass to mark a circle on the paper with the pencil. Draw without lifting the point of the compass off the paper.

Finally, we conclude this section by looking at the symmetry properties of circles. Any diameter is a line of symmetry of a circle so that a circle has infinitely many lines of symmetry. Also, a circle has infinitely many rotation symmetries, since every angle whose vertex is the center of the circle is an angle of rotation symmetry.

Practice Problems

Problem 39.7
Let ABCD be a parallelogram.
(a) Prove that $\angle A$ and $\angle B$ are supplementary.
(b) Prove that angles $\angle A$ and $\angle C$ are congruent.

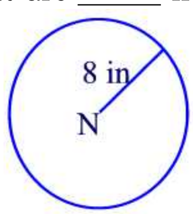
Problem 39.8
Using the figure below find the height of the trapezoid(i.e., the distance between the parallel sides) in terms of a and b.



Problem 39.9
"The diagonals of a rectangle are congruent." Why does this statement imply that the diagonals of a square must also be congruent?

Problem 39.10
You learn the theorem that the diagonals of a parallelogram bisect each other. What other quadrilaterals must have this property?

Problem 39.11
Fill in the blank to describe the following circle with center N. Circle N is the set of _____ in a plane that are _____ from ____.



Problem 39.12
A **chord** is a line segment with endpoints on a circle. True or false? Every chord is also a diameter of the circle.

Problem 39.13
The diameter of a circle divides its interior into two congruent regions. How can someone use this property in dividing a circular pizza or pie in half?

Problem 39.14
If possible draw a triangle and a circle that intersect at exactly
(a) one point (b) two points (c) three points (d) four points

Problem 39.15
If possible draw two parallelograms that intersect at exactly
(a) one point (b) two points (c) three points (d) four points

Problem 39.16
A quadrilateral has two right angles. What can you deduce about the measures of the other two angles?

40 Regular Polygons

Convex and Concave Shapes

A plane figure is said to be **convex** if every line segment drawn between any two points inside the figure lies entirely inside the figure. A figure that is not convex is called a **concave** figure. Figure 40.1 shows a set of convex figures.

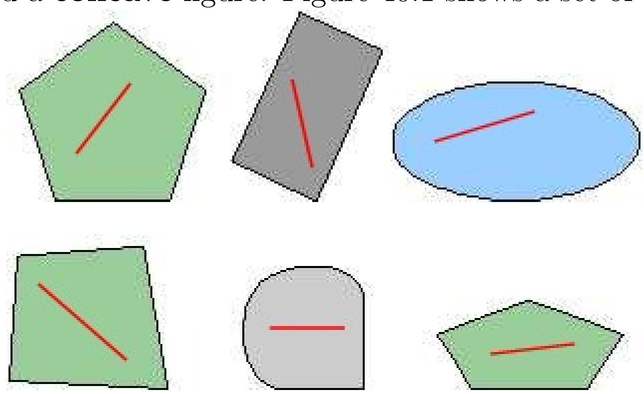


Figure 40.1

On the other hand, Figure 40.2 shows concave figures. To show that a figure is concave, it is enough to find two points within the figure whose corresponding line segment is not completely inside the figure.

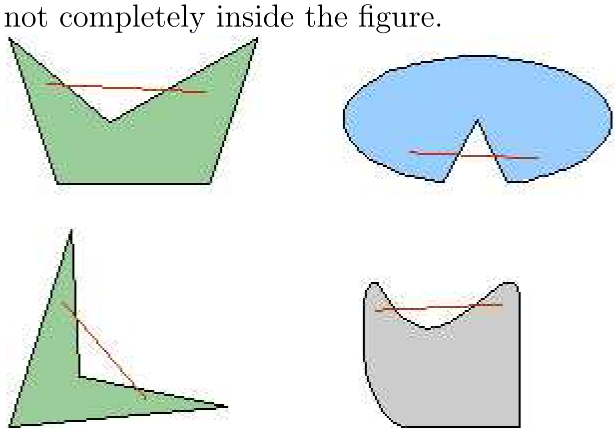


Figure 40.2

Regular Polygons

By a **closed** curve we mean a curve that starts from one point and ends in that same point. By a **simple** curve we mean a curve that does not cross itself. By a **simple closed curve** we mean a a curve in the plane that starts

and ends in the same location without crossing itself.
Several examples of curves are shown in Figure 40.3.

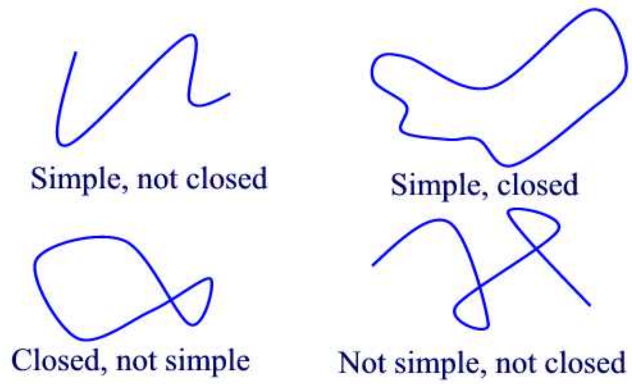


Figure 40.3

A **polygon** is a simple closed curve made up of line segments. A polygon whose line segments are congruent and whose interior angles are all congruent is called a **regular polygon**. If a regular polygon consists of n sides then we will refer to it as **regular n-gon**.
Figure 40.4 shows several tpyes of regular n-gons.

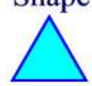
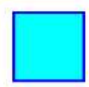
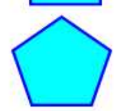
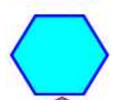
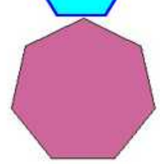
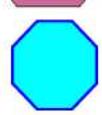
Number of Sides	Name	Shape
3	Equilateral Triangle	
4	Square	
5	Pentagon	
6	Hexagon	
7	Heptagon	
8	Octagon	

Figure 40.4

Angles of interest in a regular n-gon are the following: A **vertex angle** (also called an **interior** angle) is formed by two consecutive sides. A **central angle** is formed by the segments connecting two consecutive vertices to the center of the regular n-gon. An **exterior angle** is formed by one side together with the extension of an adjacent side as shown in Figure 40.5

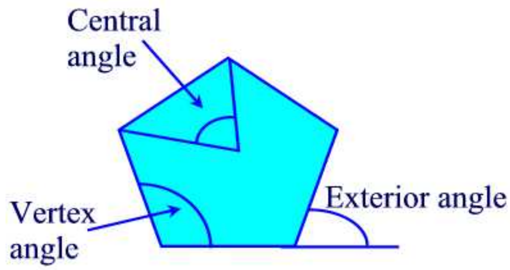
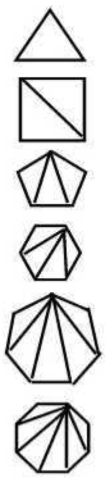


Figure 40.5

Angles Measures in Regular Polygons

Let's first find the measure of a central angle in a regular n-gon. Connecting the center of the n-gon to the n vertices we create n congruent central angles. Since the sum of the measures of the n central angles is 360° then the measure of each central angle is $\frac{360°}{n}$.

Next, we will find the measure of each interior angle of a regular n-gon. We will use the method of recognizing patterns for that purpose. Since the angles are congruent then the measure of each is the sum of the angles divided by n. Hence, we need to find the sum of the interior angles. This can be achieved by dividing the n-gon into triangles and using the fact that the sum of the three interior angles in a triangle is 180°. The table below suggests a way for computing the measure of a vertex angle in a regular n-gon for n=3,4,5,6,7,8.



n	Sum of all interior angles(in degrees)	Measure of a vertex angle
3	$1.180 = 180^0$	$180/3 = 60^0$
4	$2.180 = 360^0$	$360/4 = 90^0$
5	$3.180 = 540^0$	$540/5 = 108^0$
6	$4.180 = 720^0$	$720/6 = 120^0$
7	$5.180 = 900^0$	$900/7 = 128\frac{4}{7}^0$
8	$6.180 = 1080^0$	$1080/8 = 135^0$

So, in general, the measure of an interior angle of a regular n-gon is

$$\frac{(n-2)\cdot 180^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$$

To measure the exterior angles in a regular n-gon, notice that the interior angle and the corresponding adjacent exterior angle are supplementary. See Figure 40.6. Thus, the measure of each exterior angle is

$$180^\circ - \frac{(n-2)\cdot 180^\circ}{n} = \frac{360^\circ}{n}.$$

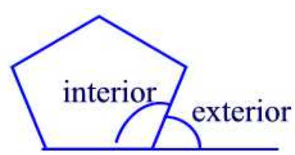


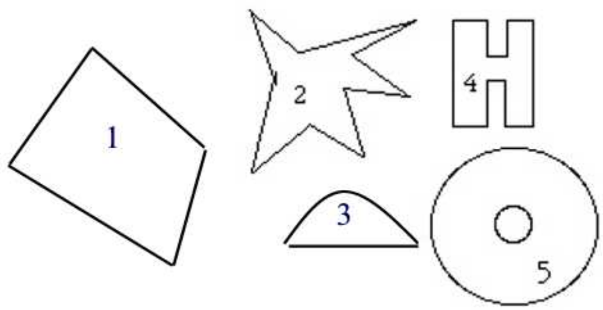
Figure 40.6

Example 40.1
(a) Find the measure of each interior angle of a regular decagon (i.e., n =10).
(b) Find the number of sides of a regular polygon, each of whose interior angles has a measure of 175°.

Solution.
(a) The measure of each angle is: $\frac{(10-2)\cdot 180^\circ}{10} = 144^\circ$.
(b) We are given that $\frac{(n-2)\cdot 180^\circ}{n} = 175^\circ$ or $180^\circ - \frac{360^\circ}{n} = 175^\circ$. This implies that $\frac{360^\circ}{n} = 180^\circ - 175^\circ = 5^\circ$. Thus, $n = \frac{360^\circ}{5^\circ} = 72$ ■

Practice Problems

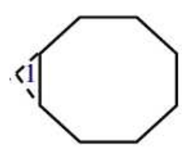
Problem 40.1
List the numerical values of the shapes that are convex .



Problem 40.2
Determine how many diagonals each of the following has:
(a) 20-gon (b) 100-gon (c) n-gon

Problem 40.3
In a regular polygon, the measure of each interior angle is 162°. How many sides does the polygon have?

Problem 40.4
Two sides of a regular octagon are extended as shown in the following figure. Find the measure of $\angle 1$.



Problem 40.5
Draw a quadrilateral that is not convex.

Problem 40.6
What is the sum of the interior angle measures of a 40-gon?

Problem 40.7
A Canadian nickel has the shape of a regular dodecagon (12 sides). How many degrees are in each interior angle?

Problem 40.8
Is a rectangle a regular polygon? Why or why not?

Problem 40.9
Find the measures of the interior, exterior, and central angles of a 12-gon.

Problem 40.10
Suppose that the measure of the interior angle of a regular polygon is $176°$. What is the measure of the central angle?

Problem 40.11
The measure of the exterior angle of a regular polygon is $10°$. How many sides does this polygon have?

Problem 40.12
The measure of the central angle of a regular polygon is $12°$. How many sides does this polygon have?

Problem 40.13
The sum of the measures of the interior angles of a regular polygon is $2880°$. How many sides does the polygon have?

Problem 40.14
How many lines of symmetry does each of the following have?

(a) a regular pentagon
(b) a regular octagon
(c) a regular hexagon.

Problem 40.15
How many rotational symmetry does a pentagon have?

41 Three Dimensional Shapes

Space figures are the first shapes children perceive in their environment. In elementary school, children learn about the most basic classes of space figures such as prisms, pyramids, cylinders, cones and spheres. This section concerns the definitions of these space figures and their properties.

Planes and Lines in Space
As a basis for studying space figures, first consider relationships among lines and planes in space. These relationships are helpful in analyzing and defining space figures.
Two planes in space are either **parallel** as in Figure 41.1(a) or intersect as in Figure 41.1(b).

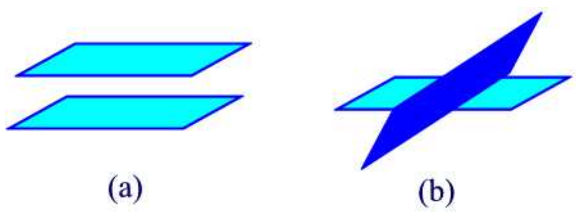


Figure 41.1

The angle formed by two intersecting planes is called the **dihedral angle**. It is measured by measuring an angle whose sides line in the planes and are perpendicular to the line of intersection of the planes as shown in Figure 41.2.

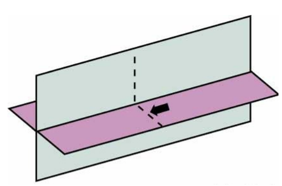


Figure 41.2

Some examples of dihedral angles and their measures are shown in Figure 41.3.

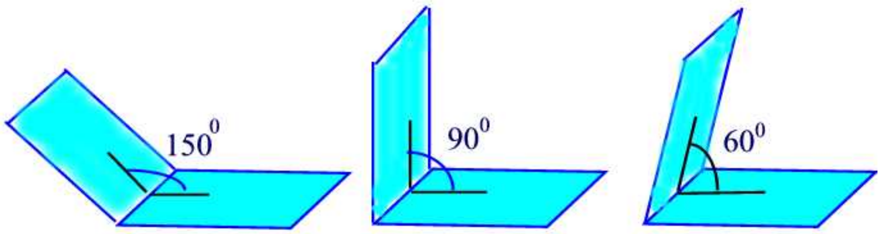


Figure 41.3

Two nonintersecting lines in space are **parallel** if they belong to a common plane. Two nonintersecting lines that do not belong to the same plane are called **skew lines.** If a line does not intersect a plane then it is said to be **parallel to the plane.** A line is said to be **perpendicular to a plane** at a point A if every line in the plane through A intersects the line at a right angle. Figures illustrating these terms are shown in Figure 41.4.

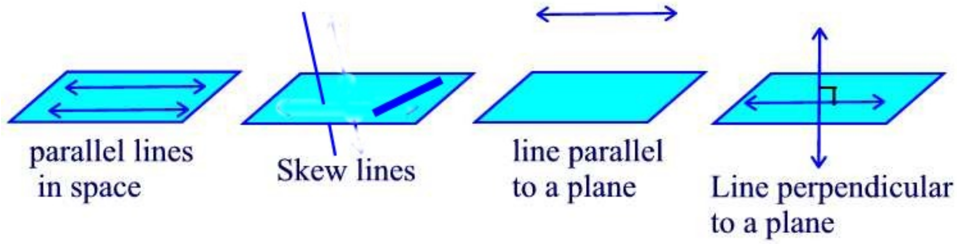


Figure 41.4

Polyhedra

To define a polyhedron, we need the terms "simple closed surface" and "polygonal region." By a **simple closed surface** we mean any surface without holes and that encloses a hollow region-its interior. An example is a **sphere,** which is the set of all points at a constant distance from a single point called the **center.** The set of all points on a simple closed surface together with all interior points is called a **solid.** For example, the shell of a hardboiled egg can be viewed as a simple closed surface whereas the shell together with the white and the yolk of the egg form a solid. A **polygonal region** is a polygon together with its interior as shown in Figure 41.5.

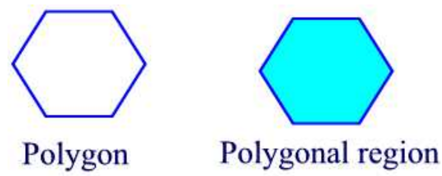


Figure 41.5

A **polyhedron** (polyhedra is the plural) is a simple closed surface bounded by polygonal regions. All of the surfaces are flat, not curved. The polygonal regions are called the **faces**. The sides of each face are called the **edges.** The vertices of the polygons are also called **vertices** of the polyhedron. Figure 41.6(a) and (b) are examples of polyhedra, but (c) and (d) are not.

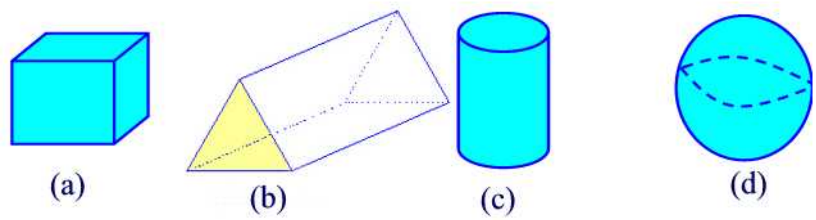


Figure 41.6

Polyhedra are named according to the number of faces. For example, a **tetrahedron** has four faces, a **pentahedron** has five faces, a **hexahedron** has six faces, and so on.

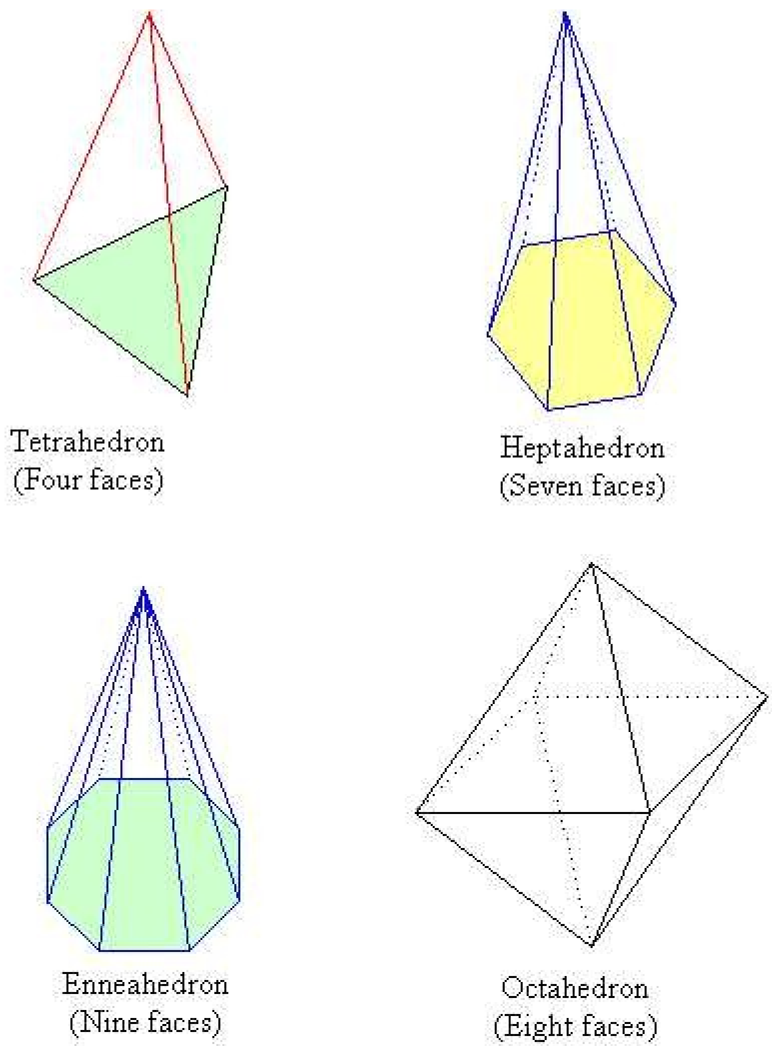


Figure 41.7

Polyhedra can be classified into several types:

• **Prisms:** A prism is a polyhedron, with two parallel faces called **bases.** The other faces are called **lateral faces** and are always parallelogram. If the lateral faces are rectangles then the prism is called a **right prism.** Otherwise, the prism is called an **oblique prism.** Figure 41.8 shows some types of prisms.

Figure 41.8

• **Pyramids:** A polyhedron is a **pyramid** if it has 3 or more triangular faces sharing a common vertex, called the **apex.** The base of a pyramid may be any polygon. Pyramids are named according to the type of polygon forming the base. Pyramids whose bases are regular polygons are called **regular pyramids**. If in addition, the lateral faces are isosceles triangles then the pyramids are called **right regular pyramids.** Otherwise, they are called **oblique regular pyramids.** See Figure 41.9.

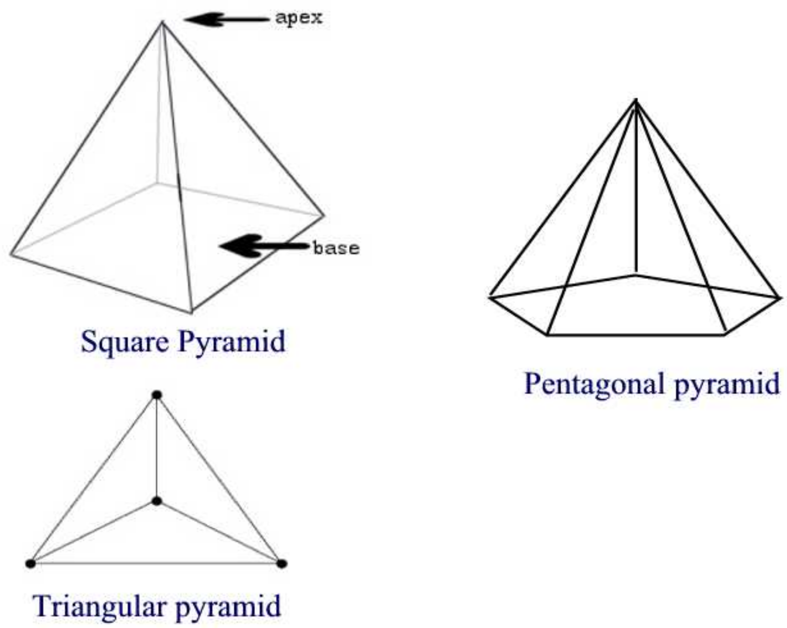


Figure 41.9

Regular Polyhedra

A polyhedron is said to be **regular** if all its faces are identical regular polygonal regions and all dihedral angles have the same measures. Regular polyhedra are also known as **platonic solids.** There are five regular polyhedra analyzed in the table below and shown in Figure 41.10.

Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Hexahedron	6	12	8
Icosahedron	20	30	12
Dodecahedron	12	30	20

Figure 41.10

Curved Shapes: Cylinders, Cones, and Spheres
Some everyday objects suggest space figures that are not polyhedra: cylinders (can of soup), cones (ice cream cone), and spheres (a basketball).

• **Cylinders:** A **cylinder** is the space figure obtained by translating the points of a simple closed region in one plane to a parallel plane as shown in Figure 41.11(a). The simple closed regions are called the **base** of the cylinder. The remaining points constitute the **lateral surface** of the cylinder.

The most common types of cylinders in our environment are right circular cylinders. The line joining the centers of the two circles is called the **axis.** If the axis is perpendicular to the circles then the cylinder is called a **right circular cylinder** (See Figure 41.11(b)). Otherwise it is called an **oblique circular cylinder.** (See Figure 41.11(c))

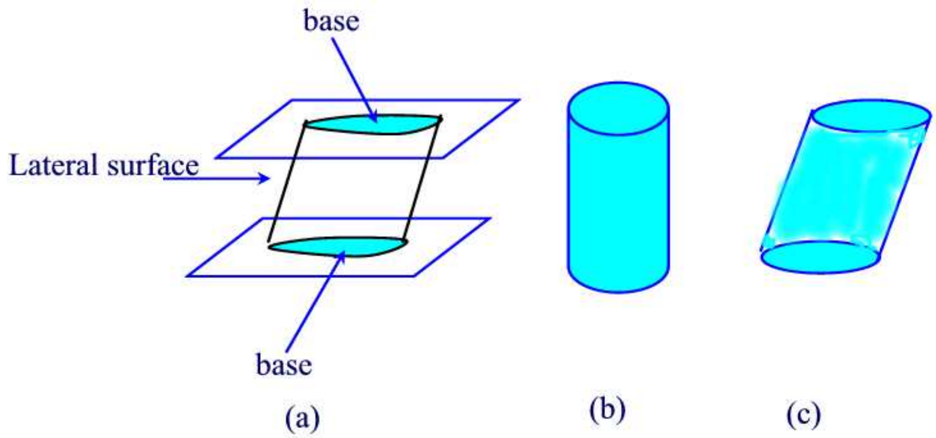


Figure 41.11

• **Cones:** Suppose we have a simple closed region in a plane and a point P not in the plane of the region. The union of line segments connecting P to each point in the simple closed region is called a **cone.** See Figure 41.12(a). P is called the **vertex**. The simple closed region is called the **base**. The points in the cone not in the base constitute the **lateral surface**. If the base is a circle then the cone is called a **circular cone**. A circular cone where the line connecting the vertex to the center of the base is perpendicular to the base is called a **right circular cone**. See Figure 41.12(b). Otherwise, the cone is called **oblique circular cone.** See Fibure 41.12(c).

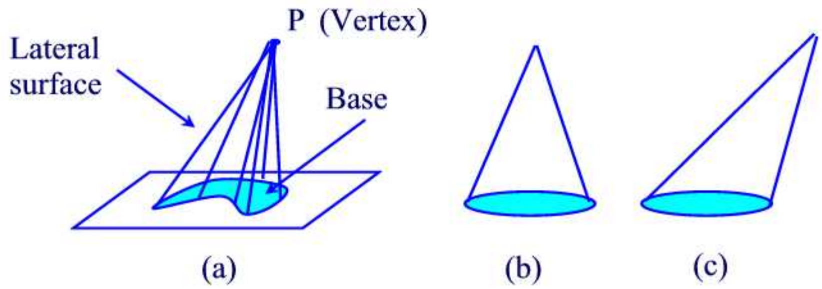


Figure 41.12

• **Spheres:** We have already encountered this concept. Recall that a sphere is the collection of all points in the space that are the same distance from a fixed point called **center.** Any line segment joining the center to a point on the surface of the sphere is called a **radius**. Any line segment going through the center and connecting two points on the sphere is called a **diameter.** See Figure 41.13.

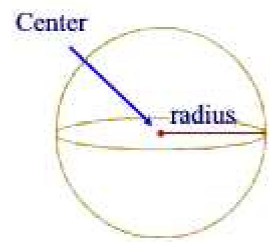
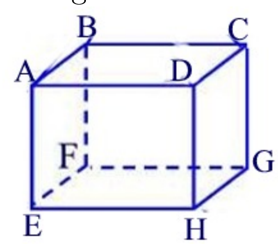


Figure 41.13

Practice Problems

Problem 41.1
Determine the measures of all dihedral angles of a right prism whose bases are regular octagons.

Problem 41.2
Consider the cube given in the figure below.



(a) How many planes are determined by the faces of the cube?
(b) Which edges of the cube are parallel to edge $\overline{AB}$?
(c) Which edges of the cube are contained in lines that are skew to the line going through A and B?

Problem 41.3
If a pyramid has an octagon for a base, how many lateral faces does it have?

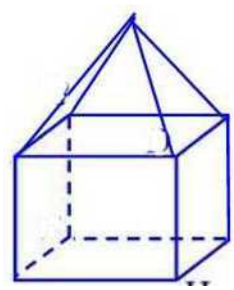
Problem 41.4
Determine the number of faces, vertices, and edges for a hexagonal pyramid.

Problem 41.5
Sketch drawings to illustrate different possible intersections of a square pyramid and a plane.

Problem 41.6
Sketch each of the following.

(a) A plane and a cone that intersect in a circle.
(b) A plane and a cylinder that intersect in a segment.
(c) Two pyramids that intersect in a triangle.

Problem 41.7
For the following figure, find the number of faces, edges, and vertices.



Problem 41.8
For each of the following figures, draw all possible intersections of a plane with
(a) Cube (b) Cylinder

Problem 41.9
A diagonal of a prism is any segment determined by two vertices that do not lie in the same face. How many diagonals does a pentagonal prism have?

Problem 41.10
For each of the following, draw a prism and a pyramid that have the given region as a base:

(a) Triangle
(b) Pentagon
(c) Regular hexagon

Problem 41.11
If possible, sketch each of the following:
(a) An oblique square prism

(b) An oblique square pyramid
(c) A noncircular right cone
(d) A noncircular cone that is not right

Problem 41.12
A simple relationship among the number of faces(F), the number of edges(E), and the number of vertices(V) of any convex polyhedron exists and is known as **Euler's formula.** Use the following table to find this relationship.

Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Hexahedron	6	12	8
Icosahedron	20	30	12
Dodecahedron	12	30	20

Problem 41.13
What is the sum of the angles at the each vertex of a
(a) Tetrahedron
(b) Octahedron
(c) Cube
(d) Icosahedron
(e) Dodecahedron

Problem 41.14
Determine for each of the following the minimum number of faces possible:
(a) Prism (b) Pyramid (c) Hexahedron

Problem 41.15
True or false?
(a) Through a given point not on plane P, there is exactly one line parallel to P.
(b) Every set of four points is contained in one plane.
(c) If a line is perpendicular to one of two parallel planes then it is perpendicular to the other.

Problem 41.16
Tell whether each of the following suggests a polygon or a polygonal region.
(a) A picture frame.
(b) A page in this book.
(c) A stop sign.

Problem 41.17
A certain prism has 20 vertices. How many faces and edges does it have?

Problem 41.18
Why are there no such things as skew planes?

Problem 41.19
A prism has a base with n sides.

(a) How many faces does it have?
(b) How many vertices does it have?
(c) How many edges does it have?

Problem 41.20
A pyramid has a base with n sides.

(a) How many faces does it have?
(b) How many vertices does it have?
(c) How many edges does it have?

42 Systems of Measurements

So far only two goemetric figures have been measured: line segments, measured by the distance between their endpoints; and angles, measured by the degrees of rotation needed to turn one side to the other. In this and coming sections we will introduce more general notions of measurement of geometric figures. All curves, not just segments will be given a length. Plane regions will be measured by area and perimeter. Space figures will be measured by surface area and volume.

We begin by discussing the general process of measurement. The two principal systems of measurements are then described: the English System and the Metric or International System.

The Measurement Process

The measurement process consists of the following steps:

(1) Select an object and an attribute of the object to measure, such as its length, area, volume, temperature, or weight.

(2) Select an appropriate unit which to measure the attribute.

(3) Determine the number of units needed to measure the attribute.

Measurements can be done using either nonstandard units or standard units. For example, the first nonstandard units of length were based on people's hands, feet, and arms. Since different people have different-sized hands, feet, and arms then these measurements are adequate for convenient but lack accuracy. This is why we need standard measuring units such as of those of the metric system. By using standard units, people around the globe communicate easily with one another about measurements. We consider two standard systems of measurements:

The US Customary System(English System)

The following are the common conversion equivalencies in the English System.

Length

$$1 \text{ in (inch)} = \frac{1}{12} \text{ ft (foot)}$$
$$1 \text{ yd (yard)} = 3 \text{ ft}$$
$$1 \text{ rd (rod)} = 16.5 \text{ ft}$$
$$1 \text{ furlong} = 660 \text{ ft}$$

$$1 \text{ mi (mile)} = 5280 \text{ ft}$$

Area

$$1 \text{ square inch} = \frac{1}{144} \text{ square feet}$$
$$1 \text{ square yard} = 9 \text{ square feet}$$
$$1 \text{ acre} = 43{,}560 \text{ square feet}$$
$$1 \text{ square mile} = 27{,}878{,}400 \text{ square feet}$$

Volume

$$1 \text{ cubic in} = \frac{1}{1728} \text{ cubic feet}$$
$$1 \text{ cubic yard} = 27 \text{ cubic feet}$$

Capacity

$$1 \text{ tablespoon} = 3 \text{ teaspoons}$$
$$1 \text{ ounce} = 2 \text{ tablespoons}$$
$$1 \text{ cup} = 8 \text{ ounces}$$
$$1 \text{ pint} = 2 \text{ cups}$$
$$1 \text{ quart} = 2 \text{ pints}$$
$$1 \text{ gallon} = 4 \text{ quarts}$$
$$1 \text{ barrel} = 31.5 \text{ gallons}$$

Weight

$$1 \text{ pound} = 16 \text{ ounces}$$
$$1 \text{ ton} = 2000 \text{ pounds}$$

Time

$$1 \text{ minute} = 60 \text{ seconds}$$
$$1 \text{ hour} = 60 \text{ minutes}$$
$$1 \text{ day} = 24 \text{ hours}$$
$$1 \text{ week} = 7 \text{ days}$$
$$1 \text{ year} \approx 365.25 \text{ days}$$

Temperature(Fahrenheit)

$$32°F = \text{freezing point of water}$$
$$212°F = \text{boiling point of water}$$

Dimensional Analysis
To convert from one unit to another, the process knwon as **dimensional analysis** can be used. This process works with unit ratios(ratios equal to 1). For example, $\frac{1\ yd}{3\ ft}$ and $\frac{5280\ ft}{1\ mi}$ are units ratios. Therefore to convert 5.25 mi to yards, we have the following

$$5.25 \text{ mi} = 5.25\ mi \times \frac{5280\ ft}{1\ mi} \times \frac{1\ yd}{3\ ft} = 9240 \text{ yd}$$

Example 42.1
If a cheetah is clocked at 60 miles per hour, what is its speed in feet per second?

Solution.

$$60\frac{mi}{hr} = 60\frac{mi}{hr} \times \frac{5280\ ft}{1\ mi} \times \frac{1\ hr}{60\ min} \times \frac{1\ min}{60\ sec} = 88\frac{ft}{sec} \blacksquare$$

Example 42.2
Convert each of the following:
(a) 219 ft = ____ yd
(b) 8432 yd = ____ mi
(c) 0.2 mi = ____ ft
(d) 64 in = ____ yd

Solution.
(a) 219 ft = 219 ft $\times \frac{1\ yd}{3\ ft}$ = 73 yd.
(b) 8432 yd = 8432 yd $\times \frac{3\ ft}{1\ yd} \times \frac{1\ mi}{5280\ ft} \approx 4.79\ mi.$
(c) 0.2 mi = 0.2 mi $\times \frac{5280\ ft}{1\ mi}$ = 1056 ft
(d) 64 in = 64 in $\times \frac{1\ ft}{12\ in} \times \frac{1\ yd}{3\ ft} \approx$ 1.78 yd $\blacksquare$

International System
Six common prefixes used in the metric system are listed below.

Prefixes	Multiple or fraction
kilo	1000
hecto	100
deca	10
deci	0.1
centi	0.01
milli	0.001

The following are the common conversion equivalencies in the Metric System.

Length

Unit	Abbreviation	Number of Meters
kilometer	km	1,000
hectometer	hm	100
dekameter	dam	10
meter	m	1
decimeter	dm	0.1
centimeter	cm	0.01
millimeter	mm	0.001

Area

Unit	Abbreviation	Number of square meters
square millimeter	mm^2	0.000001
square centimeter	cm^2	0.0001
square decimeter	dm^2	0.01
Are	a	100
Hectare	ha	10000
square kilometer	km^2	1000000

Volume

Unit	Abbreviation	Number of cubic meters
cubic meter	m^3	1
cubic decimeter	dm^3	0.001
cubic centimeter	cm^3	0.000001

Capacity

Unit	Abbreviation	Number of liters
kiloliter	kL	1,000
hectoliter	hL	100
dekaliter	daL	10
liter	L	1
deciliter	dL	0.10
centiliter	cL	0.01
milliliter(cm^3)	mL	0.001

Mass

Unit	Abbreviation	Number of grams
metric ton	t	1,000,000
kilogram	kg	1,000
hectogram	hg	100
dekagram	dag	10
gram	g	1
decigram	dg	0.10
centigram	cg	0.01
milligram	mg	0.001

Temperature(Celsius)

$$0°C = \text{freezing point of water}$$
$$100°C = \text{boiling point of water}$$

Example 42.3
Convert the following measurements to the unit shown.

(a) 1495 mm = ____ m
(b) 29.4 cm = ____ mm
(c) 38741 m = ____ km

Solution.
(a) 1495 mm = $1495\ mm \times \frac{1\ m}{1000\ mm}$ = 1.495 m
(b) 29.4 cm = $29.4\ cm \times \frac{1\ mm}{0.1\ cm}$ = 294 mm
(c) 38741 m = $38741\ m \times \frac{1\ km}{1000\ m}$ = 38.741km ■

Conversion Between Systems
Length: 1 in = 2.54 cm
Area: $1\ in^2 = 2.54^2\ cm^2$
Volume: $1\ in^3 = 2.54^3\ cm^3$
Capacity: $1\ cm^3$ = 1 mL, 1 gal ≈ 3.79 L
Weight: 1 oz = 29 g
Temperature: $F = \frac{9}{5}C + 32$

Practice Problems

Problem 42.1
A small bottle of Perrier sparking water contains 33 cl. What is the volume in ml?

Problem 42.2
Fill in the blanks.
(a) 58728 g = ____ kg
(b) 632 mg = ____ g
(c) 0.23 kg = ____ g

Problem 42.3
Convert each of the following.
(a) 100 in = ____ yd
(b) 400 yd = ____ in
(c) 300 ft = ____ yd
(d) 372 in = ____ ft

Problem 42.4
Complete the following table.

Item	m	cm	mm
Length of a piece of paper		35	
Height of a woman	1.63		
Width of a filmstrip			35
Length of a cigarette			100

Problem 42.5
List the following in decreasing order: 8 cm, 5218 mm, 245 cm, 91 mm, 6 m, 700 mm.

Problem 42.6
Complete each of the following.
(a) 10 mm = ____ cm
(b) 3 km = ____ m
(c) 35 m = ____ cm
(d) 647 mm = ____ cm

Problem 42.7
Complete the following conversions.
(a) 3 feet = ____ inches
(b) 2 miles = ____ feet
(c) 5 feet = ____ yards

Problem 42.8
Complete the following conversions.
(a) 7 yards = ____ feet
(b) 9 inches = ____ feet
(c) 500 yards = ____ miles

Problem 42.9
Complete the following conversions.
(a) 9.4 L = ____ mL
(b) 37 mg = ____ g
(c) 346 mL = ____ L

Problem 42.10
A nurse wants to give a patient 0.3 mg of a certain drug. The drug comes in a solution containing 0.5 mg per 2 mL. How many milliters should be used?

Problem 42.11
A nurse wants to give a patient 3 gm of sulfisoxable. It comes in 500 mg tablets. How many tablets should be used?

Problem 42.12
Complete the following conversions.
(a) 3 gallons = ____ quarts
(b) 5 cups = ____ pints
(c) 7 pints = ____ quarts
(d) 12 cups = ____ gallons

Problem 42.13
True or false? Expain.
(a) 1 mm is longer than 1 in.
(b) 1 m is longer than 1 km.
(c) 1 g is heavier than 1 lb.
(d) 1 gallon is more than 1 L.

Problem 42.14
Derive a conversion formula for degrees Celsius to degrees Fahrenheit.

Problem 42.15
A temperature of $-10°C$ is about
(a) $-20°F$ (b) $10°F$ (c) $40°F$ (d) $70°F$

Problem 42.16
Convert the following to the nearest degree
(a) Moderate oven ($350°F$) to degrees Celsius.
(b) $20°C$ to degrees Fahrenheit.
(c) $-5°C$ to degrees Fahrenheit.

Problem 42.17
Complete the following conversions.
(a) $1\ cm^2 =$ ____ mm^2
(b) $610\ dam^2 =$ ____ hm^2
(c) $564\ m^2 =$ ____ km^2
(d) $0.382\ km^2 =$ ____ m^2

Problem 42.18
Suppose that a bullet train is traveling 200 mph. How many feet per seconds is it traveling?

Problem 42.19
A pole vaulter vaulted 19ft$4\frac{1}{2}$in. Find the height in meters.

Problem 42.20
The area of a rectangular lot is 25375 ft^2. What is the area of the lot in acres? Use the fact that 640 acres = 1 square mile.

Problem 42.21
A vase holds 4286 grams of water. What is the capacity in liters? Recall that the density of water is 1 g/cm^3.

Problem 42.22
By using dimensional analysis, make the following conversions.
(a) 3.6 lb to oz
(b) 55 mi/hr to ft/min
(c) 35 mi/hr to in/sec
(d) $575 per day to dollars per minute.

Problem 42.23
The density of a substance is the ratio of its mass to its volume. A chunk of oak firewood weighs 2.85 kg and has a volume of 4100 cm^3. Determine the density of oak in g/cm^3, rounding to the nearest thousandth.

Problem 42.24
The speed of sound is 1100 ft/sec at sea level. Express the speed of sound in mi/hr.

Problem 42.25
What temperature is numerically the same in degrees Celsius and degrees Fahrenheit?

43 Perimeter and Area

Perimeters of figures are encountered in real life situations. For example, one might want to know what length of fence will enclose a rectangular field. In this section we will study the perimeters of polygons and circles.
By definition, the **perimeter** of a simple closed plane figure is the length of its boundary. The perimeter is always measured in units of length, such as feet or centimeters.
The perimeter of a polygon is defined to be the sum of the lengths of its sides. Figure 43.1 exhibits the perimeters of some standard plane figures. We will use p to denote the perimeter of a plane figure.

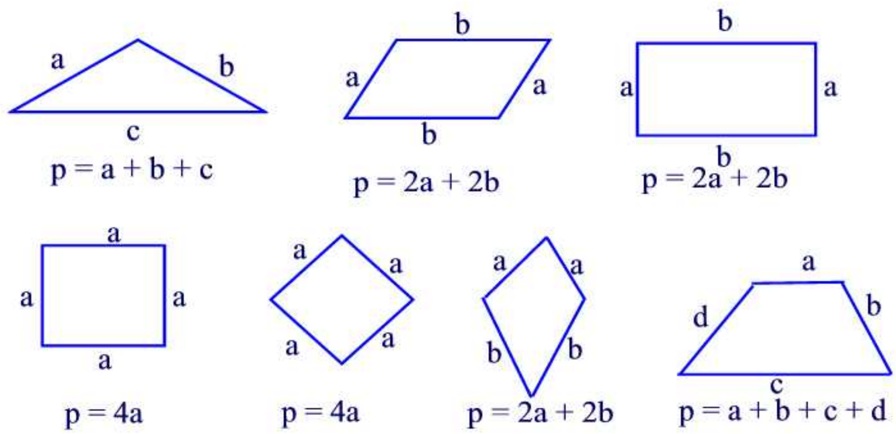


Figure 43.1

The Circumference of a Circle:
The perimeter of a circle is called its **circumference.** Using a tape measure around the circle one can find the circumference. Also, one will notice that the ratio of the circumference of a circle to its diameter is the same for all circles. This common ratio is denoted by π. Thus

$$\frac{C}{d} = \pi \text{ or } C = \pi d.$$

Since the diameter d is twice the radius r, we also have $C = 2\pi r$. It has been shown that π is an irrational number with unending decimal expansion: $\pi = 3.1415926\cdots$.

Example 43.1
A dining room table has 8 sides of equal length. If one side measures 15 inches, what is the perimeter of the table?

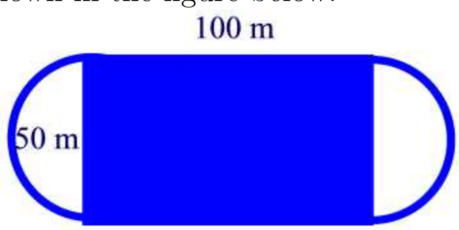
Solution.
If we denote the length of a side by a then the perimeter of the table is $p = 8a$. Since a = 15 in then p=15(8) = 120 in ■

Example 43.2
The circumference of a circle is 6π cm. Find its radius.

Solution.
Since $C = 2\pi r$ and $C = 6\pi$ then $2\pi r = 6\pi$. Solving for r we find $r = \frac{6\pi}{2\pi} = 3\ cm$ ■

Practice Problems

Problem 43.1
An oval track is made up by erecting semicircles on each end of a 50-m by 100-m rectangle as shown in the figure below.



What is the perimeter of the track?

Problem 43.2
Find each of the following:
(a) The circumference of a circle if the radius is 2 m.
(b) The radius of a circle if the circumference is 15π m.

Problem 43.3
Draw a triangle ABC. Measure the length of each side. For each of the following, tell which is greater?
(a) AB + BC or AC
(b) BC + CA or AB
(c) AB + CA or BC

Problem 43.4
Can the following be the lengths of the sides of a triangle? Why or Why not?
(a) 23 cm, 50 cm, 60 cm
(b) 10 cm, 40 cm, 50 cm
(c) 410 mm, 260 mm, 14 cm

Problem 43.5
Find the circumference of a circle with diameter 6π cm.

Problem 43.6
What happens to the circumference of a circle if the radius is doubled?

Problem 43.7
Find the length of the side of a square that has the same perimeter as a rectangle that is 66 cm by 32 cm.

Problem 43.8
A bicycle wheel has a diameter of 26 in. How far a rider travel in one full revolution of the tire? Use 3.14 for π.

Problem 43.9
A car has wheels with radii of 40 cm. How many revolutions per minute must a wheel turn so that the car travels 50 km/h?

Problem 43.10
A lot is 21 ft by 30 ft. To support a fence, an architect wants an upright post at each corner and an upright post every 3 ft in between. How many of these posts are needed?

Area
The number of units required to cover a region in the plane is known as its **area.** Usually squares are used to define a **unit square.** An area of "10 square units" means that 10 unit squares are needed to cover a flat surface. Below, we will find the area of some quadrilaterals and the area of a circle.

Area of a rectangle:
A 3-cm by 5-cm rectangle can be covered by 15 unit squares when the unit square is 1 cm^2, as shown in Figure 43.2(a). Similarly a 2.5-cm by 3.5 cm rectangle can be covered by six whole units, five half-unit squares, and one quarter unit square, giving a total area of 8.75 cm^2 as shown in Figure 43.2(b). This is also the product of the width and the length since $2.5\times3.5 = 8.75\ cm^2$. It follows that a rectangle of length L and width W has an area A given by the formula

$$A = LW.$$

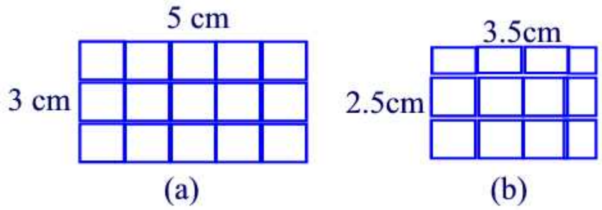


Figure 43.2

Example 43.3
The area of a rectangle of length L and width W is equal to its perimeter. Find the relationship between L and W.

Solution.
Since $A = LW$ and $p = 2(L + W)$ then $LW = 2(L + W)$. That is, $LW = 2L + 2W$. Subtracting 2L from both sides we obtain $L(W - 2) = 2W$. Thus, $L = \frac{2W}{W-2}$ ■

Area of a square:
Since a square is a rectangle with length equals width then the area of a square with side length equals to s is given by the formula

$$A = s^2.$$

Area of a triangle:
First, we consider the case of a right triangle as shown in Figure 43.3(a). Construct rectangle ABDC where ΔDCB is a copy of ΔABC as shown in Figure 43.3(b). The area of the rectangle ABDC is bh, and the area of ΔABC is one-half the area of the rectangle. Hence, the area of $\Delta ABC = \frac{1}{2}bh$.
Now, suppose we have an arbitrary triangle as shown in Figure 43.3(c). Then

$$\text{Area of } \Delta ABC = \text{area of } \Delta ADC + \text{area of } \Delta CDB$$

That is,

$$\begin{array}{rcl} A & = & \frac{1}{2}h \cdot AD + \frac{1}{2}h \cdot DB \\ & = & \frac{h}{2}(AD + DB) \\ & = & \frac{h \cdot AB}{2} \end{array}$$

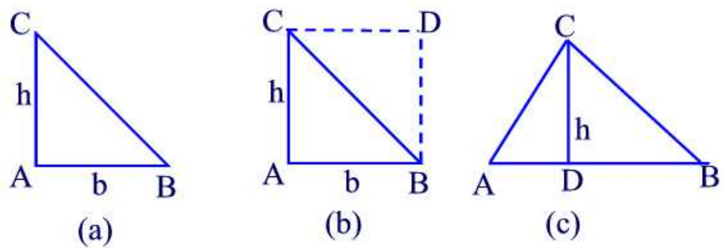


Figure 43.3

Area of a parallelogram:
Consider a parallelogram with a pair of opposite sides b units long, and these sides are h units apart as shown in Figure 43.4(a). The side with length b is called the **base** of the parallelogram and h is the **height** or **altitude.** Removing and replacing a right triangle as shown in Figure 43.4(b) results in a rectangle of length b and width h with the same area as the parallelogram. This means that the area of the parallelogram is given by the formula

$$A = bh.$$

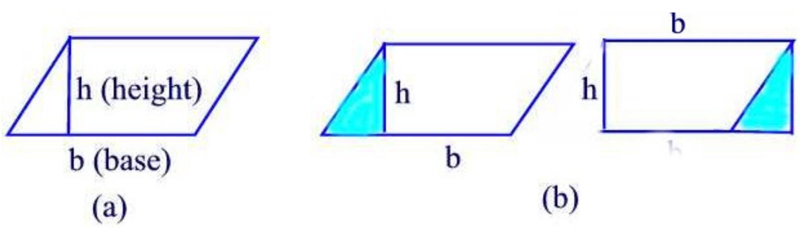
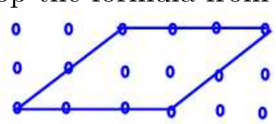


Figure 43.4

Example 43.4
Suppose a fifth grader has not yet learned the area formula for a parallelogram. How could you develop the formula from the geoboard diagram?



Solution.
The geoboard shows that the length of the base is 3 and the height is 2. Using unit parallelogram we see that 6 complete unit parallelograms cover the inside of the large parallelogram. This shows that the area is $2 \cdot 3 = 6$ ■

Area of a Rhombus:
Consider the rhombus given in Figure 43.5. Since the diagonals bisect each other at right angles then we have

$$TQ = TS = \frac{QS}{2} \text{ and } TP = TR = \frac{PR}{2}$$

But

Area of rhombus = area of triangle PQR + area of triangle PRS

Thus,

$$\begin{aligned} A &= \tfrac{1}{2}TQ \cdot PR + \tfrac{1}{2}TS \cdot PR \\ &= \tfrac{PR}{2}(TQ + TS) \\ &= \tfrac{PR \cdot QS}{2} \end{aligned}$$

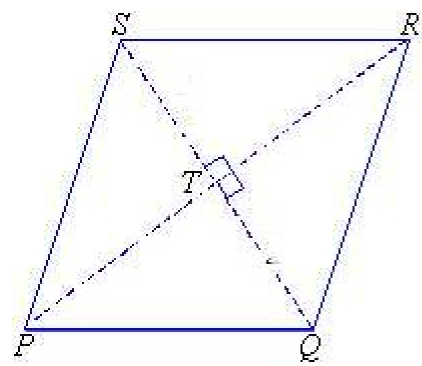


Figure 43.5

Area of a kite:
Consider the kite shown in Figure 43.6. Then

Area of kite = area of triangle DAB + area of triangle DCB

Thus,

$$\begin{aligned} A &= \tfrac{1}{2}TA \cdot BD + \tfrac{1}{2}TC \cdot BD \\ &= \tfrac{BD}{2}(TA + TC) \\ &= \tfrac{BD \cdot AC}{2} \end{aligned}$$

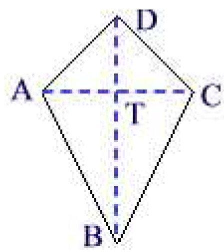


Figure 43.6

Area of a trapezoid:
Consider the trapezoid shown in Figure 43.7(a). Consider two identical trapezoids, and "turn" one around and "paste" it to the other along one side as pictured in Figure 43.7(b).

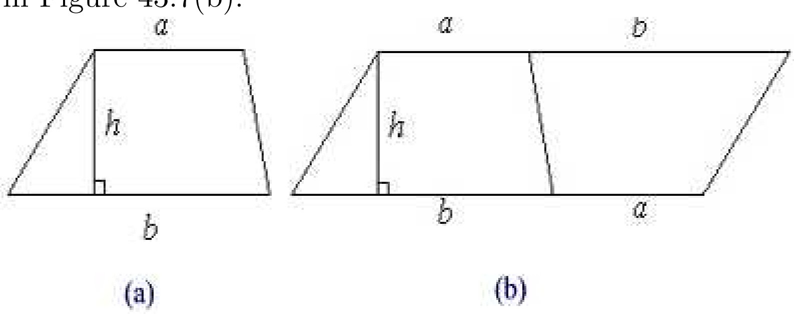


Figure 43.7

The figure formed is a parallelogram having an area of $h(a+b)$, which is twice the area of one of the trapezoids. Thus, the area of a trapezoid with height h and parallel sides a and b is given by the formula

$$A = \frac{h(a+b)}{2}.$$

Area of a circle:
We will try to discover the formula of the area of a circle, say of radius 7 cm, by executing the following steps:
Step 1: Using a compass, draw a circle of radius 7 cm. Then mark the circle's centre and draw its radius. See Figure 43.8(a).
Step 2: Place the centre of the protractor at the centre of the circle and the zero line along the radius. Then mark every 30° around the circle.
Step 3: Using a ruler and a pencil, draw lines joining each 30° mark to the

centre of the circle to form 6 diameters. The diagram thus obtained will have 12 circular sectors as shown in Figure 43.8(c).

Step 4: Colour the parts as shown in Figure 43.8(d).

Step 5: Cut out the circle and then cut along the diameters so that all parts (i.e. sectors) are separated. See Figure 43.8(e).

Step 6: Using a ruler, measure the base and the height of the approximate parallelogram obtained in Step 5. See Figure 43.8(f).

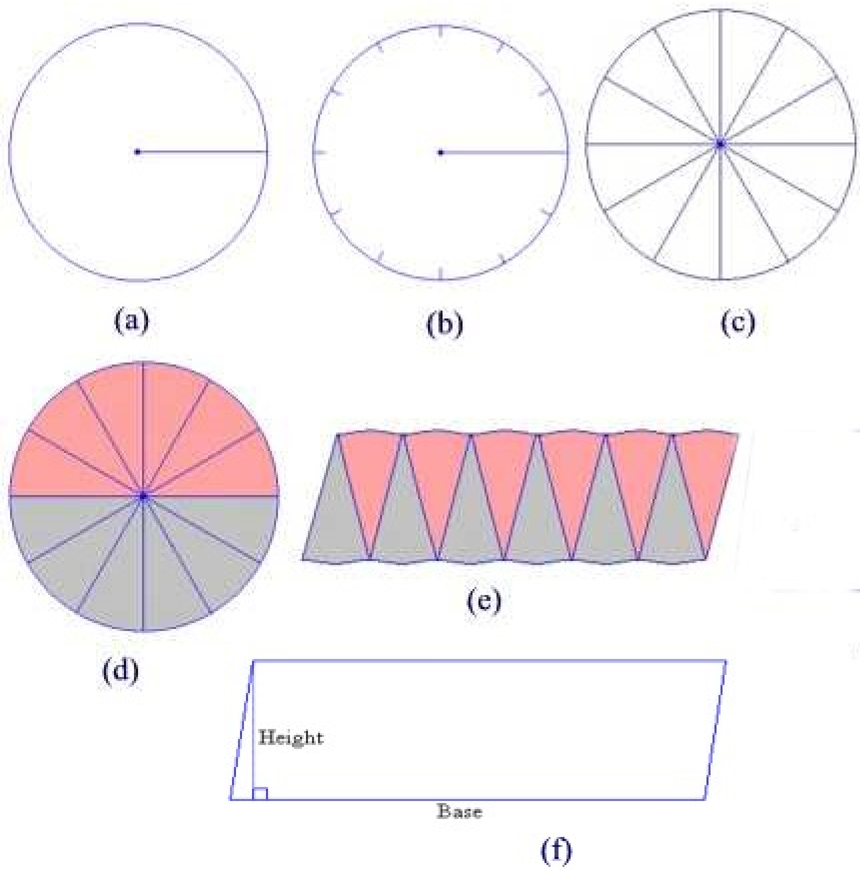


Figure 43.8

Thus, the base is approximately one-half of the circumference of the circle, that is $b = 7\pi$. The height of the parallelogram is just the radius. So $h = 7$. Hence, the area of the circle is about $49\pi = \pi 7^2$.

In general, the area of a circle of radius r is given by the formula

$$A = \pi r^2.$$

The Pythagorean Formula
Given a right triangle with legs of lengths a and b and hypotenuse of length c as shown in Figure 43.9(a). The Pythagorean formula states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse:

$$c^2 = a^2 + b^2.$$

The justification of this is as follows: Start with four copies of the same triangle. Three of these have been rotated $90°, 180°$, and $270°$, respectively as shown in Figure 43.9(b). Each has area ab/2. Let's put them together so that they form a square with side c as shown in Figure 43.9(c).
The square has a square hole with the side $(a - b)$. Summing up its area $(a - b)^2$ and the area of the four triangles $4\left(\frac{ab}{2}\right) = 2ab$ we get

$$c^2 = (a - b)^2 + 2ab = a^2 - 2ab + b^2 + 2ab = a^2 + b^2.$$

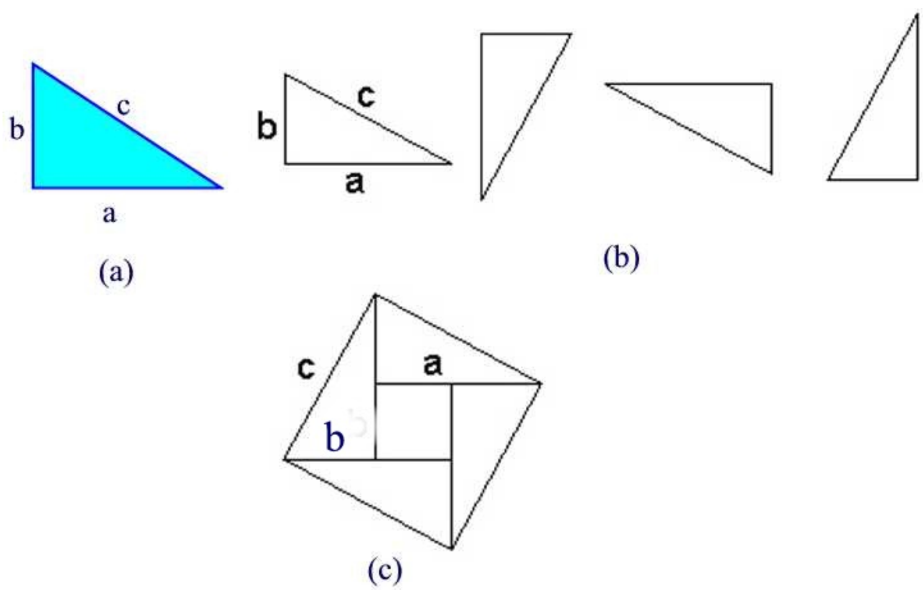


Figure 43.9

Practice Problems

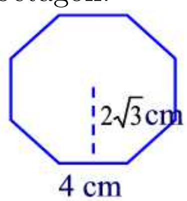
Problem 43.11
Convert each of the following:

(a) 1 cm^2 = ____ mm^2
(b) 124,000,000 m^2 = ____ km^2

Problem 43.12
Find the cost of carpeting a 6.5 m × 4.5 m rectangular room if one meter square of carpet costs \$ 13.85.

Problem 43.13
A rectangular plot of land is to be seeded with grass. If the plot is 22 m × 28 m and 1-kg bag of seed is needed for 85 m^2 of land, how many bags of seed are needed?

Problem 43.14
Find the area of the following octagon.

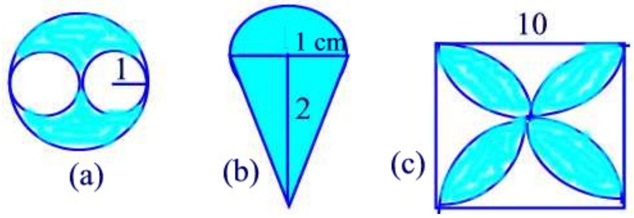


Problem 43.15
(a) If a circle has a circumference of 8π cm, what is its area?
(b) If a circle of radius r and a square with a side of length s have equal areas, express r in terms of s.

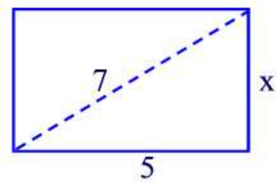
Problem 43.16
A circular flower bed is 6 m in diameter and has a circular sidewalk around it 1 m wide. Find the area of the sidewalk.

Problem 43.17
(a) If the area of a square is 144 cm^2, what is its perimeter?
(b) If the perimeter of a square is 32 cm, what is its area?

Problem 43.18
Find the area of each of the following shaded parts. Assume all arcs are circular. The unit is cm.



Problem 43.19
For the drawing below find the value of x.

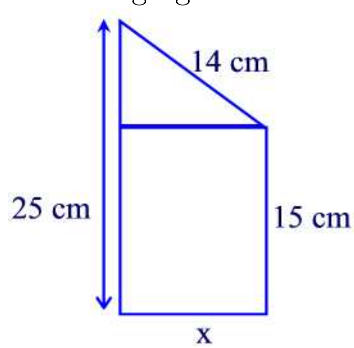


Problem 43.20
The size of a rectangular television screen is given as the length of the diagonal of the screen. If the length of the screen is 24 cm and the width is 18 cm, what is the diagonal length?

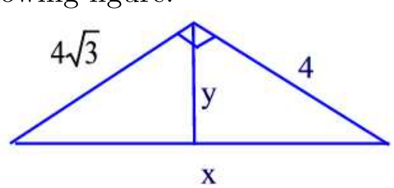
Problem 43.21
If the hypotenuse of a right triangle is 30 cm long and one leg is twice as long as the other, how long are the legs of the triangle?

Problem 43.22
A 15-ft ladder is leaning against a wall. The base of the ladder is 3 ft from the wall. How high above the ground is the top of the ladder?

Problem 43.23
Find the value of x in the following figure.



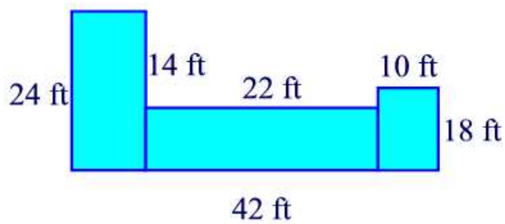
Problem 43.24
Find x and y in the following figure.



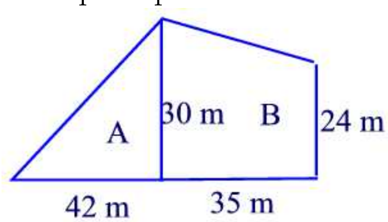
Problem 43.25
Complete the following table which concerns circles:

Radius	Diameter	Circumference	Area	
5 cm				
	24 cm			
			$17\pi\ m^2$	
		20πcm		

Problem 43.26
Suppose the housing authority has valued the house shown here at \$220 per ft^2. Find the assessed value.

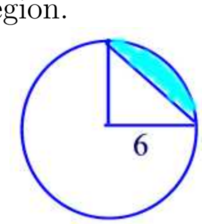


Problem 43.27
Two adjacent lots are for sale. Lot A cost \$ 20,000 and lot B costs \$27,000. Which lot has the lower cost per square meter?

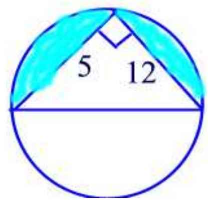


Problem 43.28
If the radius of a circle increases by 30% then by what percent does the area of the circle increase?

Problem 43.29
Find the area of the shaded region.



Problem 43.30
Find the area of the shaded region.



44 Surface Area

The **surface area** of a space figure is the total area of all the faces of the figure. In this section, we discuss the surface areas of some of the space figures introduced in Section 41.

Right Prisms

Let's find the surface area of the right prism given in Figure 44.1.

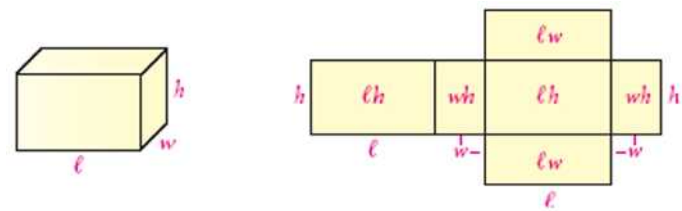


Figure 44.1

As you can see, we can dissamble the prism into six rectangles. The total area of these rectangles which is the surface area of the box is

$$SA = 2lw + 2h(w + l).$$

If we let A denote the area of the base and P the perimeter of the base then $A = lw$ and $P = 2(l + w)$. Thus,

$$SA = 2A + Ph.$$

This formula is valid for any right prism with height h, area of base A, and perimeter of base P.

Right Cylinders

To find the surface area of a right cylinder with height h and radius of base r we dissamble the cylinder into a rectangle and two circles as shown in Figure 44.2

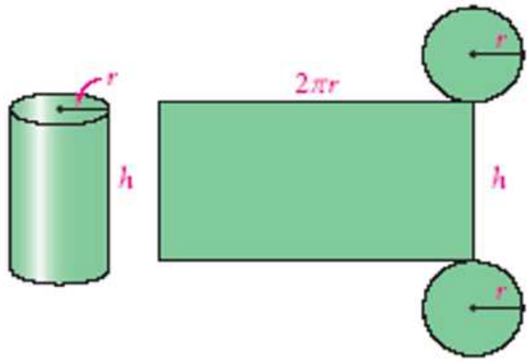


Figure 44.2

Thus, the surface area is the sum of the area of the rectangle together with the areas of the two circles. That is,

$$SA = 2\pi r^2 + 2\pi r \cdot h = 2\pi r(r + h).$$

Example 44.1
A thin can has a height of 15 cm and a base radius of 7 cm. It is filled with water. Find the total surface area in contact with the water. (Take $\pi = 22/7$)

Solution.

$$SA = 2\pi(7)(7 + 15) = 2 \cdot \frac{22}{7} \cdot 7 \cdot 22 = 968\ cm^2 \blacksquare$$

Example 44.2
The lateral surface area of a solid cylinder is 880 cm^2 and its height is 10 cm. Find the circumference and area of the base of the cylinder. Take $\pi = \frac{22}{7}$.

Solution.
Since the lateral surface of a cylinder is height $\times$ perimeter of the base then $P = \frac{880}{10} = 88\ cm$. Now, the radius of the circle is $r = \frac{88}{2\pi} = 44 \cdot \frac{7}{22} = 14\ cm$. Finally, the area of the base is $A = \pi 14^2 = \frac{22}{7} \cdot 196 = 616\ cm^2 \blacksquare$

Pyramids
The surface area of a pyramid is obtained by summing the areas of the faces.

We illustrate this for the square pyramid shown in Figure 44.3

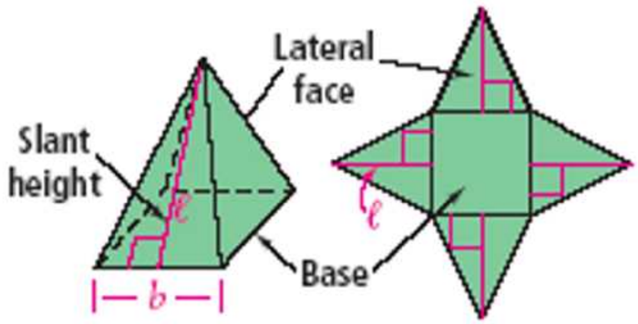


Figure 44.3

The area of the base is b^2 whereas the total area of the lateral faces is $4 \times \frac{bl}{2} = 2bl$. Thus,

$$SA = b^2 + 2bl.$$

If we let A denote the area of the base and P the perimeter of the base then

$$SA = A + \frac{1}{2}Pl.$$

This formula is valid for any right pyramid. We call l the **Slant height**.

Example 44.3
A right pyramid with slant height of 12 cm stands on a square base of sides 10 cm. Calculate the total surface area.

Solution.

$$SA = 10^2 + 2(10)(12) = 100 + 240 = 340\ cm^2 \blacksquare$$

Right Cones
First, we define the **slant height** of a right circular cone to be the length of a straight line drawn from any point on the perimeter of the base to the apex. See Figure 44.4. If the radius of the base is r and the altitude of the cone is h, then by applying the Pythagorean formula we find that the length of the slant height is given by

$$l = \sqrt{r^2 + h^2}$$

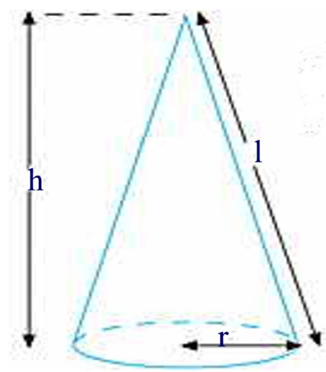


Figure 44.4

To find the surface area of the cone, we cut it along a slant height and open it out flat to obtain a circular sector shown in Figure 44.5

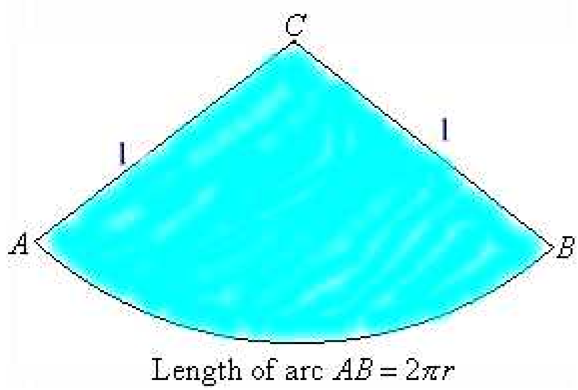


Figure 44.5

Thus, the lateral surface area is the area of the circular sector. But

$$\begin{aligned}
\frac{\text{Area of sector ABC}}{\text{Area of circle with center C}} &= \frac{\text{Arc length of AB}}{\text{Circumference of circle with center at C}} \\
\frac{\text{Area of sector ABC}}{\pi l^2} &= \frac{2\pi r}{2\pi l} \\
\text{Area of sector ABC} &= \pi r l
\end{aligned}$$

But

$$\text{Total surface area} = \text{Area of base} + \text{lateral surface area.}$$

Hence

$$\begin{aligned} SA &= \pi r^2 + \pi r l \\ &= \pi r(r + l) \\ &= \pi r(r + \sqrt{r^2 + h^2}) \end{aligned}$$

Spheres

Archimedes discovered that a cylinder that circumscribes a sphere, as shown in Figure 44.6, has a lateral surface area equal to the surface area, SA, of the sphere.

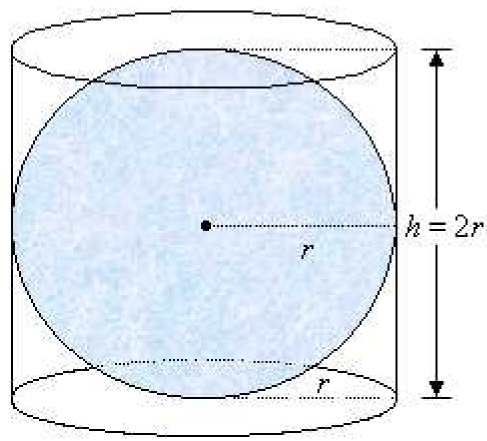


Figure 44.6

Thus,

$$\begin{aligned} SA &= 2\pi r h \\ &= 2\pi r \times 2r \\ &= 4\pi r^2 \end{aligned}$$

Example 44.4

A solid sphere has a radius of 3 m. Calculate its surface area. Round your answer to a whole number. (Take $\pi = 22/7$)

Solution.

$$\begin{aligned} SA &= 4\pi r^2 \\ &= 4 \times \frac{22}{7} \times 3^2 \\ &\approx 113\ m^2 \blacksquare \end{aligned}$$

Example 44.5

Find the radius of a sphere with a surface area of $64\pi\ m^2$.

Solution.

$$\begin{array}{rcl} SA &=& 4\pi r^2 \\ 64\pi &=& 4\pi r^2 \\ 16 &=& r^2 \\ r &=& 4\ cm \ \blacksquare \end{array}$$

Practice Problems

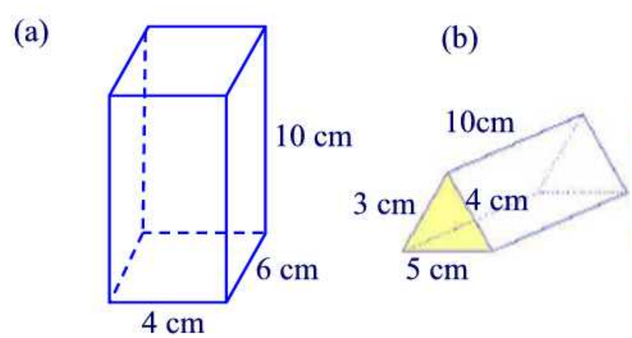
Problem 44.1
A small can of frozen orange juice is about 9.5 cm tall and has a diameter of about 5.5 cm. The circular ends are metal and the rest of the can is cardboard. How much metal and how much cardboard are needed to make a juice can?

Problem 44.2
A pyramid has a square base 10 cm on a side. The edges that meet the apex have length 13 cm. Find the slant height of the pyramid, and then calculate the total surface area of the pyramid.

Problem 44.3
An ice cream cone has a diameter of 2.5 in and a slant height of 6 in. What is the lateral surface area of the cone?

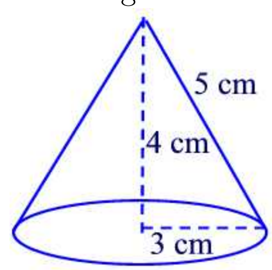
Problem 44.4
The diameter of Jupiter is about 11 times larger than the diameter of the planet Earth. How many times greater is the surface area of Jupiter?

Problem 44.5
Find the surface area of each of the right prisms below.



Problem 44.6
The Great Pyramid of Cheops is a right square pyramid with a height of 148 m and a square base with perimeter of 930 m. The slant height is 188 m. The basic shape of the Transamerica Building in San Fransisco is a right square pyramid that has a height of 260 m and a square base with a perimeter of 140 m. The altitude of slant height is 261 m. How do the lateral surface areas of the two structures compare?

Problem 44.7
Find the surface area of the following cone.



Problem 44.8
The Earth has a spherical shape of radius 6370 km. What is its surface area?

Problem 44.9
Suppose one right circular cylinder has radius 2 m and height 6 m and another has radius 6 m and height 2 m.

(a) Which cylinder has the greater lateral surface area?
(b) Which cylinder has the greater total surface area?

Problem 44.10
The base of a right pyramid is a regular hexagon with sides of length 12 m. The height of the pyramid is 9 m. Find the total surface area of the pyramid.

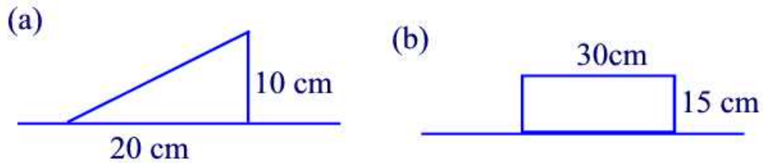
Problem 44.11
A square piece of paper 10 cm on a side is rolled to form the lateral surface area of a right circular cylinder and then a top and bottom are added. What is the surface area of the cylinder?

Problem 44.12
The top of a rectangular box has an area of 88 cm^2. The sides have areas 32 cm^2 and 44 cm^2. What are the dimensions of the box?

Problem 44.13
What happens to the surface area of a sphere if the radius is tripled?

Problem 44.14
Find the surface area of a square pyramid if the area of the base is 100 cm^2 and the height is 20 cm.

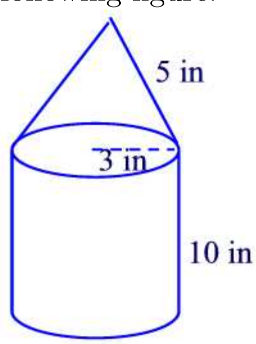
Problem 44.15
Each region in the following figure revolves about the horizontal axis. For each case, sketch the three dimensional figure obtained and find its surface area.



Problem 44.16
The total surface area of a cube is 10,648 cm^2. Find the length of a diagonal that is not a diagonal of a face.

Problem 44.17
If the length, width, and height of a rectangular prism is tripled, how does the surface area change?

Problem 44.18
Find the surface area of the following figure.



Problem 44.19
A room measures 4 meters by 7 meters and the ceiling is 3 meters high. A liter of paint covers 40 square meters. How many liters of paint will it take to paint all but the floor of the room?

Problem 44.20
Given a sphere with diameter 10, find the surface area of the smallest cylinder containing the sphere.

45 Volume

Surface area measures the area of the two-dimensional boundary of a three-dimensional figure; it is the area of the outside surface of a solid. Volume, on the other hand, is a measure of the space a figure occupies, or the space "inside" the three-dimensional figure. "How much cardboard does it take to make the box?" is a question about surface area; "How much cereal can be put into the box?" is a question about volume. Surface area is measured in squares of length units, such as square feet (ft^2), square centimeters (cm^2) or square inches (in^2). However, volume is measured in cubes of length units, such as cubic feet (ft^3), cubic centimeters (cm^3) or cubic inches (in^3).

Rectangular Prisms
Consider a rectangular box of length 4 cm , width 2 cm, and height 3 cm. Then this box occupy the same space as 24 boxes each of length 1 cm, width 1 cm, and height 1 cm. Note that $2 \times 4 \times 3 = 24$. Thus, the volume of the box is $V = 24\ cm^3$. This suggests that the volume of a rectangular prism with length l, width w, and height h is given by the formula

$$V = lwh.$$

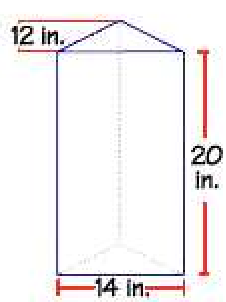
Since lw is the area B of the base of the prism then we can say that

$$\text{Volume of rectangular prism} = \text{area of base} \times \text{height}$$

This formula is valid for any prism whose base area is B and height is h

Remark 45.1
Note that a cube is a rectangular prism with $l = w = h = s$ so that its volume is $V = s^3$

Example 45.1
Find the volume of the prism given below.



Solution.
First we have to find the area of the triangle that forms the base of this prism. The area of a triangle is one-half the base of the triangle times the height of the triangle. The triangle has a base of 14 inches and a height of 12 inches. If we substitute these values into the formula, we get 84 square inches for the area of the triangle.

$$B = \frac{1}{2} \times 14 \times 12 = 84\ in^2$$

Since $h = 20\ in$ then the total volume is

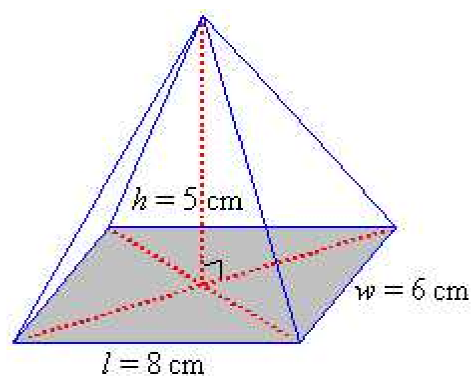
$$V = 84 \times 20 = 1680\ in^3 \blacksquare$$

Right Pyramids
Consider a pyramid with rectangular base of area B and height h. Also, consider a rectangular box of area of base B and height h. Fill the pyramid with rice and pour the content into the box. Repeat this process three times. You will notice that the cube is completely full with rice. This shows that the volume of the pyramid is one-third that of the box. That is,

$$V = \frac{1}{3}Bh.$$

This formula is true for any right pyramid.

Example 45.2
Find the volume of a rectangular-based pyramid whose base is 8 cm by 6 cm and height is 5 cm.



Solution.
The volume is
$$V = \frac{1}{3} \times 8 \times 6 \times 5 = 80\ cm^3 \blacksquare$$

Right Cylinders
Since we know how to calculate the volume of a prism, we use a regular prism to approach a cylinder with base radius r and height h: a regular prism whose base is a quadrilateral, a pentagon, a hexagon, a heptagon, an octagon, or a many-sided regular polygon. As the number of the sides increases, the perimeter of the regular polygon approaches the circumference of the circle, and the area of the base polygon approaches the base area of the cylinder. As a result, the volume of the regular prism approaches the volume of the cylinder as shown in Figure 45.2

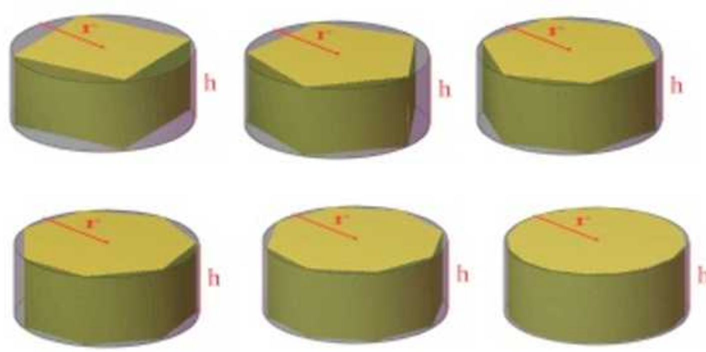


Figure 45.2

Eventually, the regular prism and the cylinder fit together perfectly when the number of the sides is large enough. Since the volume of the regular prism is the product of its base area and its height, and since its base area and the cylinder base fit together perfectly, the volume of the cylinder is also the product of its base area and its height. Since the base area is πr^2 then the volume of the cylinder is:
$$V = \pi r^2 h.$$

Example 45.3
A steel pipe is 35.0 cm long and has an inside radius of 8.0 cm and an outside radius of 10.0 cm. How much steel is needed to build the pipe? (The density

of steel is about 7.8 g/cm^3)

Solution.
Given:

$$\begin{gathered}\text{Inner radius } r_1 = 8.0 \text{ cm}\\ \text{Outer radius } r_2 = 10.0 \text{ cm}\\ \text{Steel pipe length h} = 35.0 \text{ cm}\end{gathered}$$

The volume of the pipe is the difference of two cylinders: one with the radius of 10.0 cm and the other one with the radius of 8.0 cm. That is,

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h\\ &= 3.14 \times 10.02 \times 35.0 - 3.14 \times 8.02 \times 35.0\\ &= 3956.4\ cm^3 \end{aligned}$$

The weight of the pipe:

$$w = Volume \times\ density = 30859.9\ 9 \approx 30.86\ kg \blacksquare$$

Right Circular Cones
Consider a cone with radius r and height h. Take a cylinder of radius r and height h. Fill the cone with rice til its top and pour the content into the cylinder. Repeat this process three times. After the third time you will notice that the cylinder is completely full. This shows that the volume of the cone is one third the volume of the cylinder. But the volume of the cylinder is $\pi r^2 h$. Hence, the volume of the cone is

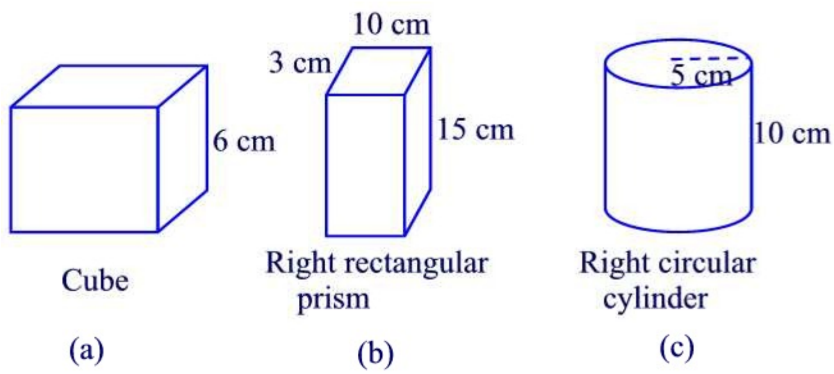
$$V = \frac{1}{3}\pi r^2 h$$

Spheres
We find the volume of a sphere as follows: Take half a sphere of radius r and fill it completely with rice. Empty the rice into a right cylinder of radius r and height r. You will notice that the rice fill up $\frac{2}{3}$ of the cylinder. This shows that the volume of the hemisphere is equal to $2/3$ the volume of the cylinder. But the volume of the cylinder is πr^3. Hence, the volume of a hemisphere is $\frac{2}{3}\pi r^3$. The volume of the sphere is twice the volume of a hemisphere and is given by the formula

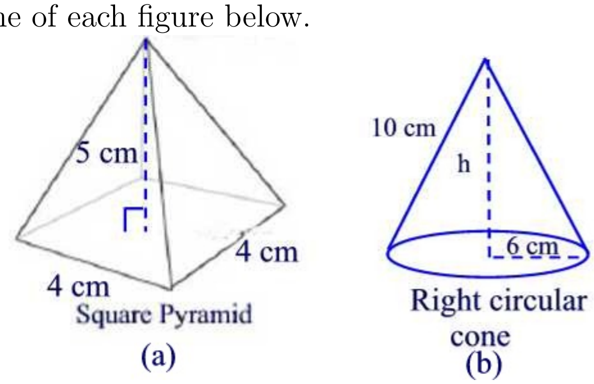
$$V = \frac{4}{3}\pi r^3.$$

Practice Problems

Problem 45.1
Find the volume of each figure below.



Problem 45.2
Find the volume of each figure below.



Problem 45.3
Maggie is planning to build a new one-story house with floor area of 2000 ft^2. She is thinking of putting in a 9-ft ceiling. If she does this, how many cubic feet of space will she have to heat or cool?

Problem 45.4
Two cubes have sides lengths 4 cm and 6 cm, respectively. What is the ratio of their volumes?

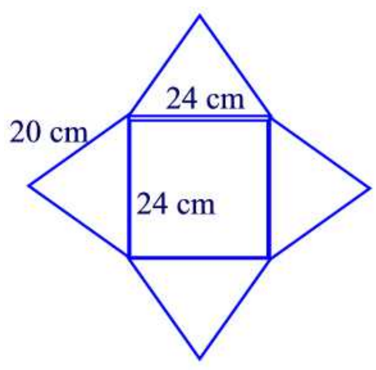
Problem 45.5
What happens to the volume of a sphere if its radius is doubled?

Problem 45.6
An olympic-sized pool in the shape of a right rectangular prism is 50 m $\times$ 25 m. If it is 2 m deep throughout, how many liters of water does it hold? Recall that 1 m^3 = 1000 L.

Problem 45.7
A standard straw is 25 cm long and 4 mm in diameter. How much liquid can be held in the straw at one time?

Problem 45.8
The pyramid of Khufu is 147 m high and its square base is 231 m on each side. What is the volume of the pyramid?

Problem 45.9
A square right regular pyramid is formed by cutting, folding, and gluing the following pattern.



(a) What is the slant height of the pyramid?
(b) What is the lateral surface area of the pyramid?
(c) Use the Pythagorean formula to find the height of the pyramid.
(d) What is the volume of the pyramid?

Problem 45.10
A cube 10 cm on a side holds 1 liter. How many liters does a cube 20 cm on a side hold?

Problem 45.11
A right circular cone has height r and a circular base of radius $2r$. Compare the volume of the cone to that of a sphere of radius r.

Problem 45.12
A store sell two types of freezers. Freezer A costs \$350 and measures 2 ft by 2 ft by 4.5 ft. Freezer B cots \$480 and measures 3 ft by 3 ft by 3.5 ft. Which freezer is the better buy?

Problem 45.13
Write a sentence that tells the difference between the surface area and volume of a prism.

Problem 45.14
A cylindrical water tank has a radius of 6.0 m. About how high must be filled to hold 400.0 m^3?

Problem 45.15
Roll an 8.5 by 11 in sheet of paper into a cylindrical tube. What is the diameter?

Problem 45.16
A cylindrical pipe has an inner radius r, an outer radius R, and length l. Find its volume.

Problem 45.17
A basketball has a diameter of 10 in. What is its volume?

Problem 45.18
A standard tennis can is a cylinder that holds three tennis ball.

(a) Which is greater the circumference of the can or its height?
(b) Find the volume of the can?

Problem 45.19
A cylindrical aquarium has a circular base with diameter 2 ft and height 3 ft. How much water does it hold, in cubic feet?

Problem 45.20
The circumference of a beach ball is 73 inches. How many cubic inches of air does the ball hold? Round your answer to the nearest cubic inch.

46 Congruence of Triangles

Two triangles are congruent if one can be moved on top of the other, so that edges and vertices coincide. The corresponding sides have the same lengths, and corresponding angles are congruent. That is, two triangles ΔABC and $\Delta A'B'C'$ are **congruent** if $m(\angle A) = m(\angle A'), m(\angle B) = m(\angle B'), m(\angle C) = m(\angle C'), AB = A'B', AC = A'C'$, and $BC = B'C'$. We write $\Delta ABC \cong \Delta A'B'C'$.

Example 46.1
Suppose that $\Delta ABC \cong \Delta A'B'C', AB = A'B', AB = 2x + 10$, and $A'B' = 4x - 20$. Find x.

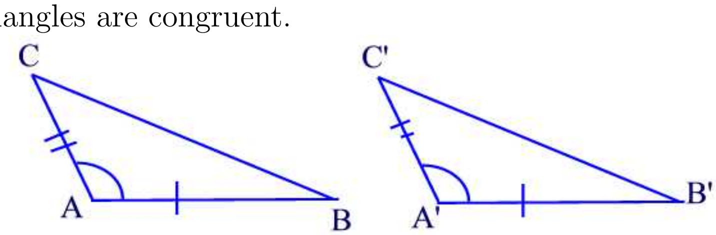
Solution.
Since $AB = A'B'$ then

$$\begin{array}{ccr} 4x - 20 & = & 2x + 10 \\ 4x - 2x & = & 10 + 20 \\ 2x & = & 30 \\ x & = & 15 \blacksquare \end{array}$$

Remark 46.1
It is very important to maintain the vertices in the proper order. Not doing so is a common mistake.

Simpler conditions can be applied to verify that two triangles are congruent. The first one involves two sides and the included angle. We adopt this result as an axiom, that is we accept this result as true by assumption, not by a proof as in the case of a theorem.

Axiom(Side-Angle-Side)
If two triangles have two sides and the included angles equal, respectively, then the triangles are congruent.

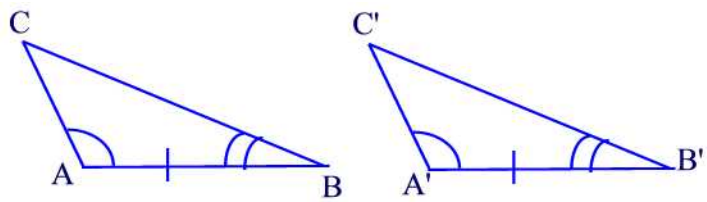


Example 46.2
In ΔABC, if $AB = AC$ then $m(\angle B) = m(\angle C)$.

Solution.
Consider the correspondence of vertices $A \leftrightarrow A, B \leftrightarrow C$, and $C \leftrightarrow B$. Under this correspondence, two sides and the included angle of ΔABC are congruent respectively to the corresponding sides and included angle of ΔACB. Hence, by SAS the triangles are congruent. Therefore, the corresponding angles are congruent and $m(\angle B) = m(\angle C)$ ■

The second congruence property that we consider involves two angles and the side included.

Theorem 46.1 *(Angle-Side-Angle)*
If two angles and the included side of a triangle are congruent, respectively, to two angles and the included side of another triangle, then the two triangles are congruent.



Proof.
We start with two triangles ΔABC and $\Delta A'B'C'$, where $m(\angle A) = m(\angle A'), AB = A'B'$, and $m(\angle B) = m(\angle B')$. We will show $BC = B'C'$ and then apply SAS theorem above. (The same sort of argument shows $AC = A'C'$.) In relating BC and B'C', there are three possibilities: BC = B'C', $BC < B'C'$, or $BC > B'C'$. If the first case holds, we are done. Now suppose $BC < B'C'$, then we can find a point G between B' and C' so that $BC = B'G$, and then $\Delta ABC \cong \Delta A'B'G$ by SAS. In particular, this implies $m(\angle C'A'B') = m(\angle CAB) = m(\angle GA'B')$. However, the segement $\overline{A'G}$ lies inside the angle $\angle C'A'B'$, so $m(\angle GA'B') < m(\angle C'A'B')$, which contradicts $m(\angle GA'B') = m(\angle C'A'B')$. So we cannot have $BC < B'C'$. A similar argument arrives at a contradiction if $BC > B'C'$, and so the only possibility is that $BC = B'C'$ ■

Example 46.3
In Figure 46.1, we have $AC = CD$. Show that $\Delta ABC \cong \Delta DEC$.

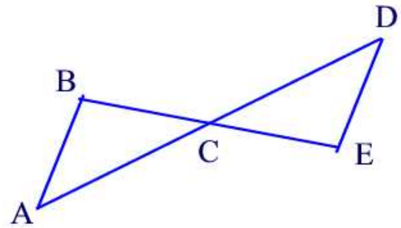
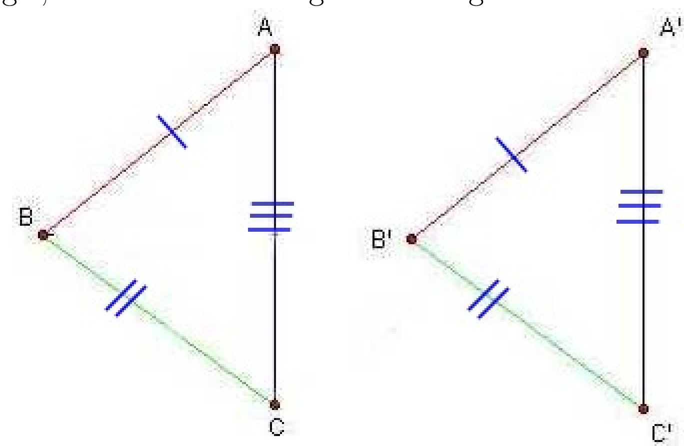


Figure 46.1

Solution.
Note that AD is a transversal line crossing two parallel lines. Then $m(\angle ACB) = m(\angle DCE)$ (vertical angles) and $m(\angle CAB) = m(\angle CDE)$ (alternate interior angles). Moreover, AC = CD so by ASA the two triangles are congruent $\blacksquare$

The third property of congruence involves the three sides of the triangles.

Theorem 46.2 *(Side-Side-Side)*
If a triangle has all three sides congruent to the corresponding sides of a second triangle, then the two triangles are congruent.



Proof.
Let B'' be the point such that $m(\angle ACB'') = m(\angle A'C'B')$ and $m(\angle CAB'') = m(\angle C'A'B')$ and $B'' \neq B$. See Figure 46.2

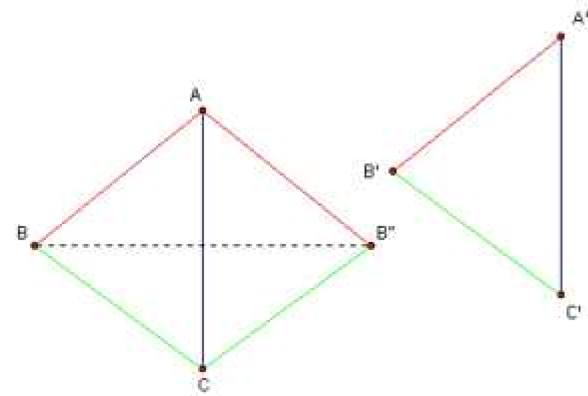


Figure 46.2

This implies by angle-side-angle congruence that $\Delta AB''C \cong \Delta A'B'C'$. Also triangles $\Delta ABB''$ and $\Delta CBB''$ are two isosceles triangles. Thus, the base angles are congruent. This implies $m(\angle ABB'') = m(\angle AB''B)$ and $m(\angle CBB'') = m(\angle CB''B)$. Thus, $m(\angle ABC) = m(\angle AB''C)$. Hence, by SAS, $\Delta ABC \cong \Delta AB''C$. Therefore triangles ABC and $A'B'C'$ are congruent $\blacksquare$

Example 46.4
In Figure 46.3, we are given that AC = CD, AB = BD. Show that $\Delta ABC \cong \Delta DBC$.

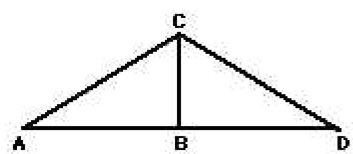
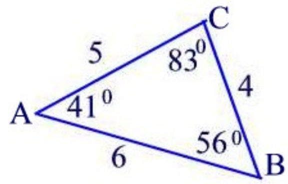


Figure 46.3

Solution.
Since AC=CD, AB=BD, and BC = CB then by SSS we have $\Delta ABC \cong \Delta DBC$ $\blacksquare$

Practice Problems

Problem 46.1
Suppose $\Delta JKL \cong \Delta ABC$, where ΔABC is shown below.

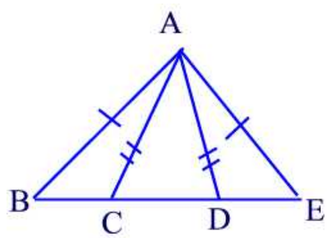


Find the following
(a) KL (b) LJ (c) $m(\angle L)$ (d) $m(\angle J)$

Problem 46.2
Using congruence of triangles show that equilateral triangles are equiangular.

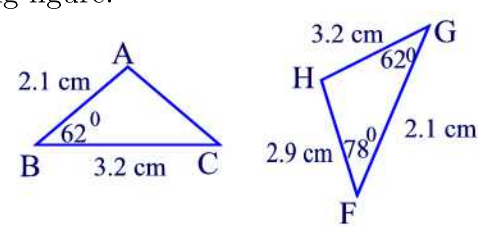
Problem 46.3
Let the diagonals of a parallelogram ABCD intersect at a point M.
(a) Show that $\Delta ABM \cong \Delta CDM$.
(b) Use part (a) to explain why M is the midpoint of both diagonals of the parallelogram.

Problem 46.4
In the figure below, AB = AE and AC=AD.



(a) Show that $m(\angle B) = m(\angle E)$
(b) Show that $m(\angle ACD) = m(\angle ADC)$
(c) Show that $\Delta ABC \cong \Delta AED$
(d) Show that $BC = DE$.

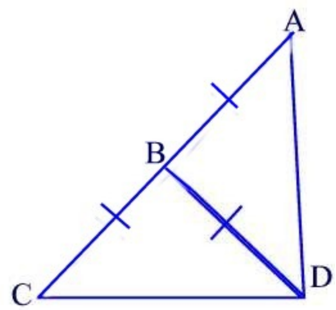
Problem 46.5
Consider the following figure.



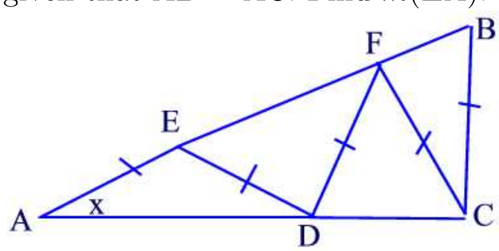
(a) Find AC
(b) Find $m(\angle H), m(\angle A)$, and $m(\angle C)$.

Problem 46.6
Show that if $\Delta ABC \cong \Delta A'B'C'$ and $\Delta A'B'C' \cong \Delta A''B''C''$ then $\Delta ABC \cong \Delta A''B''C''$.

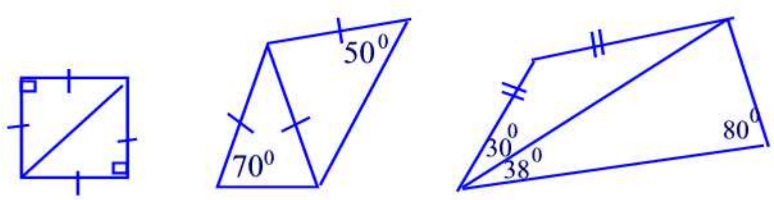
Problem 46.7
In the figure below, given that AB=BC=BD. Find $m(\angle ADC)$.



Problem 46.8
In the figure below given that AB = AC. Find $m(\angle A)$.

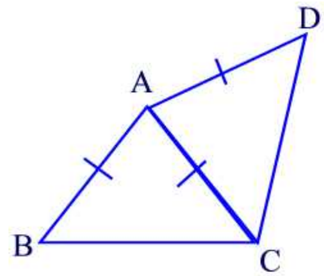


Problem 46.9
Find all missing angle measures in each figure.



(a) (b) (c)

Problem 46.10
An eighth grader says that AB=AC=AD , as shown in the figure below, then $m(\angle B) = m(\angle C) = m(\angle D)$. Is this right? If not, what would you tell the child?

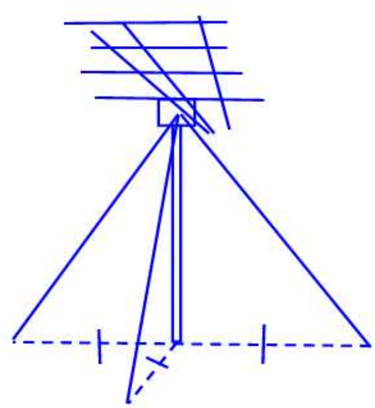


Problem 46.11
What type of figure is formed by joining the midpoints of a rectangle?

Problem 46.12
If two triangles are congruent what can be said about their perimeters? areas?

Problem 46.13
In a pair of right triangles, suppose two legs of one are congruent to respectively to two legs of the other. Explain whether the triangles are congruent and why.

Problem 46.14
A rural homeowner had his television antenna held in place by three guy wires, as shown in the following figure. If the distance to each of the stakes from the base of the antenna are the same, what is true about the lengths of the wires? Why?



Problem 46.15
For each of the following, determine whether the given conditions are sufficient to prove that $\Delta PQR \cong \Delta MNO$. Justify your answer.
(a) PQ = MN, PR = MO, $m(\angle P) = m(\angle M)$
(b) PQ = MN, PR = MO, QR = NO
(c) PQ = MN, PR = MO, $m(\angle Q) = m(\angle N)$

Problem 46.16
Given that $\Delta RST \cong \Delta JLK$, complete the following statements.
(a) $\Delta TRS \cong \Delta$____
(b) $\Delta SRT \cong \Delta$____
(c) $\Delta TSR \cong \Delta$____
(d) $\Delta JKL \cong \Delta$____

Problem 46.17
You are given ΔRST and ΔXYZ with $m(\angle S) = m(\angle Y)$.
(a) To show $\Delta RST \cong \Delta XYZ$ by the SAS congruence property, what more would you need to know?
(b) To show that $\Delta RST \cong \Delta XYZ$ by the ASA congruence property, what more would you need to know?

Problem 46.18
You are given ΔABC and ΔGHI with AB = GH. To show that $\Delta ABC \cong \Delta GHI$ by the SSS congruence property, what more would you need to know?

Problem 46.19
Suppose that ABCD is a kite with AB = AD and BC = DC. Show that the diagonal $\overline{AC}$ divides the kite into two congruent triangles.

Problem 46.20
(a) Show that the diagonal of a parallelogram divides it into two congruent triangles.
(b) Use part (a) to show that the opposite sides of a parallelogram are congruent.
(c) Use part (a) to show that the opposite angles of a parallelogram are congruent.

47 Similar Triangles

An overhead projector forms an image on the screen which has the same shape as the image on the transparency but with the size altered. Two figures that have the same shape but not necessarily the same size are called **similar**. In the case of triangles, this means that the two triangles will have the same angles and their sides will be in the same proportion (for example, the sides of one triangle might all be 3 times the length of the sides of the other). More formally, we have the following definition:
Two triangles ΔABC and ΔDEF are similar, written $\Delta ABC \sim \Delta DEF$ if and only if
(i) $m(\angle A) = m(\angle D), m(\angle B) = m(\angle E)$, and $m(\angle C) = m(\angle F)$

(ii) $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

The common ratio in (ii) is called the **scaled-factor.** An example of two similar triangles is shown in Figure 47.1

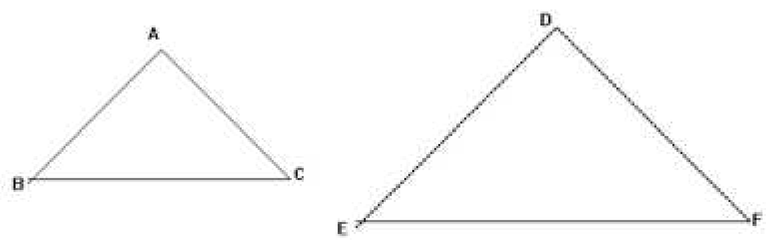


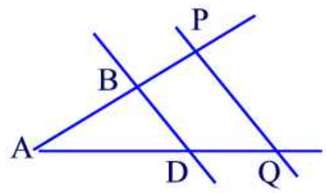
Figure 47.1

Example 47.1
Suppose in Figure 47.1 that EF = 6 cm, BC = 2cm and AB = 3cm. What is the value of DE?

Solution.
Since DE = 3AB and AB = 3 then DE = 9 cm ■

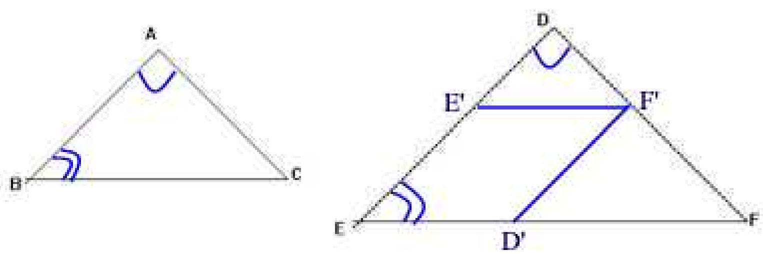
The proofs of various properties of similar triangles depend upon certain properties of parallel lines. Mainly, we need the following theorem which we state without proof.

Theorem 47.1
Let PQ be a line intersecting an angle $\angle BAD$. Then $\frac{AP}{AB} = \frac{AQ}{AD}$ if and only if the lines PQ and BD are parallel.



When we attempted to prove two triangles to be congruent we had a few tests SSS, SAS, ASA. In a similar way we have a few tests to help us determine whether two triangles are similar.
If the measures of two angles of a triangle are given, then the measure of the third angle is known automatically. Thus, the shape of the triangle is completely determined. Since similar triangles have the same shape, we have the following similarity condition.

Theorem 47.2 *(AA Test)*
If two angles of one triangle are congruent with the corresponding two angles of another triangle, then the two triangles are similar.



Proof.
Let $ABC \leftrightarrow DEF$ denote a correspondence of ΔABC and ΔDEF in which $m(\angle A) = m(\angle D)$ and $m(\angle B) = m(\angle E)$. Then $m(\angle A) + m(\angle B) = m(\angle D) + m(\angle E)$. But $180° - m(\angle C) = m(\angle A) + m(\angle B)$ and $180° - m(\angle F) = m(\angle D) + m(\angle E)$. Hence, $m(\angle C) = m(\angle F)$.
Now, let F' be a point on $\overline{DF}$ and E' be a point on $\overline{DE}$ such that $DF' = AC$ and $DE' = AB$. Then by SAS test for congruent triangles we have $\Delta ABC \cong DE'F'$. Hence, $m(\angle C) = m(\angle DF'E')$ and $m(\angle DF'E') = m(\angle DFE)$. This last equality implies that $FE || F'E'$ by the Corresponding Angles Theorem.

Hence, Theorem 47.1 implies

$$\frac{DF'}{DF} = \frac{DE'}{DE}.$$

Since $DF' = AC$ and $DE' = AB$ then we have

$$\frac{AC}{DF} = \frac{AB}{DE}.$$

Now, let D' be a point on $\overline{FE}$ such that $FD' = CB$. Then by repeating the above argument we find

$$\frac{AC}{DF} = \frac{BC}{FE}.$$

It follows that

$$\frac{AC}{DF} = \frac{BC}{FE} = \frac{AB}{DE}.$$

This shows $\Delta ABC \sim \Delta DEF$ ■

Example 47.2
Consider Figure 47.2.

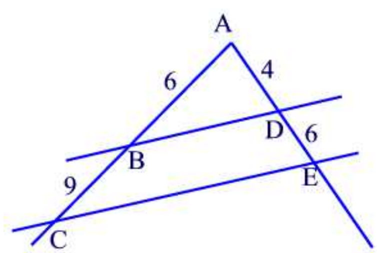


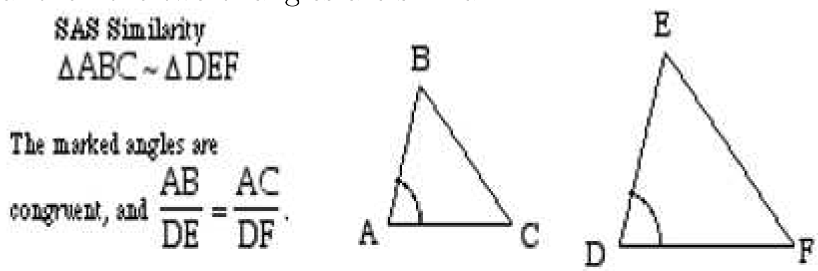
Figure 47.2

Show that $\frac{BD}{CE} = \frac{AD}{AE} = \frac{AB}{AC} = \frac{2}{5}$

Solution.
Since $m(\angle A) = m(\angle A)$ and $m(\angle ADB) = m(\angle AEC)$ then by the AA test we have $\Delta ABD \sim \Delta ACE$. Hence,

$$\frac{BD}{CE} = \frac{AD}{AE} = \frac{AB}{AC} = \frac{6}{15} = \frac{2}{5} \blacksquare$$

Theorem 47.3 *(SAS Test)*
If two sides of one triangle are proportional to the two corresponding sides of the second triangle and the angles between the two sides of each triangle are equal then the two triangles are similar.



Proof.
Let E' be a point on the line segement $\overline{DE}$ such that $DE' = AB$. Also, let F' be a point on $\overline{DF}$ such that $DF' = AC$. See Figure 47.3.

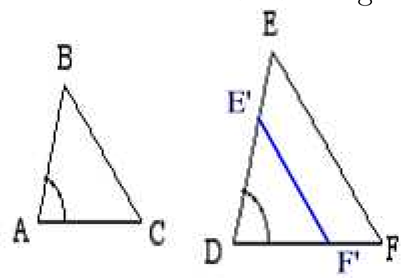


Figure 47.3

Since

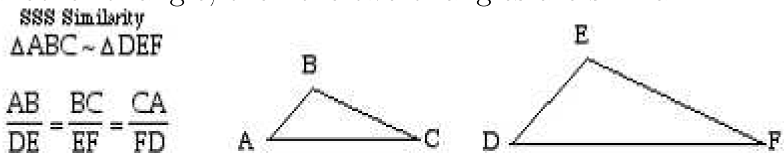
$$\frac{AB}{DE} = \frac{AC}{DF}$$

then

$$\frac{DE'}{DE} = \frac{DF'}{DF}.$$

By Theorem 47.1, the lines EF and $E'F'$ are parallel. Thus, $m(\angle DEF) = m(\angle DE'F')$ and $m(\angle DFE) = m(\angle DF'E')$. By the AA similarity theorem we have $\Delta DE'F' \sim \Delta DEF$. But by the SAS congruence test, we have $\Delta ABC \cong \Delta DE'F'$ and in particular $\Delta ABC \sim \Delta DE'F'$ since two congruent triangles are also similar. Hence, $\Delta ABC \sim \Delta DEF$ ∎

Theorem 47.4 *(SSS Test)*
If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.



Proof.
Let E' be a point on the line segement $\overline{DE}$ such that $DE' = AB$. Also, let F' be a point on $\overline{DF}$ such that $DF' = AC$. See Figure 47.4.

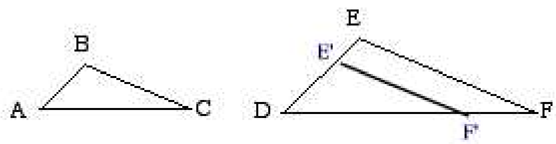
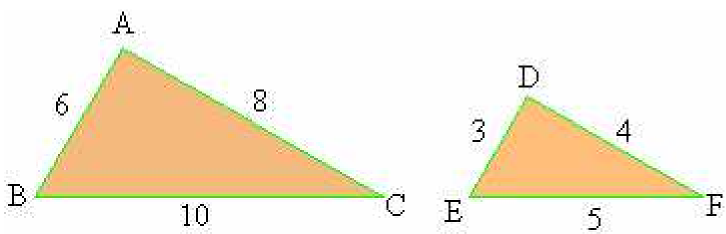


Figure 47.4

Since
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$
then
$$\frac{DE'}{DE} = \frac{DF'}{DF} = \frac{BC}{EF}.$$
By Theorem 47.1, the lines EF and $E'F'$ are parallel. By SAS similarity theorem , we have $\Delta DE'F' \sim \Delta DEF$. Thus,
$$\frac{DE'}{DE} = \frac{DF'}{DF} = \frac{E'F'}{EF}.$$
But $\frac{DE'}{DE} = \frac{BC}{EF}$. Hence, $E'F' = BC$. By the SSS congruence theorem we have $\Delta ABC \cong \Delta DE'F'$. Hence, $\Delta ABC \sim \Delta DE'F'$. Since $\Delta DE'F' \sim \Delta DEF$ then $\Delta ABC \sim \Delta DEF$ ■

Example 47.3
Show that the triangles in the figure below are similar.



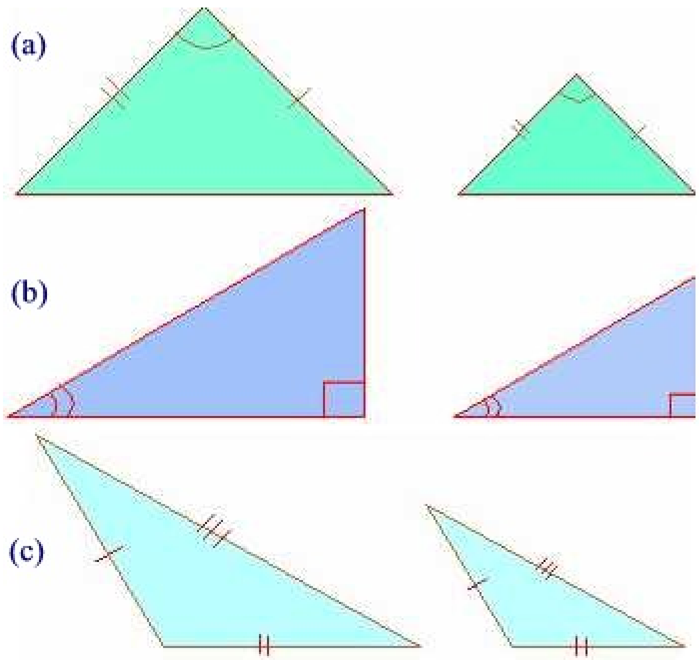
Solution.
Since $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = 2$ then by Theorem 47.4 $\Delta ABC \sim \Delta DEF$ ■

Practice Problems

Problem 47.1
Which of the following triangles are always similar?
(a) right triangles
(b) isosceles triangles
(c) equilateral triangles

Problem 47.2
Show that if $\Delta ABC \sim \Delta A'B'C'$ and $\Delta A'B'C' \sim \Delta A''B''C''$ then $\Delta ABC \sim \Delta A''B''C''$.

Problem 47.3
Each pair of triangles is similar. By which test can they be proved to be similar ?



Problem 47.4
Suppose $\Delta ABC \sim \Delta DEF$ with scaled factor k.
(a) Compare the perimeters of the two triangles.
(b) Compare the areas of the two triangles.

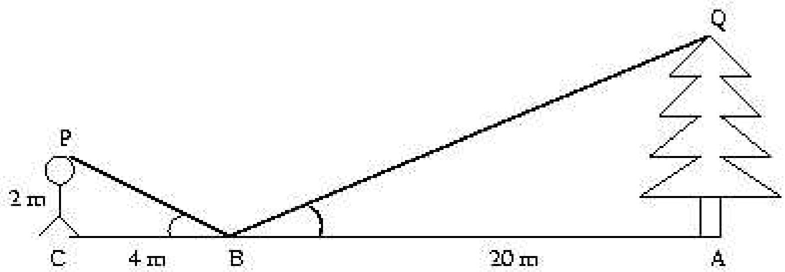
Problem 47.5
Areas of two similar triangles are 144 sq.cm. and 81 sq.cm. If one side of the first triangle is 6 cm then find the corresponding side of the second triangle.

Problem 47.6
The side of an equilateral triangle ΔABC is 5 cm. Find the length of the side of another equilateral triangle ΔPQR whose area is four times area of ΔABC.

Problem 47.7
The corresponding sides of two similar triangles are 4 cm and 6 cm. Find the ratio of the areas of the triangles.

Problem 47.8
A clever outdoorsman whose eye-level is 2 meters above the ground, wishes

to find the height of a tree. He places a mirror horizontally on the ground 20 meters from the tree, and finds that if he stands at a point C which is 4 meters from the mirror B, he can see the reflection of the top of the tree. How high is the tree?

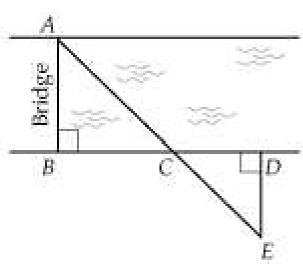


Problem 47.9
A child 1.2 meters tall is standing 11 meters away from a tall building. A spotlight on the ground is located 20 meters away from the building and shines on the wall. How tall is the child's shadow on the building?

Problem 47.10
On a sunny day, Michelle and Nancy noticed that their shadows were different lengths. Nancy measured Michelle's shadow and found that it was 96 inches long. Michelle then measured Nancy's shadow and found that it was 102 inches long.

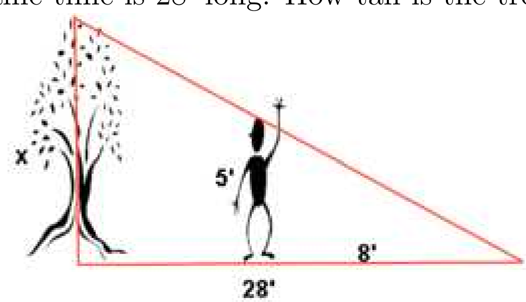
(a) Who do you think is taller, Nancy or Michelle? Why?
(b) If Michelle is 5 feet 4 inches tall, how tall is Nancy?
(c) If Nancy is 5 feet 4 inches tall, how tall is Michelle?

Problem 47.11
An engineering firm wants to build a bridge across the river shown below. An engineer measures the following distances: BC = 1,200 feet, CD = 40 feet, and DE = 20 feet.

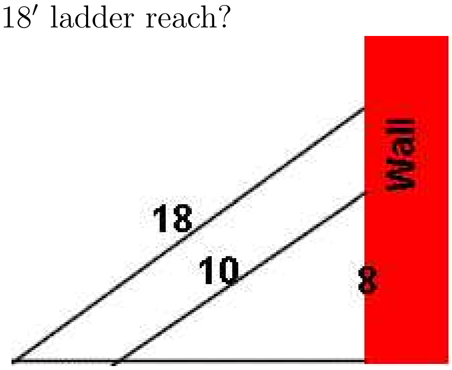


(a) Prove ΔABC is similar to ΔEDC.
(b) Railings cost \$4 per foot. How much will it cost to put railings on both sides of the bridge?

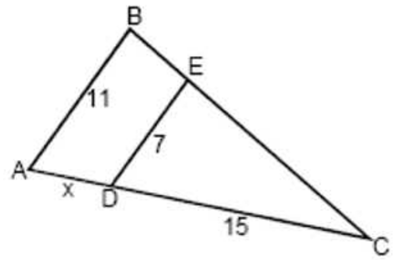
Problem 47.12
At a certain time of the day, the shadow of a $5'$ boy is $8'$ long. The shadow of a tree at this same time is $28'$ long. How tall is the tree?



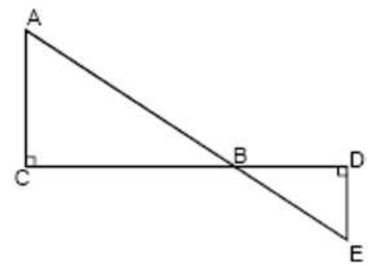
Problem 47.13
Two ladders are leaned against a wall such that they make the same angle with the ground. The $10'$ ladder reaches $8'$ up the wall. How much further up the wall does the $18'$ ladder reach?



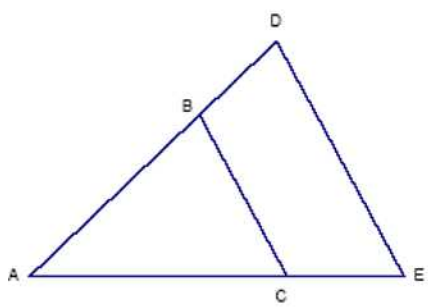
Problem 47.14
Given that lines DE and AB are parallel in the figure below, determine the value of x, i.e. the distance between points A and D.



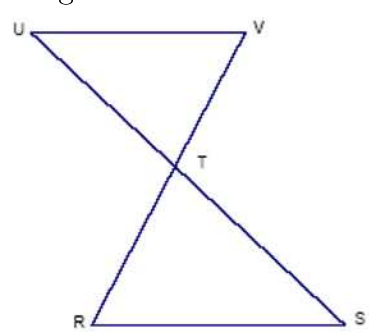
Problem 47.15
In the figure below, lines AC and DE are vertical, and line CD is horizontal. Show that $\Delta ABC \sim \Delta EBD$.



Problem 47.16
Find a pair of similar triangles in each of these figures:

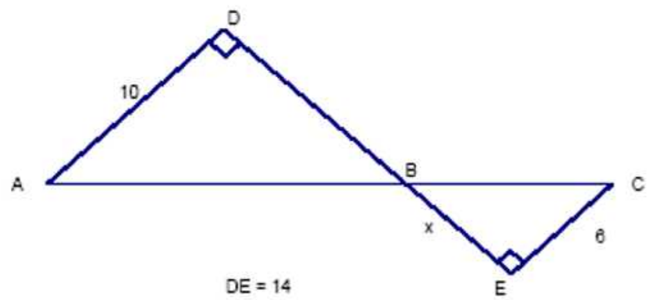


$\overline{BC} \parallel \overline{DE}$

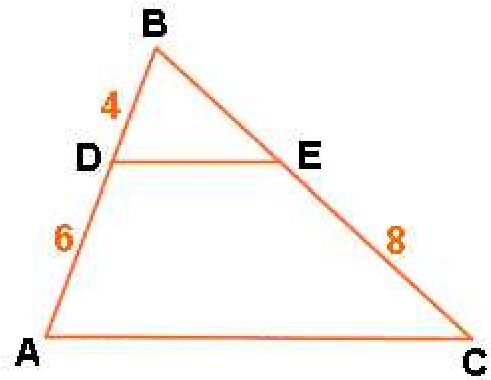


$\overline{UV} \parallel \overline{RS}$

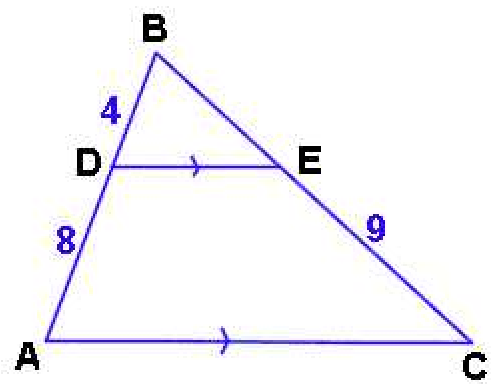
Problem 47.17
Find x :



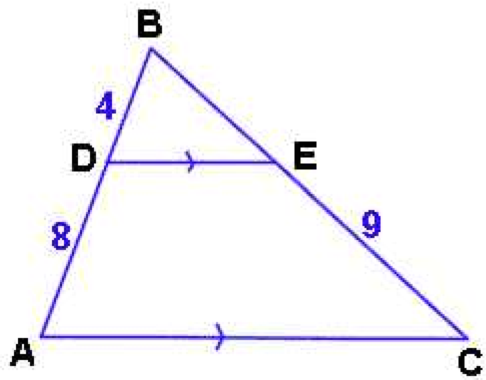
Problem 47.18
In the diagram, DE is parallel to AC. Also, BD = 4, DA = 6 and EC = 8. Find BC to the nearest tenth.



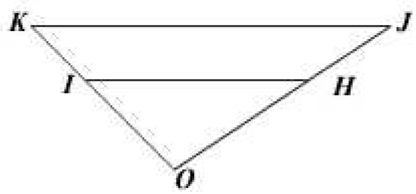
Problem 47.19
Find BC.



Problem 47.20
Find BE



Problem 47.21
Copy and complete the given table. It is given that $\frac{OH}{HJ} = \frac{OI}{IK}$.



OH	HJ	OJ	OI	IK	OK
15	5	?	45	?	?

48 Solutions to Practice Problems

Problem 25.1
Which of the following are integers?
(a) -11 (b) 0 (c) $\frac{3}{4}$ (d) $\frac{-9}{3}$

Solution.
(a), (b), and (d) are integers. Note that $\frac{-9}{3} = -3$. The number in (c) is the decimal number 0.75 $\blacksquare$

Problem 25.2
Let $N = \{-1, -2, -3, \cdots\}$. Find
(a) $N \cup W$
(b) $N \cap \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers or positive integers and W is the set of whole numbers.

Solution.
(a) $N \cup W = \mathbb{Z}$.
(b) $N \cap \mathbb{N} = \emptyset$ $\blacksquare$

Problem 25.3
What number is 5 units to the left of -95?

Solution.
The number -100 is five units to the left of the number -95 on the number line model $\blacksquare$

Problem 25.4
A and B are 9 units apart on the number line. A is twice as far from 0 as B. What are A and B?

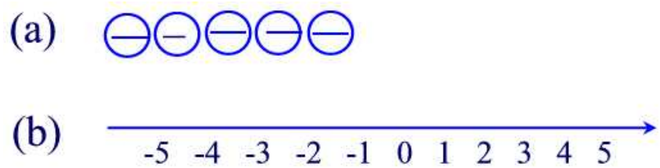
Solution.
We use the method of guessing.

B	A	dist(A,O)	dist(B,O)	dist(A,O) = 2dist(B,O)?
1	-8	8	1	NO
2	-7	7	2	NO
3	-6	6	3	YES

A second answer is B to be -3 and A to be 6. A third answer is $A = 18$ and $B = 9$ $\blacksquare$

Problem 25.5
Represent -5 using signed counters and number line.

Solution.



Problem 25.6
In terms of distance, explain why $|-4| = 4$.

Solution.
Because -4 is located 4 units to the left of zero $\blacksquare$

Problem 25.7
Find the additive inverse (i.e. opposite) of each of the following integers.
(a) 2 (b) 0 (c) -5 (d) m (e) $-m$

Solution.
(a) -2 (b) 0 (c) 5 (d) $-m$ (e) m $\blacksquare$

Problem 25.8
Evaluate each of the following.
(a) $|-5|$ (b) $|10|$ (c) $-|-4|$ (d) $-|5|$

Solution.
(a) $|-5| = 5$ (b) $|10| = 10$ (c) $-|-4| = -4$ (d) $-|5| = -5$ $\blacksquare$

Problem 25.9
Find all possible integers x such that $|x| = 2$.

Solution.
We are looking for numbers that are 2 units away from zero. In this case, either $x = -2$ or $x = 2$ $\blacksquare$

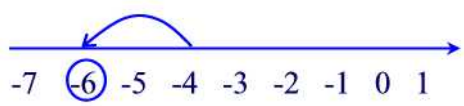
Problem 25.10
Let W be the set of whole numbers, $\mathbb{Z}^+$ the set of positive integers (i.e. $\mathbb{Z}^+ = \mathbb{N}$), $\mathbb{Z}^-$ the set of negative integers, and $\mathbb{Z}$ the set of all integers. Find each of the following.
(a) $W \cup \mathbb{Z}$ (b) $W \cap \mathbb{Z}$ (c) $\mathbb{Z}^+ \cup \mathbb{Z}^-$ (d) $\mathbb{Z}^+ \cap \mathbb{Z}^-$ (e) $W - \mathbb{Z}^+$

Solution.
(a) $W \cup \mathbb{Z} = \mathbb{Z}$
(b) $W \cap \mathbb{Z} = W$
(c) $\mathbb{Z}^+ \cup \mathbb{Z}^- = \{\cdots, -3, -2, -1, 1, 2, 3, \cdots\}$
(d) $\mathbb{Z}^+ \cap \mathbb{Z}^- = \emptyset$
(e) $W - \mathbb{Z}^+ = \{0\}$

Problem 25.11
What is the opposite or additive inverse of each of the following? (a and b are integers)
(a) $a + b$ (b) $a - b$

Solution.
(a) $-(a + b) = (-a) + (-b)$
(b) $-(a - b) = (-a) - (-b) = (-a) + b$ ■

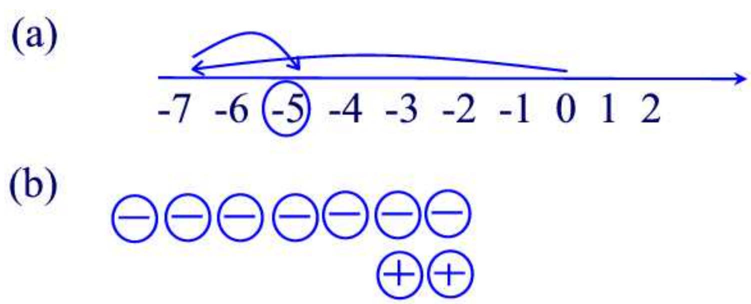
Problem 25.12
What integer addition problem is shown on the number line?



Solution.
$(-4) + (-2) = -6$ ■

Problem 25.13
(a) Explain how to compute $-7 + 2$ with a number line.
(b) Explain how to compute $-7 + 2$ with signed counters.

Solution.



Problem 25.14
In today's mail, you will receive a check for \$86, a bill for \$30, and another bill for \$20. Write an integer addition equation that gives the overall gain or loss.

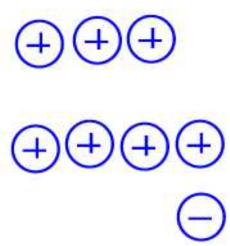
Solution.
$86 + (-30) + (-20)$ ■

Problem 25.15
Compute the following without a calculator.
(a) $-54 + 25$ (b) $(-8) + (-17)$ (c) $400 + (-35)$

Solution.
(a) $-54 + 25 = -(54 - 25) = -29$
(b) $(-8) + (-17) = -(8 + 17) = -25$
(c) $400 + (-35) = 400 - 35 = 365$ ■

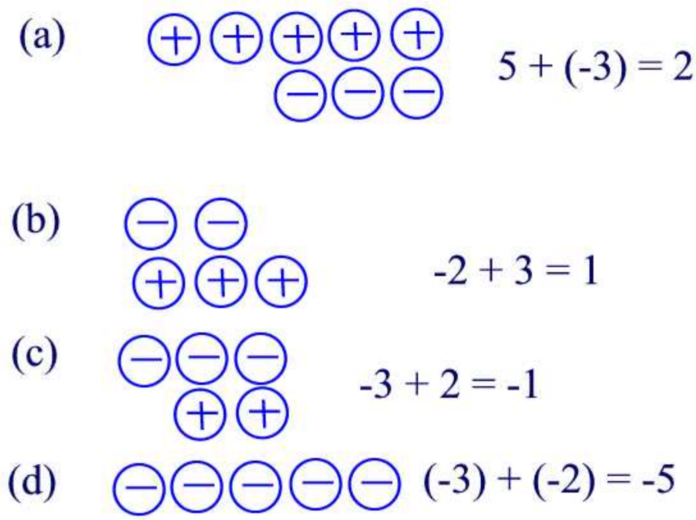
Problem 25.16
Show two ways to represent the integer 3 using signed counters.

Solution.



Problem 25.17
Illustrate each of the following addition using the signed counters.
(a) $5 + (-3)$ (b) $-2 + 3$ (c) $-3 + 2$ (d) $(-3) + (-2)$

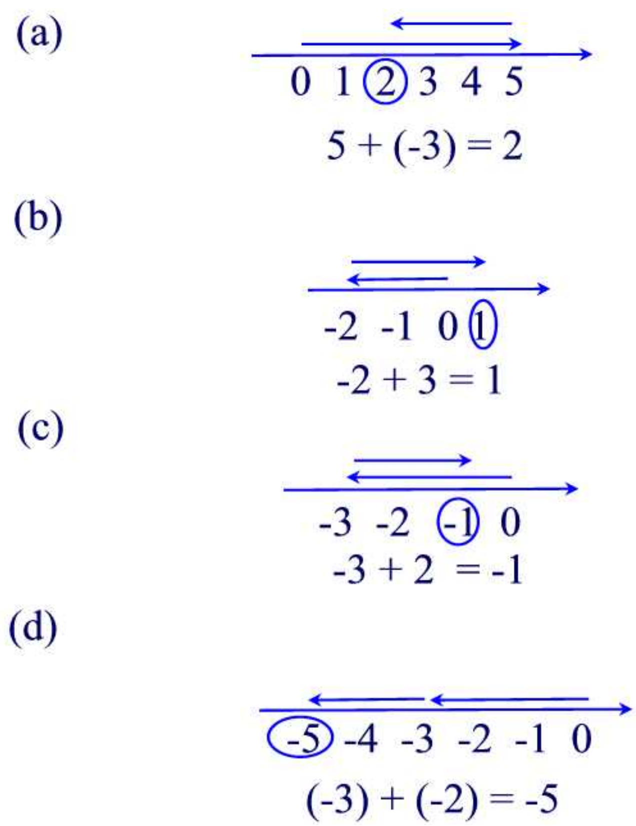
Solution.



Problem 25.18

Demonstrate each of the additions in the previous problem using number line model.

Solution.



Problem 25.19
Write an addition problem that corresponds to each of the following sentences and then answer the question.
(a) The temperature was $-10°C$ and then it rose by $8°C$. What is the new temperature?
(b) A certain stock dropped 17 points and the following day gained 10 points. What was the net change in the stock's worth?
(c) A visitor to a casino lost \$200, won \$100, and then lost \$50. What is the change in the gambler's net worth?

Solution.
(a) $(-10) + 8 = -(10 - 8) = -2°C$
(b) $(-17) + 10 = -(17 - 10) = -7$

(c) $(-200)+100+(-50) = ((-200)+100)+(-50) = (-(200-100))+(-50) = (-100)+(-50) = -(100+50) = \$-150$ ■

Problem 25.20
Compute each of the following: (a) $|(-9)+(-5)|$ (b) $|7+(-5)|$ (c) $|(-7)+6|$

Solution.
(a) $|(-9)+(-5)| = |-(9+5)| = |-14| = 14$
(b) $|7+(-5)| = |7-5| = |2| = 2$
(c) $|(-7)+6| = |-(7-6)| = |-1| = 1$ ■

Problem 25.21
If a is an element of $\{-3,-2,-1,0,1,2\}$ and b is an element of $\{-5,-4,-3,-2,-1,0,1\}$, find the smallest and largest values for the following expressions.
(a) $a+b$ (b) $|a+b|$

Solution.

a	b	a+b	$\|a+b\|$
-3	-5	-8	8
-3	-4	-7	7
-3	-3	-6	6
-3	-2	-5	5
-3	-1	-4	4
-3	0	-3	3
-3	1	-2	2

a	b	a+b	$\|a+b\|$
-2	-5	-7	7
-2	-4	-6	6
-2	-3	-5	5
-2	-2	-4	4
-2	-1	-3	3
-2	0	-2	2
-2	1	-1	1

a	b	a+b	$\|a+b\|$
-1	-5	-6	6
-1	-4	-5	5
-1	-3	-4	4
-1	-2	-3	3
-1	-1	-2	2
-1	0	-1	1
-1	1	0	0

a	b	a+b	$\|a+b\|$
0	-5	-5	5
0	-4	-4	4
0	-3	-3	3
0	-2	-2	2
0	-1	-1	1
0	0	0	0
0	1	1	1

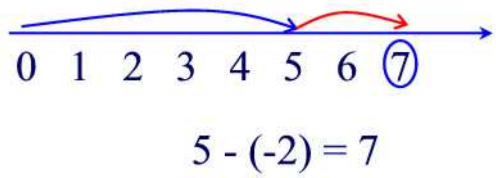
a	b	a+b	$\|a+b\|$
1	-5	-4	4
1	-4	-3	3
1	-3	-2	2
1	-2	-1	1
1	-1	0	0
1	0	1	1
1	1	2	2

a	b	a+b	$\|a+b\|$
2	-5	-3	3
2	-4	-2	2
2	-3	-1	1
2	-2	0	0
2	-1	1	1
2	0	2	2
2	1	3	3

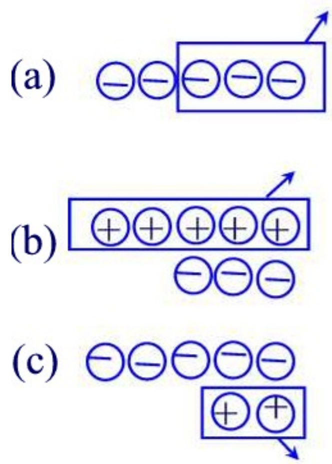
(a) The largest value of $a+b$ is 3 and the smallest value is -8
(b) The largest value of $|a+b|$ is 8 and the smallest value is 0 ■

Problem 25.22
Explain how to compute $5 - (-2)$ using a number line.

Solution.



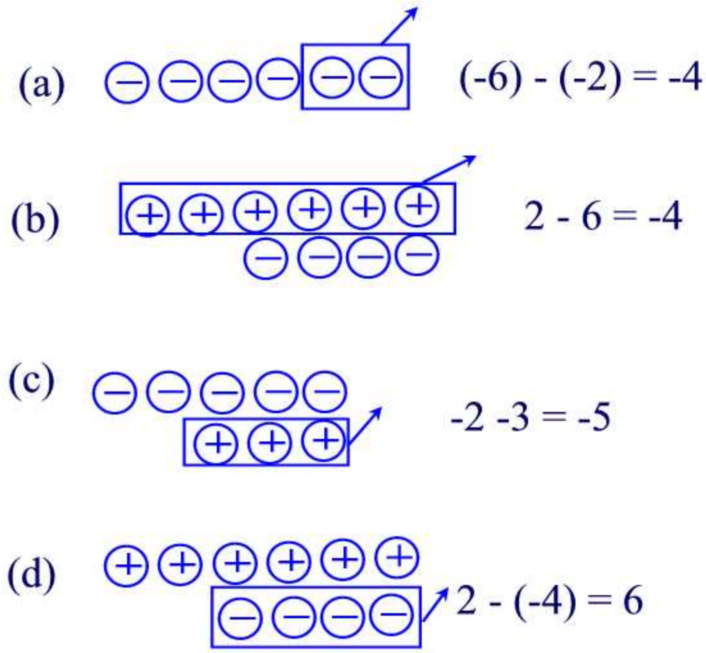
Problem 25.23
Tell what subtraction problem each picture illustrates.



Solution.
(a) $(-5) - (-3) = -2$
(b) $2 - 5 = -3$
(c) $(-3) - 2 = -5$ ■

Problem 25.24
Explain how to compute the following with signed counters.
(a) $(-6) - (-2)$ (b) $2 - 6$ (c) $-2 - 3$ (d) $2 - (-4)$

Solution.



Problem 25.25
(a) On a number line, subtracting 3 is the same as moving _____ units to the _____.
(b) On a number line, adding -3 is the same as moving _____ units to the _____.

Solution.
(a) On a number line, subtracting 3 is the same as moving three units to the left.
(b) On a number line, adding -3 is the same as moving three units to the left $\blacksquare$

Problem 25.26
An elevetor is at an altitude of -10 feet. The elevator goes down 30 ft.
(a) Write an integer equation for this situation.
(b) What is the new altitude?

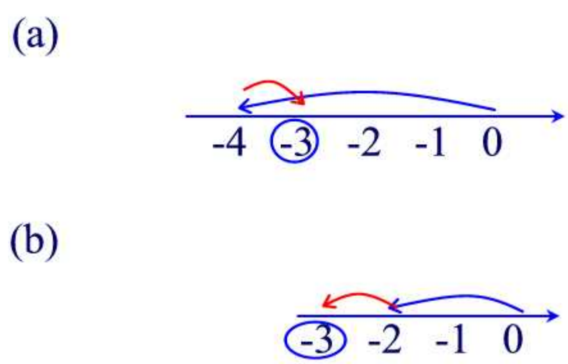
Solution.
(a) $(-10) - 30$
(b) $(-10) - 30 = (-10) + (-30) = -(10 + 30) = -40\ ft$ $\blacksquare$

Problem 25.27
Compute each of the following using $a - b = a + (-b)$.
(a) $3 - (-2)$ (b) $-3 - 2$ (c) $-3 - (-2)$

Solution.
(a) $3 - (-2) = 3 + [-(-2)] = 3 + 2 = 5$
(b) $-3 - 2 = (-3) + (-2) = -(3 + 2) = -5$
(c) $-3 - (-2) = (-3) + [-(-2)] = (-3) + 2 = -1$ ■

Problem 25.28
Use number-line model to find the following.
(a) $-4 - (-1)$ (b) $-2 - 1$.

Solution.



Problem 25.29
Compute each of the following.
(a) $|5 - 11|$ (b) $|(-4) - (-10)|$ (c) $|8 - (-3)|$ (d) $|(-8) - 2|$

Solution.
(a) $|5 - 11| = |-6| = 6$
(b) $|(-4) - (-10)| = |6| = 6$
(c) $|8 - (-3)| = |11|11$
(d) $|(-8) - 2| = |-10| = 10$ ■

Problem 25.30
Find x.
(a) $x + 21 = 16$ (b) $(-5) + x = 7$ (c) $x - 6 = -5$ (d) $x - (-8) = 17$.

Solution.
(a) $x + 21 = 16 \Longrightarrow x = 16 - 21 = -5$
(b) $(-5) + x = 7 \Longrightarrow x = 7 + 5 = 12$
(c) $x - 6 = -5 \Longrightarrow x = 6 - 5 = 1$
(d) $x - (-8) = 17 \Longrightarrow x = (-8) + 17 = 9$ ■

Problem 25.31
Which of the following properties hold for integer subtraction. If a property does not hold, disprove it by a counterexample.
(a) Closure (b) Commutative (c) Associative (d) Identity

Solution.
(a) Hold
(b) Does not hold since $2 - 3 \neq 3 - 2$
(c) Does not hold since $(1 - 2) - 3 \neq 1 - (2 - 3)$
(d) Does not hold since $1 - 0 \neq 0 - 1$ ■

Problem 26.1
Use patterns to show that $(-1)(-1) = 1$.

Solution.
First, we compute the following

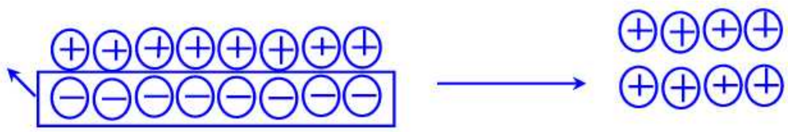
$$\begin{array}{rcr} (-1)\times 3 & = & -3 \\ (-1)\times 2 & = & -2 \\ (-1)\times 1 & = & -1 \\ (-1)\times 0 & = & 0 \end{array}$$

Thus, everytime the second factor decreases by 1 the results increase by 1. Hence, continuing the pattern we find

$$\begin{array}{rcr} (-1)\times 3 & = & -3 \\ (-1)\times 2 & = & -2 \\ (-1)\times 1 & = & -1 \\ (-1)\times 0 & = & 0 \\ (-1)\times(-1) & = & 1 \ \blacksquare \end{array}$$

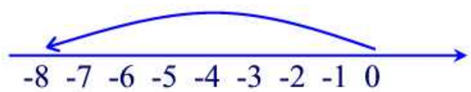
Problem 26.2
Use signed counters to show that $(-4)(-2) = 8$.

Solution.



Problem 26.3
Use number line to show that $(-4)2 = -8$.

Solution.



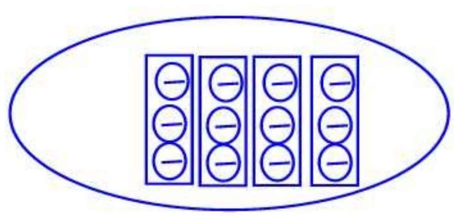
Problem 26.4
Change $3 \times (-2)$ into a repeated addition and then compute the answer.

Solution.
$3 \times (-2) = (-2) + (-2) + (-2) = -6$ ■

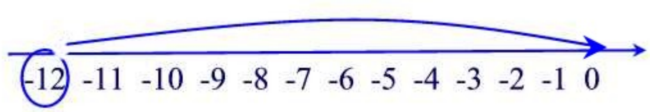
Problem 26.5
(a) Compute $4 \times (-3)$ with repeated addition.
(b) Compute $4 \times (-3)$ using signed counters.
(c) Compute $4 \times (-3)$ using number line.

Solution.
(a) $4 \times (-3) = (-3) + (-3) + (-3) + (-3) = -12$

(b)



(c)



Problem 26.6
Show how $(-2) \times 4$ can be found by extending a pattern in a whole-number multiplication.

Solution.
We compute the following

$$\begin{array}{rcl} 3 \times 4 & = & 12 \\ 2 \times 4 & = & 8 \\ 1 \times 4 & = & 4 \\ 0 \times 4 & = & 0 \end{array}$$

It follows that each time the first factor is decreased by 1, the corresponding product is decreased by 4. Thus,

$$\begin{array}{rcl} 3 \times 4 & = & 12 \\ 2 \times 4 & = & 8 \\ 1 \times 4 & = & 4 \\ 0 \times 4 & = & 0 \\ (-1) \times 4 & = & -4 \\ (-2) \times 4 & = & -8 \blacksquare \end{array}$$

Problem 26.7
Mike lost 3 pounds each week for 4 weeks.
(a) What was the total change in his weight?
(b) Write an integer equation for this situation.

Solution.
(a) He lost a total of 12 pounds.
(b) $(-3) \times 4 = -12$ pounds $\blacksquare$

Problem 26.8
Compute the following without a calculator.
(a) $3 \times (-8)$
(b) $(-5) \times (-8) \times (-2) \times (-3)$

Solution.
(a) $3 \times (-8) = -(3 \times 8) = -24$
(b) $(-5) \times (-8) \times (-2) \times (-3) = ((-5) \times (-8)) \times ((-2) \times (-3)) = (5 \times 8) \times (2 \times 3) = 40 \times 6 = 240$ $\blacksquare$

Problem 26.9
Extend the following pattern by writing the next three equations.

$$\begin{aligned} 6 \times 3 &= 18 \\ 6 \times 2 &= 12 \\ 6 \times 1 &= 6 \\ 6 \times 0 &= 0 \end{aligned}$$

Solution.

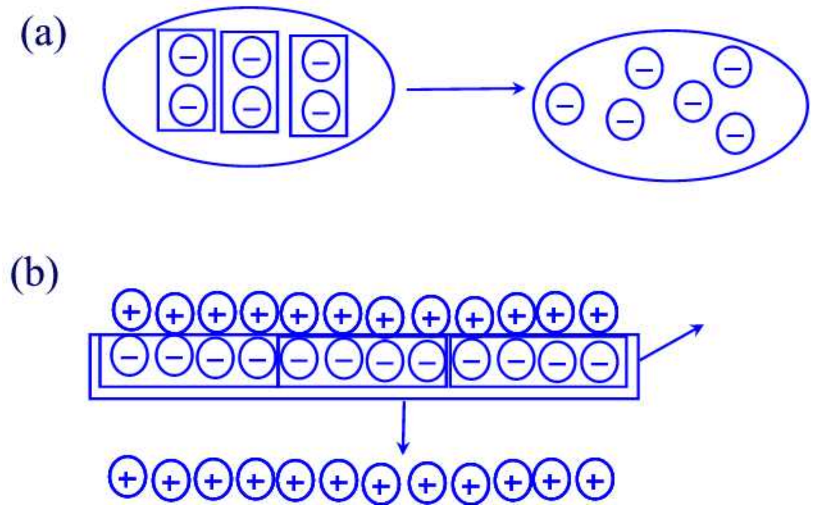
$$\begin{aligned} 6 \times 3 &= 18 \\ 6 \times 2 &= 12 \\ 6 \times 1 &= 6 \\ 6 \times 0 &= 0 \\ 6 \times (-1) &= -6 \\ 6 \times (-2) &= -12 \\ 6 \times (-3) &= -18 \blacksquare \end{aligned}$$

Problem 26.10
Find the following products.
(a) $6(-5)$ (b) $(-2)(-16)$ (c) $-(-3)(-5)$ (d) $-3(-7-6)$.

Solution.
(a) $6(-5) = -(6 \times 5) = -30$
(b) $(-2)(-16) = 2 \times 16 = 32$
(c) $-(-3)(-5) = -(3 \times 5) = -15$
(d) $-3(-7-6) = (-3) \times (-13) = 3 \times 13 = 39$ $\blacksquare$

Problem 26.11
Represent the following products using signed counters and give the results.
(a) $3 \times (-2)$ (b) $(-3) \times (-4)$

Solution.



Problem 26.12
In each of the following steps state the property used in the equations.

$$\begin{array}{rcl} a(b-c) & = & a[b+(-c)] \\ & = & ab+a(-c) \\ & = & ab+[-(ac)] \\ & = & ab-ac \end{array}$$

Solution.

$$\begin{array}{rcl} a(b-c) & = & a[b+(-c)] \ \ \textit{adding the opposite} \\ & = & ab+a(-c) \ \ \textit{distributivity} \\ & = & ab+[-(ac)] \ \ \textit{Theorem } 26.1(a) \\ & = & ab-ac \ \ \textit{adding the opposite} \ \blacksquare \end{array}$$

Problem 26.13
Extend the meaning of a whole number exponent

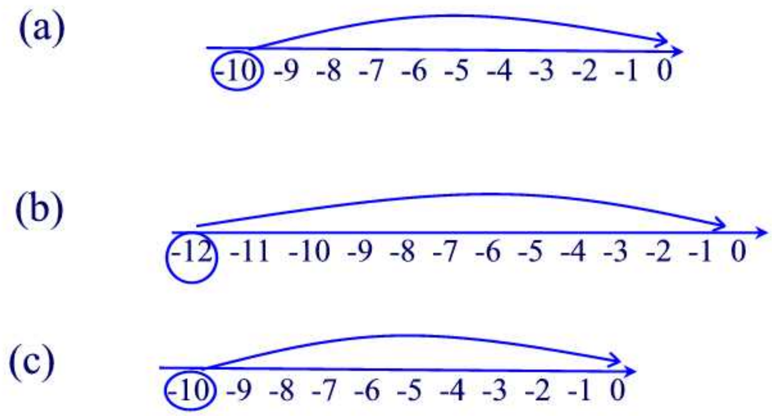
$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \ factors}$$

where a is any integer. Use this definition to find the following values.
(a) $(-2)^4$ (b) -2^4 (c) $(-3)^5$ (d) -3^5

Solution.
(a) $(-2)^4 = (-2)(-2)(-2)(-2) = 16$
(b) $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$
(c) $(-3)^5 = (-3)(-3)(-3)(-3) = 81$
(d) $-3^5 = -(3 \times 3 \times 3 \times 3) = -81$ ■

Problem 26.14
Illustrate the following products on an integer number line.
(a) $2 \times (-5)$ (b) $3 \times (-4)$ (c) $5 \times (-2)$

Solution.



Problem 26.15
Expand each of the following products.
(a) $-6(x+2)$ (b) $-5(x-11)$ (c) $(x-3)(x+2)$

Solution.
(a) $-6(x+2) = -6x + (-6)(2) = -6x + (-12) = -6x - 12$
(b) $-5(x-11) = -5x - (-5)(11) = -5x - (-55) = -5x + 55$
(c) $(x-3)(x+2) = x(x+2) - 3(x+2) = x^2 + 2x - (3x+6) = x^2 + 2x - 3x - 6 = x^2 - x - 6$ ■

Problem 26.16
Name the property of multiplication of integers that is used to justify each of the following equations.
(a) $(-3)(-4) = (-4)(-3)$
(b) $(-5)[(-2)(-7)] = [(-5)(-2)](-7)$

(c) $(-5)(-7)$ is a unique integer
(d) $(-8) \times 1 = -8$
(e) $4 \cdot [(-8) + 7] = 4 \cdot (-8) + 4 \cdot 7$

Solution.
(a) Commutativty
(b) Associativity
(c) Closure
(d) Identity element
(e) Distributivity ■

Problem 26.17
If $3x = 0$ what can you conclude about the value of x?

Solution.
Since $3x = 0 = 3(0)$ then by the Theorem 26.5 we have $x = 0$ ■

Problem 26.18
If a and b are negative and c is positive, determine whether the following are positive or negative.
(a) $(-a)(-c)$ (b) $(-a)(b)$ (c) $(c-b)(c-a)$ (d) $a(b-c)$

Solution.
(a) Since a is negative then $-a$ is positive. Also, since c is positive then $-c$ is negative. Thus, $(-a)(-b)$ is negative.
(b) Since $-a$ is positive and b is negative then $(-a)(b)$ is negative.
(c) Since $(c-b)(c-a) = c^2 - ac - bc + ab = c^2 + (-a)c + (-b)c + ab$ and c^2 positive, $(-a)c$ positive, $(-b)c$ positive, and ab is positive then $(c-b)(c-a)$ is positive.
(d) $a(b-c) = ab - ac = ab + (-a)c$. But ab is positive as well as $(-a)c$. Hence, $a(b-c)$ is positive ■

Problem 26.19
Is the following equation true? $a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$.

Solution.
False. Let $a = 2, b = 3$, and $c = 4$. Then $a \cdot (b \cdot c) = 2 \cdot (3 \cdot 4) = 24$ and $(a \cdot b) \cdot (a \cdot c) = (2 \cdot 3) \cdot (2 \cdot 4) = 6 \cdot 8 = 48$ ■

Problem 26.20
Perform these divisions.
(a) $36 \div 9$ (b) $(-36) \div 9$ (c) $36 \div (-9)$ (d) $(-36) \div (-9)$ (e) $165 \div (-11)$
(f) $275 \div 11$

Solution.
(a) If $c = 36 \div 9$ then $9c = 36$. Consequently, $c = 4$.
(b) If $c = (-36) \div 9$ then $9c = -36$ and consequently $c = -4$.
(c) If $c = 36 \div (-9)$ then $-9c = 36$ so that $c = -4$.
(d) If $c = (-36) \div (-9)$ then $-9c = -36$ and so $c = 4$.
(e) If $c = 165 \div (-11)$ then $-11c = 165$ and so $c = -15$.
(f) If $c = 275 \div 11$ then $11c = 275$ and so $c = 25$ ■

Problem 26.21
Write another multiplication equation and two division equations that are equivalent to the equation

$$(-11) \cdot (-25, 753) = 283, 283.$$

Solution.
Multiplication equation: $11 \cdot 25753 = 283283$
Division equation: $283283 \div (-11) = -25753$
Division equation: $283283 \div (-25753) = -11$ ■

Problem 26.22
Write two multiplication equations and another division equation that are equivalent to the equation

$$(-1001) \div 11 = -91.$$

Solution.
Multiplication equation: $11 \times (-91) = -1001$
Multiplication equation: $(-11) \times 91 = -1001$
Division equation: $1001 \div -11 = -91$ ■

Problem 26.23
Use the definition of division to find each quotient, if possible. If a quotient is not defined, explain why.
(a) $(-40) \div (-8)$ (b) $(-143) \div 11$ (c) $0 \div (-5)$ (d) $(-5) \div 0$ (e) $0 \div 0$

Solution.
(a) If $c = (-40) \div (-8)$ then $-8c = -40$ and so $c = 5$.
(b) If $c = (-143) \div 11$ then $11c = -143$ and so $c = -13$.
(c) If $c = 0 \div (-5)$ then $-5c = 0$ and so $c = 0$.
(d) If $c = (-5) \div 0$ then $0 \times c = -5$. There is no such c. This says that $(-5) \div 0$ is undefined.
(e) If $c = 0 \div 0$ then $0 \times c = 0$. Any integer works for c. By the uniqueness of c the division is undefined ■

Problem 26.24
Find all integers x (if possible) that make each of the following true.
(a) $x \div 3 = -12$ (b) $x \div (-3) = -2$ (c) $x \div (-x) = -1$ (d) $0 \div x = 0$
(e) $x \div 0 = 1$

Solution.
(a) If $x \div 3 = -12$ then $x = (-12)(3) = -36$.
(b) If $x \div (-3) = -2$ then $x = (-2)(-3) = 8$.
(c) If $x \div (-x) = -1$ then $x = (-1)(-x) = x$ so any integer works for x.
(d) If $0 \div x = 0$ then $x \times 0 = 0$. Any integer works for x.
(e) If $x \div 0 = 1$ then $x = 0$ ■

Problem 26.25
Write two division equations that are equivalent to $3 \times (-2) = -6$.

Solution.
(i) $(-6) \div (-2) = 3$ (ii) $(-6) \div 3 = -2$ ■

Problem 26.26
Explain how to compute $-10 \div 2$ using signed counters.

Solution.
Take a set of 10 negative counters and divide them into 2 equal groups of five negative counters each. So $(-10) \div 2 = -5$ ■

Problem 26.27
Rewrite each of the following as an equivalent multiplication problem, and give the solution.
(a) $(-54) \div (-6)$ (b) $32 \div (-4)$

Solution.
(a) If $c = (-54) \div (-6)$ then $-6c = -54$ and so $c = 9$.
(b) If $c = 32 \div (-4)$ then $-4c = 32$ and so $c = -8$ ■

Problem 26.28
A store lost \$480,000 last year.
(a) What was the average net change per month?
(b) Write an integer equation for this situation.

Solution.
(a) \$ - 40,000.
(b) $(-480,000) \div 12 = -40,000$ ■

Problem 26.29
Compute the following, using the correct rules for order of operations.
(a) $-2^2 - 3$ (b) $-5 + (-4)^2 \times (-2)$

Solution.
(a) $-2^2 - 3 = -4 - 3 = -7$
(b) $-5 + (-4)^2 \times (-2) = -5 + 16 \times (-2) = -5 - 32 = -37$ ■

Problem 26.30
A stock change as follows for 5 days:-2,4,6,3,-1. What is the average daily change in price?

Solution.
We have $(-2)+4+6+3+(-1) = 10$ so the average daily change is $10 \div 5 = 2$ ■

Problem 26.31
Compute: $-2 \div (-2) + (-2) - (-2)$

Solution.

$$-2 \div (-2) + (-2) - (-2) = 1 + (-2) + 2 = 1$$ ■

Problem 26.32
For what integers a and b does $a \div b = b \div a$?

Solution.
Either $a = b$ or $a = -b$ ■

Problem 26.33
Find each quotient, if possible.
(a) $[144 \div (-12)] \div (-3)$ (b) $144 \div [-12 \div (-3)]$

Solution.
(a) $[144 \div (-12)] \div (-3) = (-12) \div (-3) = 4$
(b) $144 \div [-12 \div (-3)] = 144 \div 4 = 36$ ■

Problem 26.34
Compute the following writing the final answer in terms of positive exponents.
(a) $4^{-2} \cdot 4^6$
(b) $\frac{6^3}{6^{-7}}$
(c) $(3^{-4})^{-2}$

Solution.
(a) $4^{-2} \cdot 4^6 = 4^{-2+6} = 4^4$
(b) $\frac{6^3}{6^{-7}} = 6^{3-(-7)} = 6^{10}$
(c) $(3^{-4})^{-2} = 3^{(-4)(-2)} = 3^8$ ■

Problem 26.35
Express each of the following in scientific notation.
(a) 0.0004 (b) 0.0000016 (c) 0.000000000000071

Solution.
(a) $0.0004 = 4 \times 10^{-4}$
(b) $0.0000016 = 1.6 \times 10^{-6}$
(c) $0.000000000000071 = 7.1 \times 10^{-14}$ ■

Problem 26.36
Hair on the human body can grow as fast as 0.0000000043 meter per second.
(a) At this rate, how much would a strand of hair grow in one month of 30 days? Express your answer in scientific notation.
(b) About how long would it take for a strand of hair to grow to be 1 meter in length?

Solution.
(a) $0.0000000043 \times 30 \times 24 \times 3600 = 0.0111456 = 1.11456 \times 10^{-2}$ meters.
(b) It takes $\frac{1}{0.0000000043} \approx 232558139.534$ seconds or $2.32558139534 \times 10^8$ ■

Problem 26.37
Compute each of these to three significant figures using scientific notation.
(a) $(2.47 \times 10^{-5}) \cdot (8.15 \times 10^{-9})$
(b) $(2.47 \times 10^{-5}) \div (8.15 \times 10^{-9})$

Solution.
(a) $(2.47 \times 10^{-5}) \cdot (8.15 \times 10^{-9}) = 2.47 \times 8.15 \times 10^{-14} \approx 20.131 \times 10^{-14} = 2.0131 \times 10^{-13}$
(b) $(2.47 \times 10^{-5}) \div (8.15 \times 10^{-9}) = (2.47 \div 8.15) \times 10^{4} \approx 0.3030 \times 10^{4} = 3.030 \times 10^{5}$ ■

Problem 26.38
Convert each of the following to standard notation.
(a) 6.84×10^{-5} (b) 3.12×10^{7}

Solution.
(a) $6.84 \times 10^{-5} = 0.0000684$
(b) $3.12 \times 10^{7} = 31200000$ ■

Problem 26.39
Write each of the following in scientific notation.
(a) 413,682,000 (b) 0.000000231 (c) 100,000,000

Solution.
(a) $413,682,000 = 4.13682 \times 10^{8}$
(b) $0.000000231 = 2.31 \times 10^{-7}$
(c) $100,000,000 = 1.0 \times 10^{8}$ ■

Problem 26.40
Evaluate each of the following.
(a) $-5^2 + 3(-2)^2$
(b) $-2 + 3 \cdot 5 - 1$
(c) $10 - 3 \cdot 7 - 4(-2) \div 2 + 3$

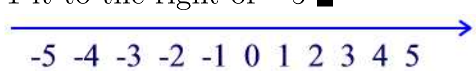
Solution.
(a) $-5^2 + 3(-2)^2 = -25 + 3 \times 4 = -25 + 12 = -13$
(b) $-2 + 3 \cdot 5 - 1 = -2 + 15 - 1 = 12$
(c) $10 - 3 \cdot 7 - 4(-2) \div 2 + 3 = 10 - 21 + 4 + 3 = -4$ ■

Problem 26.41
Evaluate each of the following, if possible.
(a) $(-10 \div (-2))(-2)$
(b) $(-10 \cdot 5) \div 5$
(c) $-8 \div (-8+8)$
(d) $(-6+6) \div (-2+2)$
(e) $|-24| \div 4 \cdot (3-15)$

Solution.
(a) $(-10 \div (-2))(-2) = 5(-2) = -10$
(b) $(-10 \cdot 5) \div 5 = -50 \div 5 = -10$
(c) $-8 \div (-8+8) = -8 \div 0$ is undefined
(d) $(-6+6) \div (-2+2) = 0 \div 0$ is undefined
(e) $|-24| \div 4 \cdot (3-15) = 24 \div 4 \cdot (-12) = 6 \cdot (-12) = -72$ ■

Problem 26.42
Use the number-line approach to verify each of the following.
(a) $-4 < 1$ (b) $-4 < -2$ (c) $-1 > -5$

Solution.
(a) $-4 < 1$ since -4 is to the left of 1.
(b) $-4 < -2$ since $-$ is to the left of -2.
(c) $-1 > -5$ since -1 it to the right of -5 ■



Problem 26.43
Order each of the following lists from smallest to largest.
(a) $\{-4, 4, -1, 1, 0\}$
(b) $\{23, -36, 45, -72, -108\}$

Solution.
(a) $-4 < -1 < 0 < 1 < 4$
(b) $-108 < -72 < -36 < 23 < 45$ ■

Problem 26.44
Replace the blank by the appropriate symbol.
(a) If $x > 2$ then $x + 4$____6
(b) If $x < -3$ then $x - 6$____ -9

Solution.
(a) If $x > 2$ then $x + 4 > 6$
(b) If $x < -3$ then $x - 6 < -9$ ■

Problem 26.45
Determine whether each of the following statements is correct.
(a) $-3 < 5$ (b) $6 < 0$ (c) $3 \leq 3$ (d) $-6 > -5$ (e) $2 \times 4 - 6 \leq -3 \times 5 + 1$

Solution.
(a) True since -3 is to the left of 5 on the number line.
(b) False since 6 is to the right of 0 on the number line.
(c) True since 3 is equal to 3.
(d) False since -6 is to the left of -5 on the number line.
(e) False since the left-hand side is positive and the right-hand side is negative ■

Problem 26.46
What different looking inequality means the same as $a < b$?

Solution.

$$b > a \blacksquare$$

Problem 26.47
Use symbols of inequalities to represent "at most" and "at least".

Solution.
"at most:" $\leq$
"at least:" $\geq$ ■

Problem 26.48
For each inequality, determine which of the numbers $-5, 0, 5$ satisfies the inequality.
(a) $x > -5$ (b) $5 < x$ (c) $-5 > x$

Solution.
(a) 0 and 5
(b) None
(c) None ■

Problem 26.49
Write the appropriate inequality symbol in the blank so that the two inequalities are true.
(a) If $x \leq -3$ then $-2x$____6
(b) If $x + 3 > 9$ then x____6

Solution.
(a) If $x \leq -3$ then $-2x \geq 6$
(b) If $x + 3 > 9$ then $x > 6$ ■

Problem 26.50
How do you know when to reverse the direction of an inequality symbol?

Solution.
Whenever the inequality is multiplied by a negative number ■

Problem 26.51
Show that each of the following inequality is true by using the addition approach.
(a) $-4 < -2$ (b) $-5 < 3$ (c) $-17 > -23$

Solution.
(a) $-2 = (-4) + 2$
(b) $3 = (-5) + 8$
(c) $-17 = (-23) + 6$ ■

Problem 26.52
A student makes a connection between debts and negative numbers. So the number -2 represents a debt of \$2. Since a debt of \$10 is larger than a debt of \$5 then the student writes $-10 > -5$. How would you convince him/her that this inequality is false?

Solution.
It is a true that a debt of \$ 10 is larger than a debt of \$ 5. However, if we use the number line model we see that -10 is to the left of -5 so that $-10 < -5$ ■

Problem 26.53
At an 8% sales tax rate, Susan paid more than \$1500 sales tax when she purchased her new Camaro. Describe this situation using an inequality with p denoting the price of the car.

Solution.

$$0.08p \geq 1500 \blacksquare$$

Problem 26.54
Show that if a and b are positive integers with $a < b$ then $a^2 < b^2$. Does this result hold for any integers?

Solution.
Multiplying $a < b$ by a and b respectively and using the fact that both numbers are positive we obtain $a^2 < ab$ and $ab < b^2$. By transitivity, we have $a^2 < b^2$. This is false if both a and b are not positive. For example,$-2 < 1$ but $(-2)^2 > 1^2$ $\blacksquare$

Problem 26.55
Elka is planning a rectangular garden that is twice as long as it is wide. If she can afford to buy at most 180 feet of fencing, then what are the possible values for the width?

Solution.
Let L denote the length and W denote the width. Then $L = 2W$. According to the problem, we have $2L+2W \leq 180$. Thus, $4W+2W \leq 180$ or $6W \leq 180$. Hence, $W \leq 30$ $\blacksquare$

Problem 27.1
Show that each of the following numbers is a rational number.
(a) -3 (b) $4\frac{1}{2}$ (c) -5.6 (d) 25%

Solution.
(a) $-3 = \frac{-3}{1}$
(b) $4\frac{1}{2} = 4 + \frac{1}{2} = \frac{9}{2}$
(c) $-5.6 = -\frac{56}{10} = \frac{-28}{5}$
(d) $25\% = \frac{25}{100} = \frac{1}{4}$ $\blacksquare$

Problem 27.2
Which of the following are equal to -3?

$$\frac{-3}{1}, \frac{3}{-1}, \frac{3}{1}, -\frac{3}{1}, \frac{-3}{-1}, -\frac{-3}{1}, -\frac{-3}{-1}.$$

Solution.

$$-3 = \frac{-3}{1} = \frac{3}{-1} = -\frac{3}{1} = -\frac{-3}{-1} \blacksquare$$

Problem 27.3
Determine which of the following pairs of rational numbers are equal.
(a) $\frac{-3}{5}$ and $\frac{63}{-105}$
(b) $\frac{-18}{-24}$ and $\frac{45}{60}$.

Solution.
(a) Since $(-3) \times (-105) = 5 \times 63$ then $\frac{-3}{5} = \frac{63}{-105}$
(b) Since $(-24) \times 45 = (-18) \times 60$ then $\frac{-18}{-24} = \frac{45}{60}$ $\blacksquare$

Problem 27.4
Rewrite each of the following rational numbers in simplest form.
(a) $\frac{5}{-7}$ (b) $\frac{21}{-35}$ (c) $\frac{-8}{-20}$ (d) $\frac{-144}{180}$

Solution.
(a) Already in simpest form.
(b) $\frac{21}{-35} = \frac{3\times 7}{(-5)\times 7} = \frac{3}{-5}$
(c) $\frac{-8}{-20} = \frac{2\times(-4)}{5\times(-4)} = \frac{2}{5}$
(d) $\frac{-144}{180} = \frac{-2^4\times 3^2}{2^2\times 3^2\times 5} = \frac{-2^2}{5} = -\frac{4}{5}$ $\blacksquare$

Problem 27.5
How many different rational numbers are given in the following list?

$$\frac{2}{5}, 3, \frac{-4}{-10}, \frac{39}{13}, \frac{7}{4}$$

Solution.

$$\frac{2}{5} \neq 3 \neq \frac{7}{4} \blacksquare$$

Problem 27.6
Find the value of x to make the statement a true one.

(a) $\frac{-7}{25} = \frac{x}{500}$ (b) $\frac{18}{3} = \frac{-5}{x}$

Solution.
(a) $\frac{-7}{25} = \frac{-140}{500}$ so $x = -140$
(b) $6 = \frac{-5}{x}$ so $x = \frac{-5}{6}$ ■

Problem 27.7
Find the prime factorizations of the numerator and the denominator and use them to express the fraction $\frac{247}{-77}$ in simplest form.

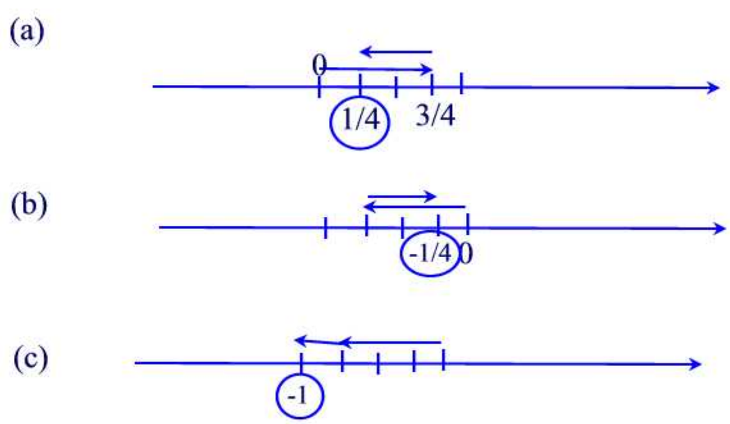
Solution.
Fraction is already in simplest form since

$$\frac{247}{-77} = \frac{19 \times 13}{(-7) \times 11} \blacksquare$$

Problem 27.8
(a) If $\frac{a}{b} = \frac{a}{c}$, what must be true?
(b) If $\frac{a}{c} = \frac{b}{c}$, what must be true?

Solution.
(a) $b = c$
(b) $a = b$ ■

Problem 27.9
Use number line model to illustrate each of the following sums.
(a) $\frac{3}{4} + \frac{-2}{4}$ (b) $\frac{-3}{4} + \frac{2}{4}$ (c) $\frac{-3}{4} + \frac{-1}{4}$

Solution.


Problem 27.10
Perform the following additions. Express your answer in simplest form.
(a) $\frac{6}{8} - \frac{-25}{100}$ (b) $\frac{-57}{100} + \frac{13}{10}$

Solution.
(a) $\frac{6}{8} - \frac{-25}{100} = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$
(b) $\frac{-57}{100} + \frac{13}{10} = \frac{-57}{100} + \frac{130}{100} = \frac{73}{100}$ ■

Problem 27.11
Perform the following subtractions. Express your answer in simplest form.
(a) $\frac{137}{214} - \frac{-1}{3}$ (b) $\frac{-23}{100} - \frac{198}{1000}$

Solution.
(a) $\frac{137}{214} - \frac{-1}{3} = \frac{411}{642} + \frac{214}{624} = \frac{625}{624}$
(b) $\frac{-23}{100} - \frac{198}{1000} = \frac{-230}{1000} - \frac{198}{1000} = \frac{-428}{1000}$ ■

Problem 27.12
Compute the following differences.
(a) $\frac{2}{3} - \frac{-9}{8}$ (b) $\left(-2\frac{1}{4}\right) - 4\frac{2}{3}$

Solution.
(a) $\frac{2}{3} - \frac{-9}{8} = \frac{16}{24} + \frac{27}{24} = \frac{43}{24}$
(b) $\left(-2\frac{1}{4}\right) - 4\frac{2}{3} = \frac{-9}{4} - \frac{14}{3} = \frac{-27}{12} - \frac{56}{12} = \frac{-83}{12}$ ■

Problem 27.13
Multiply the following rational numbers. Write your answers in simplest form.

(a) $\frac{3}{5} \cdot \frac{-10}{21}$ (b) $\frac{-6}{11} \cdot \frac{-33}{18}$ (c) $\frac{5}{12} \cdot \frac{48}{-15} \cdot \frac{-9}{8}$

Solution.
(a) $\frac{3}{5} \cdot \frac{-10}{21} = \frac{3}{5} \cdot \frac{(-2)\cdot 5}{3\cdot 7} = \frac{-2}{7}$
(b) $\frac{-6}{11} \cdot \frac{-33}{18} = \frac{-6}{11} \cdot \frac{(-3)\cdot 11}{6\cdot 3} = 1$
(c) $\frac{5}{12} \cdot \frac{48}{-15} \cdot \frac{-9}{8} = \frac{5}{12} \cdot \frac{4\cdot 12}{(-3)\cdot 5} \cdot \frac{(-3)\cdot 3}{4\cdot 2} = \frac{3}{2}$ ■

Problem 27.14
Find the following quotients. Write your answers in simplest form.

(a) $\frac{-8}{9} \div \frac{2}{9}$ (b) $\frac{12}{15} \div \frac{-4}{3}$ (c) $\frac{-13}{24} \div \frac{-39}{-48}$

Solution.
(a) $\frac{-8}{9} \div \frac{2}{9} = \frac{-8}{9} \cdot \frac{9}{2} = -4$
(b) $\frac{12}{15} \div \frac{-4}{3} = \frac{4}{3} \cdot \frac{3}{-4} = -1$
(c) $\frac{-13}{24} \div \frac{-39}{-48} = \frac{-13}{24} \cdot \frac{16}{13} = \frac{-24}{16} = -\frac{3}{2}$ ■

Problem 27.15
State the property that justifies each statement.

(a) $\left(\frac{5}{7} \cdot \frac{7}{8}\right) \cdot \frac{-8}{3} = \frac{5}{7} \cdot \left(\frac{7}{8} \cdot \frac{-8}{3}\right)$
(b) $\frac{1}{4}\left(\frac{8}{3} + \frac{-5}{4}\right) = \frac{1}{4} \cdot \frac{8}{3} + \frac{1}{4} \cdot \frac{-5}{4}$

Solution.
(a) Associativity
(b) Distributivity ■

Problem 27.16
Compute the following and write your answers in simplest form.

(a) $\frac{-40}{27} \div \frac{-10}{9}$ (b) $\frac{21}{25} \div \frac{-3}{5}$ (c) $\frac{-10}{9} \div \frac{-9}{8}$

Solution.
(a) $\frac{-40}{27} \div \frac{-10}{9} = \frac{-40}{27} \cdot \frac{9}{-10} = \frac{4}{3}$
(b) $\frac{21}{25} \div \frac{-3}{5} = \frac{21}{25} \cdot \frac{5}{-3} = -\frac{7}{5}$
(c) $\frac{-10}{9} \div \frac{-9}{8} = \frac{-10}{9} \cdot \frac{8}{-9} = \frac{80}{81}$ ■

Problem 27.17
Find the reciprocals of the following rational numbers.

(a) $\frac{4}{-9}$ (b) 0 (c) $\frac{-3}{2}$ (d) $\frac{-4}{-9}$

Solution.
(a) $\frac{-9}{4}$
(b) Does not exist
(c) $\frac{2}{-3}$
(d) $\frac{-9}{-4} = \frac{9}{4}$ ■

Problem 27.18
Compute: $\left(\frac{-4}{7} \cdot \frac{2}{-5}\right) \div \frac{2}{-7}$.

Solution.

$$\begin{aligned}\left(\tfrac{-4}{7}\cdot\tfrac{2}{-5}\right)\div\tfrac{2}{-7} &= \tfrac{8}{35}\div\tfrac{2}{-7}\\ &= \tfrac{8}{35}\cdot\tfrac{-7}{2}\\ &= -\tfrac{4}{5}\ \blacksquare\end{aligned}$$

Problem 27.19
If $\frac{a}{b}\cdot\frac{-4}{7}=\frac{2}{3}$ what is $\frac{a}{b}$?

Solution.

$$\frac{a}{b}=\frac{2}{3}\div\frac{-4}{7}=\frac{2}{3}\cdot\frac{7}{-4}=-\frac{7}{6}\ \blacksquare$$

Problem 27.20
Compute $-4\frac{1}{2}\times-5\frac{2}{3}$

Solution.

$$-4\frac{1}{2}\times-5\frac{2}{3}=\frac{-9}{2}\cdot\frac{-17}{3}=\frac{51}{2}\ \blacksquare$$

Problem 27.21
Compute $-17\frac{8}{9}\div5\frac{10}{11}$

Solution.

$$-17\frac{8}{9}\div5\frac{10}{11}=\frac{-161}{9}\div\frac{65}{11}=\frac{-161}{9}\cdot\frac{11}{65}=-\frac{1771}{585}\ \blacksquare$$

Problem 27.22
Compute each of the following:

(a) $-\left(\frac{3}{4}\right)^2$ (b) $\left(-\frac{3}{4}\right)^2$ (c) $\left(\frac{3}{4}\right)^2\cdot\left(\frac{3}{4}\right)^7$

Solution.
(a) $-\left(\frac{3}{4}\right)^2=-\frac{9}{16}$
(b) $\left(-\frac{3}{4}\right)^2=\frac{9}{16}$
(c) $\left(\frac{3}{4}\right)^2\cdot\left(\frac{3}{4}\right)^7=\left(\frac{3}{4}\right)^{2+7}=\left(\frac{3}{4}\right)^9\ \blacksquare$

Problem 27.23
True or false: (a) $\frac{-2}{3}<\frac{-3}{7}$ (b) $\frac{15}{-9}>\frac{-13}{4}$.

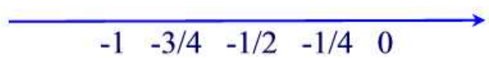
Solution.
(a) First we reduce to same denominator: $\frac{-2}{3} = \frac{-14}{21}$ and $\frac{-3}{7} = \frac{-9}{21}$. Since $\frac{-14}{21} < \frac{-9}{21}$ then the given statement is true.
(b) We have $\frac{15}{-9} = \frac{-60}{36}$ and $\frac{-13}{4} = \frac{-96}{36}$. Since $\frac{-96}{36} < \frac{-60}{36}$ then the statement is true ■

Problem 27.24
Show that $\frac{-3}{4} < \frac{-1}{4}$ using the addition approach.

Solution.
Since $\frac{-3}{4} + \frac{2}{4} = \frac{-1}{4}$ then $\frac{-3}{4} < \frac{-1}{4}$ ■

Problem 27.25
Show that $\frac{-3}{4} < \frac{-1}{4}$ by using a number line.

Solution.



Problem 27.26
Put the appropriate symbol, $<, =, >$ between each pair of numbers to make a true statement.

(a) $-\frac{5}{6}$_____ $-\frac{11}{12}$

(b) $-\frac{1}{3}$_____ $\frac{5}{4}$

(c) $\frac{-12}{15}$_____ $\frac{36}{-45}$

(d) $-\frac{3}{12}$_____ $\frac{-4}{20}$

Solution.
(a) $-\frac{5}{6} = -\frac{10}{12} > -\frac{11}{12}$

(b) $-\frac{1}{3} < \frac{5}{4}$

(c) $\frac{-12}{15} = \frac{-36}{45} = \frac{36}{-45}$

(d) $-\frac{3}{12} = -\frac{1}{4} = -\frac{5}{20} < \frac{-4}{20}$ ■

Problem 27.27
Find three rational numbers between $\frac{1}{4}$ and $\frac{2}{5}$.

Solution.
Since $\frac{1}{4} < \frac{2}{5}$ then by the density property we have

$$\frac{1}{4} < \frac{1}{2}\left(\frac{1}{4} + \frac{2}{5}\right) < \frac{2}{5}$$

or

$$\frac{1}{4} < \frac{1}{2}\left(\frac{5}{20} + \frac{8}{20}\right) < \frac{2}{5}$$
$$\frac{1}{4} < \frac{13}{40} < \frac{2}{5}$$

Similarly,

$$\frac{1}{4} < \frac{1}{2}\left(\frac{1}{4} + \frac{13}{40}\right) < \frac{13}{40} < \frac{1}{2}\left(\frac{13}{40} + \frac{2}{5}\right) < \frac{2}{5} \blacksquare$$

Problem 27.28
The properties of rational numbers are used to solve inequalities. For example,

$$\begin{array}{rcl} x + \frac{3}{5} & < & \frac{-7}{10} \\ x + \frac{3}{5} + \left(-\frac{3}{5}\right) & < & \frac{-7}{10} + \left(-\frac{3}{5}\right) \\ x & < & -\frac{13}{10} \end{array}$$

Solve the inequality

$$-\frac{2}{5}x + \frac{1}{5} > -1.$$

Solution.
Subtract $\frac{1}{5}$ from both sides

$$-\frac{2}{5}x > -1 - \frac{1}{5} = -\frac{6}{5}$$

Now multiply both sides by $-\frac{5}{2}$ to obtain

$$x < 3 \blacksquare$$

Problem 27.29
Solve each of the following inequalities.

(a) $x - \frac{6}{5} < \frac{-12}{7}$

(b) $\frac{2}{5}x < -\frac{7}{8}$

(c) $\frac{-3}{7}x > \frac{8}{5}$

Solution.
(a)
$$\begin{array}{ccc} x - \frac{6}{5} & < & \frac{-12}{7} \\ x - \frac{6}{5} + \frac{6}{5} & < & \frac{-12}{7} + \frac{6}{5} \\ x & < & -\frac{18}{35} \end{array}$$
(b)
$$\begin{array}{ccc} \frac{2}{5}x & < & -\frac{7}{8} \\ \frac{5}{2} \cdot \frac{2}{5}x & < & \frac{5}{2} \cdot \frac{-7}{8} \\ x & < & -\frac{35}{16} \end{array}$$
(c)
$$\begin{array}{ccc} \frac{-3}{7}x & > & \frac{8}{5} \\ \frac{7}{-3} \cdot \frac{-3}{7}x & < & \frac{7}{-3} \cdot \frac{8}{5} \\ x & < & -\frac{56}{15} \blacksquare \end{array}$$

Problem 27.30
Verify the following inequalities.

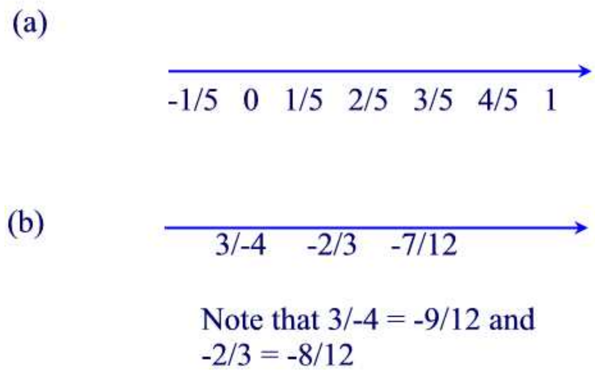
(a) $\frac{-4}{5} < \frac{-3}{4}$ (b) $\frac{1}{10} < \frac{1}{4}$ (c) $\frac{19}{-60} > \frac{-1}{3}$

Solution.
(a) Since $\frac{-4}{5} = \frac{-16}{20}$ and $\frac{-3}{4} = \frac{-15}{20}$ then $\frac{-4}{5} < \frac{-3}{4}$
(b) Since $\frac{1}{10} = \frac{2}{20}$ and $\frac{1}{4} = \frac{5}{20}$ then $\frac{1}{10} < \frac{1}{4}$
(c) Since $\frac{-1}{3} = \frac{-20}{60}$ then $\frac{19}{-60} > \frac{-1}{3}$ $\blacksquare$

Problem 27.31
Use the number-line approach to arrange the following rational numbers in increasing order:

(a) $\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}$
(b) $\frac{-7}{12}, \frac{-2}{3}, \frac{3}{-4}$

Solution.



Problem 27.32
Find a rational number between $\frac{5}{12}$ and $\frac{3}{8}$.

Solution.

$$\frac{3}{8} < \frac{1}{2}\left(\frac{5}{12} + \frac{3}{8}\right) < \frac{5}{12}$$

or

$$\frac{3}{8} < \frac{1}{2}\left(\frac{10}{24} + \frac{9}{24}\right) < \frac{5}{12}$$

$$\frac{3}{8} < \frac{19}{48} < \frac{5}{12} \blacksquare$$

Problem 27.33
Complete the following, and name the property that is used as a justification.

(a) If $\frac{-2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{5}$ then $\frac{-2}{3}$____$\frac{7}{5}$.

(b) If $\frac{-3}{5} < \frac{-6}{11}$ then $\left(\frac{-3}{5}\right) \cdot \left(\frac{2}{3}\right)$ ____ $\left(\frac{-6}{11}\right) \cdot \left(\frac{2}{3}\right)$

(c) If $\frac{-4}{7} < \frac{7}{4}$ then $\frac{-4}{7} + \frac{5}{8} < \frac{7}{4} +$ ____

(d) If $\frac{-3}{4} > \frac{11}{3}$ then $\left(\frac{-3}{4}\right) \cdot \left(\frac{-5}{7}\right)$ ____ $\left(\frac{11}{3}\right) \cdot \left(\frac{-5}{7}\right)$

(e) There is a rational number ____ any two unequal rational numbers.

Solution.
(a) If $\frac{-2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{5}$ then $\frac{-2}{3} < \frac{7}{5}$. (Transitivity)

(b) If $\frac{-3}{5} < \frac{-6}{11}$ then $\left(\frac{-3}{5}\right) \cdot \left(\frac{2}{3}\right) < \left(\frac{-6}{11}\right) \cdot \left(\frac{2}{3}\right)$. (Multiplication property)

(c) If $\frac{-4}{7} < \frac{7}{4}$ then $\frac{-4}{7} + \frac{5}{8} < \frac{7}{4} + \frac{5}{8}$ (Addition property)

(d) If $\frac{-3}{4} < \frac{11}{3}$ then $\left(\frac{-3}{4}\right) \cdot \left(\frac{-5}{7}\right) > \left(\frac{11}{3}\right) \cdot \left(\frac{-5}{7}\right)$ (Multiplication by a negative rational)

(e) There is a rational number between any two unequal rational numbers. (Density property) ■

Problem 28.1
Given a decimal number, how can you tell whether the number is rational or irrational?

Solution.
If the number is terminating or nonterminating and repeating then the decimal number is rational. Otherwise, the number is irrational ■

Problem 28.2
Write the following repeating decimal numbers as a fraction.
(a) $0.\overline{8}$ (b) $0.\overline{37}$ (c) $0.\overline{02714}$

Solution.
(a) Let $x = 0.\overline{8}$. Then $10x = 8.\overline{8} = 8 + x$. Solving for x we find $x = \frac{8}{9}$.
(b) Let $x = 0.\overline{37}$. Then $100x = 37.\overline{37} = 37 + x$. Solving for x we find $x = \frac{37}{99}$
(c) Let $x = 0.\overline{02714}$. Then $100000x = 27140.\overline{02714} = 2714 + x$. Solving for x we find $x = \frac{2714}{99999}$ ■

Problem 28.3
Which of the following describe 2.6?
(a) A whole number
(b) An integer
(c) A rational number
(d) An irrational number
(e) A real number

Solution.
(c) and (e). Note that $2.6 = \frac{26}{10} = \frac{13}{5}$ ■

Problem 28.4
Classify the following numbers as rational or irrational.
(a) $\sqrt{11}$ (b) $\frac{3}{7}$ (c) π (d) $\sqrt{16}$

Solution.
(a) Irrational, nonterminating and nonrepeating
(b) Rational
(c) Irrational, nonterminating and nonrepeating
(d) $\sqrt{16} = 4$, a rational ■

Problem 28.5
Classify the following numbers as rational or irrational.
(a) $0.34938661\cdots$ (b) $0.\overline{26}$ (c) $0.565665666\cdots$

Solution.
(a) Irrational, nontermintaing and nonrepeating.
(b) Rational, nonterminating and repeating.
(c) Irrational, nonterminating and nonrepeating ■

Problem 28.6
Match each word in column A with a word in column B.

A	B
Terminating	Rational
Repeating	Irrational
Infinite nonrepeating	

Solution.
(Terminating, Rational), (Repeating, Rational), (Infinite nonrepeating, Irrational) ■

Problem 28.7
(a) How many whole numbers are there between -3 and 3 (not including 3 and -3)?
(b) How many integers are there between -3 and 3?
(c) How many real numbers are there between -3 and 3?

Solution.
(a) 0, 1, 2
(b) $-2, -1, 0, 1, 2$
(c) infinite numbers ■

Problem 28.8
Prove that $\sqrt{3}$ is irrational.

Solution.
Suppose that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some integers a and $b \neq 0$. But then $\frac{a^2}{b^2} = 3$ so that $a^2 = 4b^2$. If a has an even number of prime factors then a^2 has an even number of prime factors. If a has an odd number of prime factors then a^2 has an even number of prime factors. So, a^2 and b^2 have both even number of prime factors. But $3 \cdot b^2$ has an odd number of prime factors. So we have that a^2 has an even number and an odd number of prime factors. This cannot happen by the Fundamental Theorem of Arithmetic which says that every positive integer has a unique number of prime factors. In conclusion, $\sqrt{3}$ cannot be written in the form $\frac{a}{b}$ so it is not rational ■

Problem 28.9
Show that $1 + \sqrt{3}$ is irrational.

Solution.
Suppose that $1+\sqrt{3} = c$ is rational. Then $\sqrt{3} = c-1$. Since both 1 and c are rationals then $c - 1$ is also rational. This shows that $\sqrt{3}$ is rational. But we just proved in the previous problem that $\sqrt{3}$ is irrational. Hence, $c = 1+\sqrt{3}$ must be irrational ■

Problem 28.10
Find an irrational number between $0.\overline{37}$ and $0.\overline{38}$

Solution.
$\frac{\sqrt{2}}{3} - \frac{1}{10}$ ■

Problem 28.11
Write an irrational number whose digits are twos and threes.

Solution.
One answer is $0.232233222333\cdots$ ■

Problem 28.12
Classify each of the following statement as true or false. If false, give a counter example.

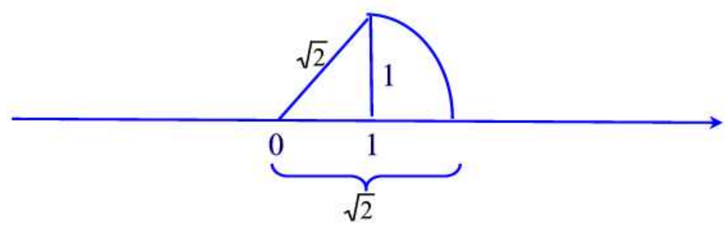
(a) The sum of any rational number and any irrational number is a rational number.
(b) The sum of any two irrational numbers is an irrational number.
(c) The product of any two irrational numbers is an irrational number.
(d) The difference of any two irrational numbers is an irrational number.

Solution.
(a) False: $1 + \sqrt{2}$ is irrational.
(b) False: $\sqrt{2} + (-\sqrt{2}) = 0$ which is rational.
(c) False: $\sqrt{2} \cdot \sqrt{2} = 2$ which is rational.
(d) False: $\sqrt{2} - \sqrt{2} = 0$ which is rational ■

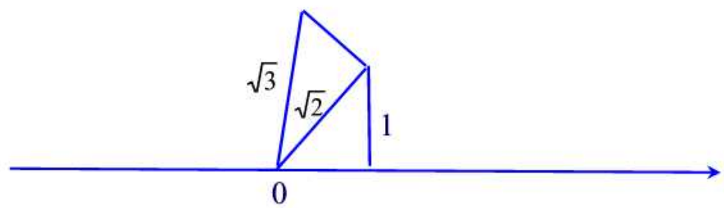
Problem 28.13
Construct the lengths $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \cdots$ as follows.
(a) First construct a right triangle with both legs of length 1. What is the length of the hypotenuse?
(b) This hypotenuse is the leg of the next right triangle. The other leg has length 1. What is the length of the hypotenuse of this triangle?
(c) Continue drawing right triangles, using the hypotenuse of the proceeding triangle as a leg of the next triangle until you have constructed one with length $\sqrt{6}$.

Solution.

(a)



(b)



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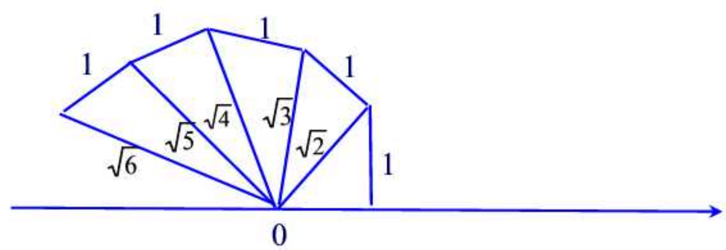


Problem 28.14

Arrange the following real numbers in increasing order.

0.56, $0.\overline{56}$, $0.5\overline{66}$, $0.56565556\cdots$, $0.\overline{566}$

Solution.

$$0.56 < 0.56565556\cdots < 0.\overline{56} < 0.\overline{566} < 0.5\overline{66} \blacksquare$$

Problem 28.15

Which property of real numbers justify the following statement

$$2\sqrt{3} + 5\sqrt{3} = (2+5)\sqrt{3} = 7\sqrt{3}$$

Solution.

Distributivity of addition with respect to multiplication $\blacksquare$

Problem 28.16
Find $x : x + 2\sqrt{2} = 5\sqrt{2}$.

Solution.
Subtract $2\sqrt{2}$ from both sides to obtain

$$x = 3\sqrt{2} \blacksquare$$

Problem 28.17
Solve the following equation: $2x - 3\sqrt{6} = 3x + \sqrt{6}$

Solution.

$$\begin{array}{ccc} 2x - 3\sqrt{6} & = & 3x + \sqrt{6} \\ -2x - \sqrt{6} + 2x - 3\sqrt{6} & = & 3x + \sqrt{6} - 2x - \sqrt{6} \\ -4\sqrt{6} & = & x \blacksquare \end{array}$$

Problem 28.18
Solve the inequality: $\frac{3}{2}x - 2 < \frac{5}{6}x + \frac{1}{3}$

Solution.

$$\begin{array}{ccc} \frac{3}{2}x - 2 & < & \frac{5}{6}x + \frac{1}{3} \\ \frac{3}{2}x - 2 - \frac{5}{6}x + 2 & < & \frac{5}{6}x + \frac{1}{3} - \frac{5}{6}x + 2 \\ \frac{2}{3}x & < & \frac{7}{3} \\ \frac{3}{2} \cdot \frac{2}{3}x & < & \frac{3}{2} \cdot \frac{7}{3} \\ x & < & \frac{7}{2} \blacksquare \end{array}$$

Problem 28.19
Determines for what real number x, if any, each of the following is true:
(a) $\sqrt{x} = 8$ (b) $\sqrt{x} = -8$ (c)$\sqrt{-x} = 8$ (d) $\sqrt{-x} = -8$

Solution.
(a) $\sqrt{x} = 8 \Longrightarrow x = 8^2 = 64$.
(b) Since $\sqrt{x} \geq 0$ then the given equation has no real solutions.
(c) $\sqrt{-x} = 8 \Longrightarrow -x = 64 \Longrightarrow x = -64$.
(d) Since $\sqrt{-x} \geq 0$ then the given equation has no real solutions $\blacksquare$

Problem 28.20
Express the following values without exponents.
(a)$25^{\frac{1}{2}}$ (b) $32^{\frac{1}{5}}$ (c) $9^{\frac{5}{2}}$ (d) $(-27)^{\frac{4}{3}}$

Solution.
(a)$25^{\frac{1}{2}} = 5$
(b) $32^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$
(c) $9^{\frac{5}{2}} = (3^2)^{\frac{5}{2}} = 3^5 = 405$
(d) $(-27)^{\frac{4}{3}} = [(-3)^3]^{\frac{4}{3}} = (-3)^4 = 81$ ■

Problem 28.21
Write the following radicals in simplest form if they are real numbers.
(a) $\sqrt[3]{-27}$ (b) $\sqrt[4]{-16}$ (c) $\sqrt[5]{32}$

Solution.
(a) $\sqrt[3]{-27} = -3$
(b) $\sqrt[4]{-16}$ undefined since the radicand is negative and the index is even.
(c) $\sqrt[5]{32} = 2$ ■

Problem 28.22
A student uses the formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$ to show that -1 = 1 as follows:

$$-1 = (\sqrt{-1})^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

What's wrong with this argument?

Solution.
$\sqrt{-1}$ is not a real number since $\sqrt{a}$ is a real number when $a \geq 0$ ■

Problem 28.23
Give an example where the following statement is true: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Solution.
$a = b = 0$ is such an example. Another example is $a = 1, b = 0$ ■

Problem 28.24
Give an example where the following statement is false: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Solution.
Take $a = b = 1$. In this case. $\sqrt{1+1} = \sqrt{2} \neq \sqrt{1} + \sqrt{1} = 2$ ■

Problem 28.25
Find an example where the following statement is false: $\sqrt{a^2+b^2} = a + b$.

Solution.
Take $a = b = 1$ and see the previous solution $\blacksquare$

Problem 28.26
Express the following values without using exponents:
(a) $(3^{10})^{\frac{3}{5}}$ (b) $81^{-\frac{5}{4}}$

Solution.
(a) $(3^{10})^{\frac{3}{5}} = 3^{10\cdot\frac{3}{5}} = 3^6 = 729$
(b) $81^{-\frac{5}{4}} = (3^4)^{-\frac{5}{4}} = 3^{4\cdot\frac{-5}{4}} = 3^{-5} = \frac{1}{81}$ $\blacksquare$

Problem 28.27
If possible, find the square root of each of the following without using a calculator.
(a) 225 (b) 169 (c) -81 (d) 625

Solution.
(a) $225 = 15^2$ so $\sqrt{225} = 15$
(b) $169 = 13^2$ so $\sqrt{169} = 13$
(c) $\sqrt{-81}$ is undefined since the radicand is negative.
(d) $625 = 25^2$ so $\sqrt{625} = 25$ $\blacksquare$

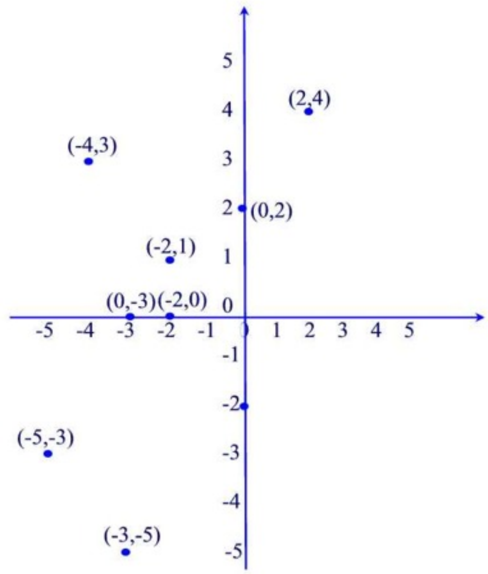
Problem 28.28
Write each of the following in the form $a\sqrt{b}$ where a and b are integers and b has the least value possible.
(a) $\sqrt{180}$ (b) $\sqrt{363}$ (c) $\sqrt{252}$

Solution.
(a) $\sqrt{180} = \sqrt{36 \cdot 5} = 6\sqrt{5}$
(b) $\sqrt{363} = \sqrt{3 \cdot 11^2} = 11\sqrt{3}$
(c) $\sqrt{252} = \sqrt{2^2 \cdot 3^2 \cdot 7} = 6\sqrt{7}$ $\blacksquare$

Problem 29.1
Plot the points whose coordinates are given on a Cartesian coordinate system.

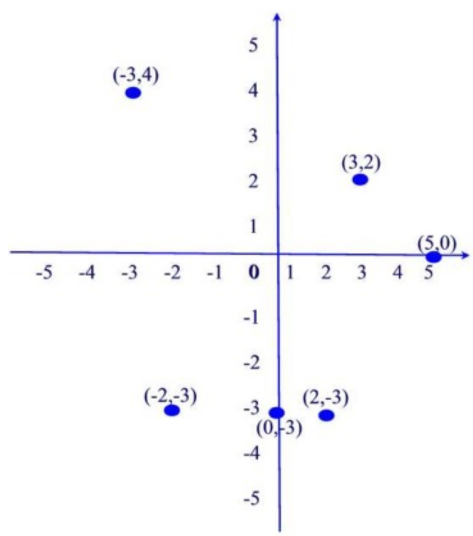
(a) $(2, 4), (0, -3), (-2, 1), (-5, -3)$.
(b) $(-3, -5), (-4, 3), (0, 2), (-2, 0)$.

Solution.



Problem 29.2
Plot the following points using graph papers.
(a) (3,2) (b) (5,0) (c) (0,-3) (d) (-3,4) (e) (-2,-3) (f) (2,-3)

Solution.



Problem 29.3
Complete the following table.

(x,y)	$x > 0, y > 0$	$x < 0, y > 0$	$x > 0, y < 0$	$x < 0, y < 0$
Quadrant				

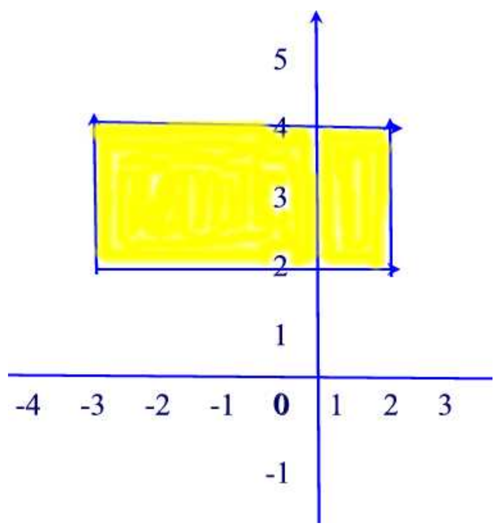
Solution.

(x,y)	$x > 0, y > 0$	$x < 0, y > 0$	$x > 0, y < 0$	$x < 0, y < 0$
Quadrant	I	II	IV	III

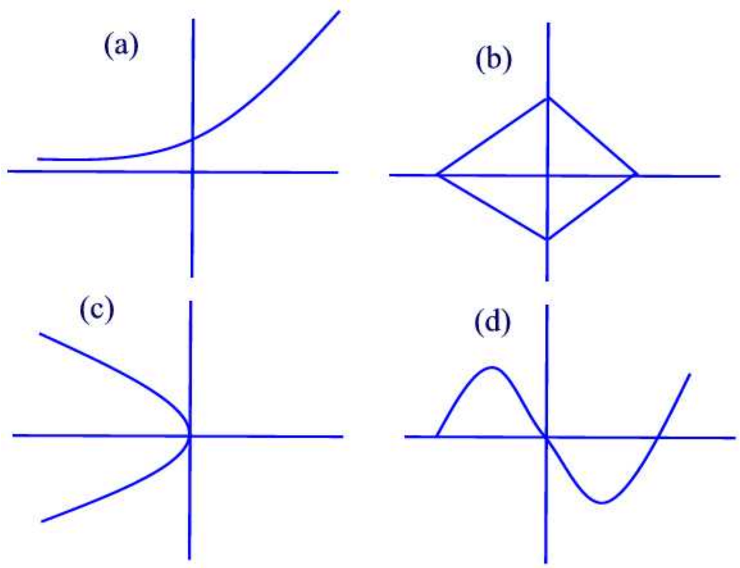
Problem 29.4
In the Cartesian plane, shade the region consisting of all points (x, y) that satisfy the two conditions

$$-3 \leq x \leq 2 \text{ and } 2 \leq y \leq 4$$

Solution.

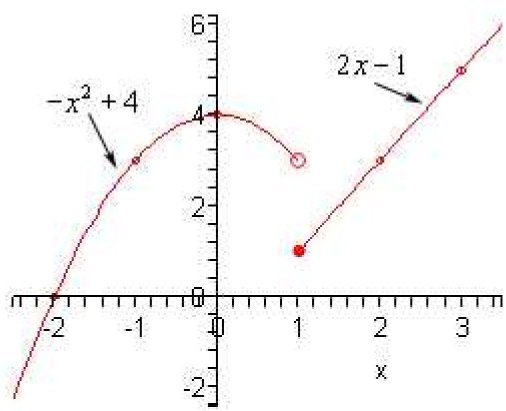


Problem 29.5
Determine which of the following graphs represent a functions.



Solution.
(a) and (d) represent functions. (b) and (c) fail the vertical line test ■

Problem 29.6
Consider the function f whose graph is given below.



(a) Complete the following table

x	-2	-1	0	1	2	3
f(x)						

(b) Find the domain and range of f.
(c) For which values of x is $f(x) = 2.5$?

Solution.
(a)

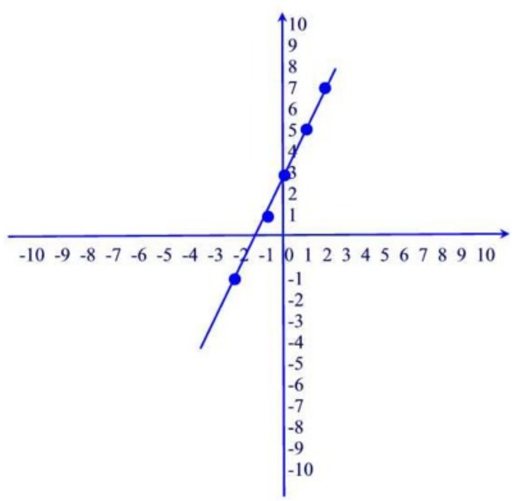
x	-2	-1	0	1	2	3
f(x)	0	3	4	1	3	5

(b) $Dom(f) = \mathbb{R}$, and $Range(f) = \mathbb{R}$.
(c) We have to solve the two equations $-x^2 + 4 = 2.5$ and $2x - 1 = 2.5$. Solving the first equation we find $x^2 = 4 - 2.5 = 1.5$ so $x = \sqrt{1.5}$ and solving the second equation we find $x = \frac{3.5}{2} = 1.75$ ■

Problem 29.7
Make a table of five values of the function $f(x) = 2x + 3$ and then use the points to sketch the graph of $f(x)$.

Solution.

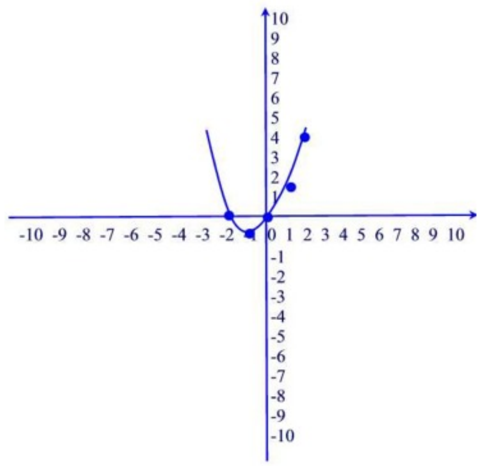
x	-2	-1	0	1	2
f(x)	-1	1	3	5	7



Problem 29.8
Make a table of five values of the function $f(x) = \frac{1}{2}x^2 + x$ and then use the points to sketch the graph of $f(x)$.

Solution.

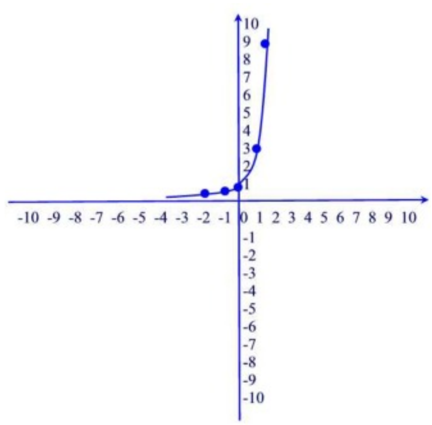
x	-2	-1	0	1	2
f(x)	0	-1/2	0	3/2	4



Problem 29.9
Make a table of five values of the function $f(x) = 3^x$ and then use the points to sketch the graph of $f(x)$.

Solution.

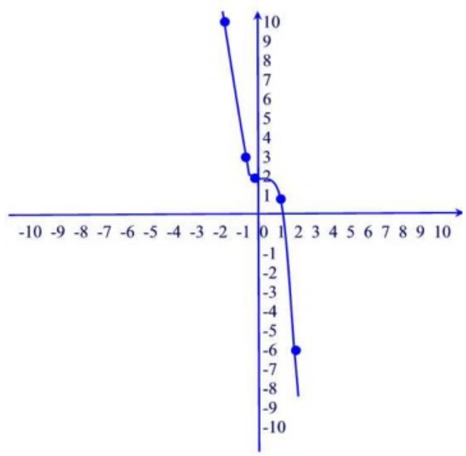
x	-2	-1	0	1	2
f(x)	1/9	1/3	1	3	9



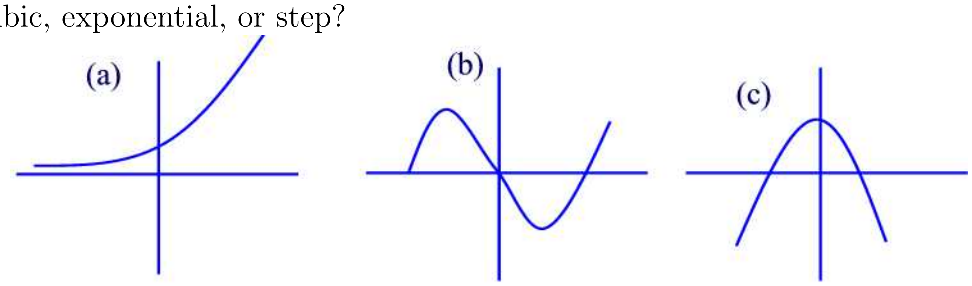
Problem 29.10
Make a table of five values of the function $f(x) = 2 - x^3$ and then use the points to sketch the graph of $f(x)$.

Solution.

x	-2	-1	0	1	2
f(x)	10	3	2	1	-6



Problem 29.11
Which type of function best fits each of the following graphs: linear, quadratic, cubic, exponential, or step?



Solution.
(a) Exponential (b) Cubic (c) Quadratic ■

Problem 29.12
Suppose that a function f is given by a table. If the output changes by a fixed amount each time the input changes by a constant then the function is linear. Determine whether each of the following functions below are linear.

x	0	2	4	6
f(x)	20	40	80	160

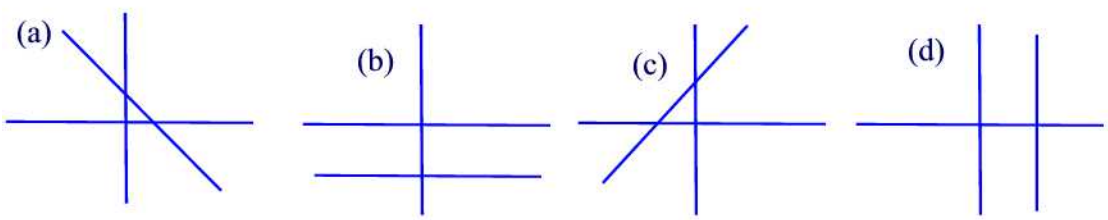
x	10	20	30	40
g(x)	6	12	18	24

Solution.
(a) f is not linear since every time the independent variable increases by 2 the dependent variable is not changing by a fixed amount.
(b) g is linear since every time x increases by 10, g increases by 6 ■

Problem 29.13
Show how to solve the equation $2x + 3 = 11$ using a calculator.

Solution.
You graph the two lines $Y_1 = 2x + 3$ and $Y_2 = 11$ on the same coordinate system and then you find the point of intersection which is the answer to the problem ■

Problem 29.14
In the linear function $f(x) = mx + b$ the parameter m is called the **slope.** The slope of the line determines whether the line rises, falls, is vertical or horizontal. Classify the slope of each line as positive, negative, zero, or undefined.



Solution.
(a) Negative (b) Zero (c) Positive (d) Undefined ■

Problem 29.15
Algebraically, one finds the slope of a line given two points (x_1, y_1) and (x_2, y_2) on the line by using the formula

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Would the ratio

$$\frac{y_1 - y_2}{x_1 - x_2}$$

give the same answer? Explain.

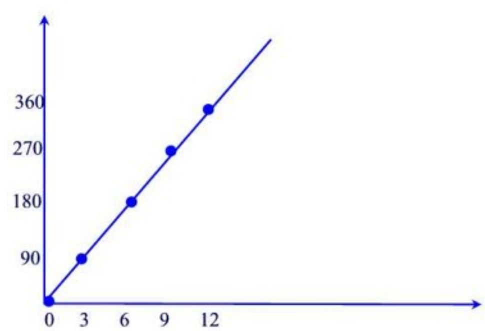
Solution.
Yes since one obtains an equivalent fraction by multiplying the numerator and the denominator by -1 ■

Problem 29.16
Water is being pumped into a tank. Reading are taken every minutes.

Time(min)	0	3	6	9	12
Quarts of water	0	90	180	270	360

(a) Plot the 5 points.
(b) What is the slope of the line joining the 5 points?
(c) Estimate how much water is in the tank after 1 minute.
(d) At what rate is the water being pumped in?

Solution.

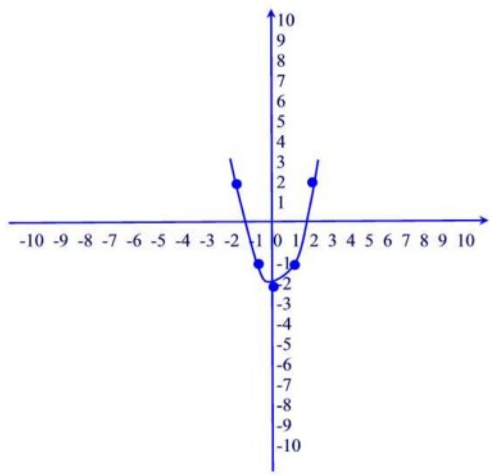


(a)
(b) $\frac{90-0}{3-0} = 30$
(c) Since the equation of the line is $Q(t) = 30t$ then $Q(1) = 30$ quarts.
(d) 30 quarts per minute ■

Problem 29.17
(a) Graph $f(x) = x^2 - 2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Solution.
(a) We have

x	-2	-1	0	1	2
f(x)	2	-1	-2	-1	2

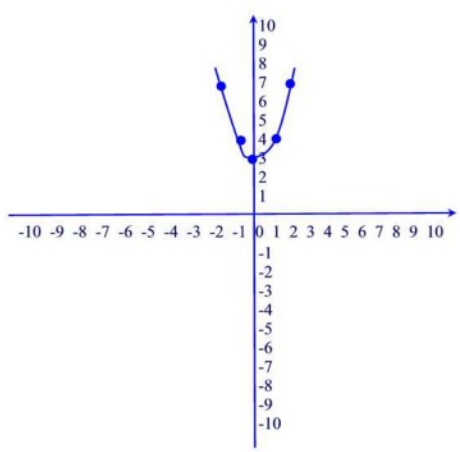


(b) A vertical translation of the graph of $y = x^2$ downward by two units ■

Problem 29.18
(a) Graph $f(x) = x^2 + 3$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Solution.
(a) We have

x	-2	-1	0	1	2
f(x)	7	4	3	4	7

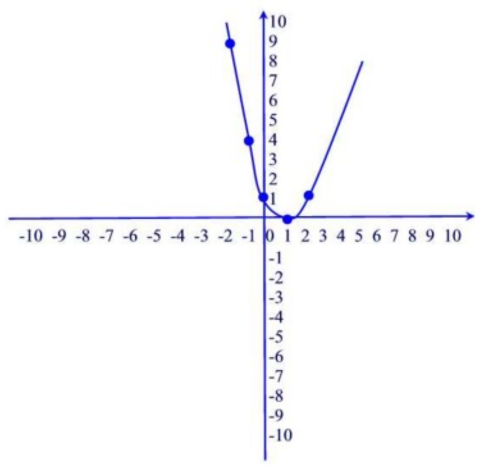


(b) A vertical translation of the graph of $y = x^2$ upward by three units ■

Problem 29.19
(a) Graph $f(x) = (x-1)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Solution.
(a) We have

x	-2	-1	0	1	2
f(x)	9	4	1	0	1

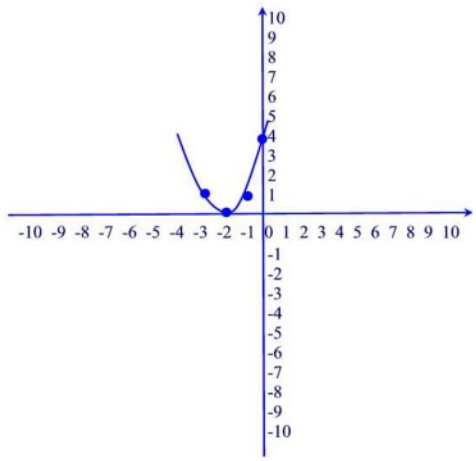


(b) A horizontal translation of the graph of $y = x^2$ to the right by one unit ■

Problem 29.20
(a) Graph $f(x) = (x+2)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
(b) How is this graph related to the graph of $y = x^2$?

Solution.
(a) We have

x	-3	-2	-1	0	1
f(x)	1	0	1	4	9



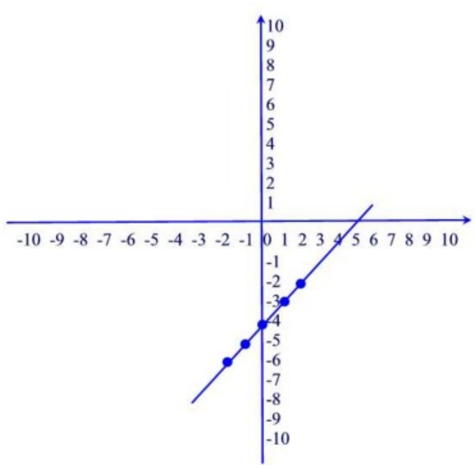
(b) A horizontal translation of the graph of $y = x^2$ to the left by two units ∎

Problem 29.21
Graph each equation by plotting points that satisfy the equation.

(a) $x - y = 4$.
(b) $y = -2|x - 3|$.
(c) $y = \frac{1}{2}(x-1)^2$.

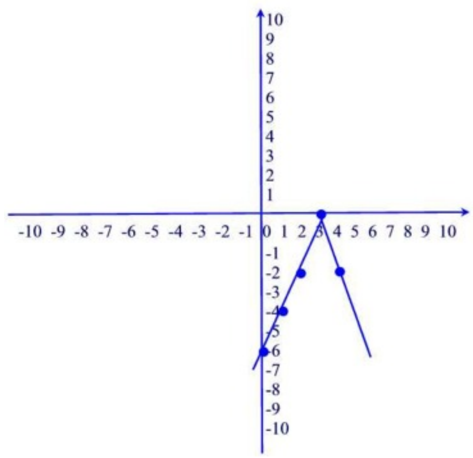
Solution.
(a) We have

x	-2	-1	0	1	2
f(x)	-6	-5	-4	-3	-2



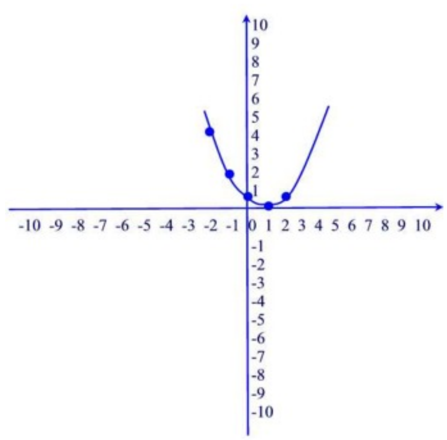
(b) We have

x	0	1	2	3	4
f(x)	-6	-4	-2	0	-2



(c) We have

x	-2	-1	0	1	2
f(x)	4.5	2	0.5	0	0.5



Problem 29.22
Find the x- and y-intercepts of each equation.

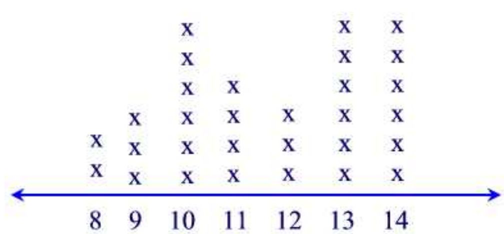
(a) $2x + 5y = 12.$
(b) $x = |y| - 4.$
(c) $|x| + |y| = 4.$

Solution.
(a) x-intercept: $y = 0 \Longrightarrow 2x = 12 \Longrightarrow x = 6.$
y-intercept: $x = 0 \Longrightarrow 5y = 12 \Longrightarrow y = \frac{12}{5}.$
(b) x-intercept: $y = 0 \Longrightarrow x = -4.$
y-intercept: $x = 0 \Longrightarrow |y| = 4 \Longrightarrow y = \pm 4.$
(c) x-intercept: $y = 0 \Longrightarrow |x| = 4 \Longrightarrow x = \pm 4.$
y-intercept: $x = 0 \Longrightarrow |y| = 4 \Longrightarrow y = \pm 4$ ■

Problem 30.1
Following are the ages of the 30 students at Washington High School who participated in the city track meet. Draw a line plot to represent these data.

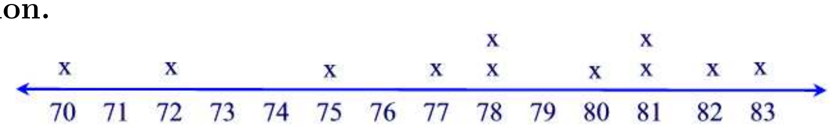
10	10	11	10	13	8	10	13	14	9
14	13	10	14	11	9	13	10	11	12
11	12	14	13	12	8	13	14	9	14

Solution.



Problem 30.2
The height (in inches) of the players on a professional basketball team are 70, 72, 75, 77, 78, 78, 80, 81,81,82, and 83. Make a line plot of the heights.

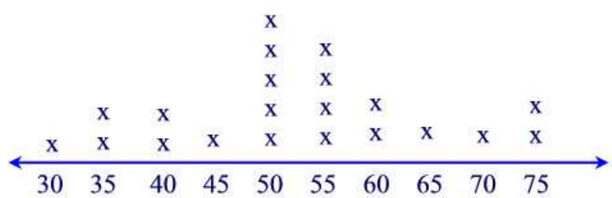
Solution.



Problem 30.3
Draw a line Plot for the following dataset.

50	35	70	55	50	30	40
65	50	75	60	45	35	75
60	55	55	50	40	55	50

Solution.



Problem 30.4
Given below the score of a 26 fourth graders.

64	82	85	99	96	81	97	80	81	80	84	87	98
75	86	88	82	78	81	86	80	50	84	88	83	82

Make a stem-and-leaf display of the scores.

Solution.

5	0
6	4
7	5 8
8	0 0 0 1 1 1 2 2 2 3 4 4 5 6 6 7 8 8
9	6 7 8 9

Problem 30.5
Each morning, a teacher quizzed his class with 20 geography questions. The class marked them together and everyone kept a record of their personal scores. As the year passed, each student tried to improve his or her quiz marks. Every day, Elliot recorded his quiz marks on a stem and leaf plot. This is what his marks looked like plotted out:

0	3 6 5
1	0 1 4 3 5 6 5 6 8 9 7 9
2	0 0 0 0

What is his most common score on the geography quizzes? What is his highest score? His lowest score? Are most of Elliot's scores in the 10s, 20s or under 10?

Solution.
20 is his most common score. His highest score is 20. His lowest score is 3. His most scores are in the 10s ■

Problem 30.6
A teacher asked 10 of her students how many books they had read in the last 12 months. Their answers were as follows:

$$12, 23, 19, 6, 10, 7, 15, 25, 21, 12$$

Prepare a stem and leaf plot for these data.

Solution.

$$\begin{array}{r|lllll} 0 & 6 & 7 \\ 1 & 0 & 2 & 2 & 5 & 9 \\ 2 & 1 & 3 & 5 \end{array}$$

Problem 30.7
Make a back-to-back stem and leaf plot for the following test scores:

Class 1:

100	96	93	92	92	92	90	90
89	89	85	82	79	75	74	73
73	73	70	69	68	68	65	61
35							

Class 2:

79	85	56	79	84	64	44	57
69	85	65	81	73	51	61	67
71	89	69	77	82	75	89	92
74	70	75	88	46			

Solution.

5	3	
	4	4 6
	5	1 6 7
9 8 8 5 1	6	1 4 5 7 9 9
9 5 4 3 3 3 0	7	0 1 3 4 5 5 7 9 9
9 9 5 2	8	1 2 4 5 5 8 9 9
6 3 2 2 2 0 0	9	2
0	10	

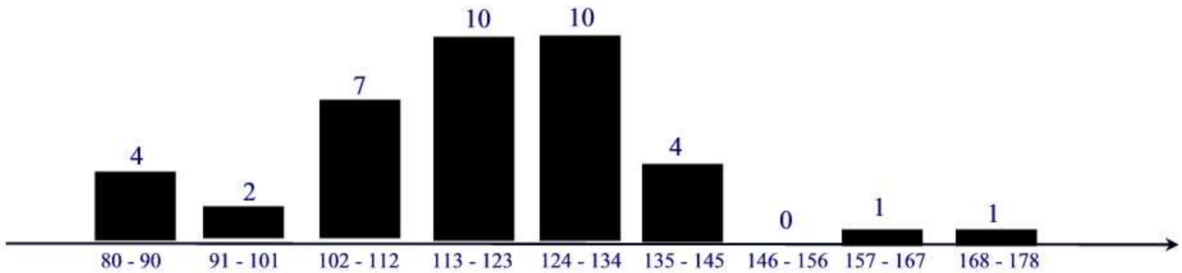
Problem 30.8
Suppose a sample of 38 female university students was asked their weights in pounds. This was actually done, with the following results:

130	108	135	120	97	110
130	112	123	117	170	124
120	133	87	130	160	128
110	135	115	127	102	130
89	135	87	135	115	110
105	130	115	100	125	120
120	120				

(a) Suppose we want 9 class intervals. Find CW.
(b) Construct a frequency distribution.
(c) Construct the corresponding histogram.

Solution.
(a) $CW = \frac{170-87}{9} \approx 10$
(b)

Class	Frequency
80 - 90	4
91 - 101	2
102 - 112	7
113 - 123	10
124 - 134	10
135 - 145	4
146 - 156	0
157 - 167	1
168 - 178	1



Problem 30.9
The table below shows the response times of calls for police service measured in minutes.

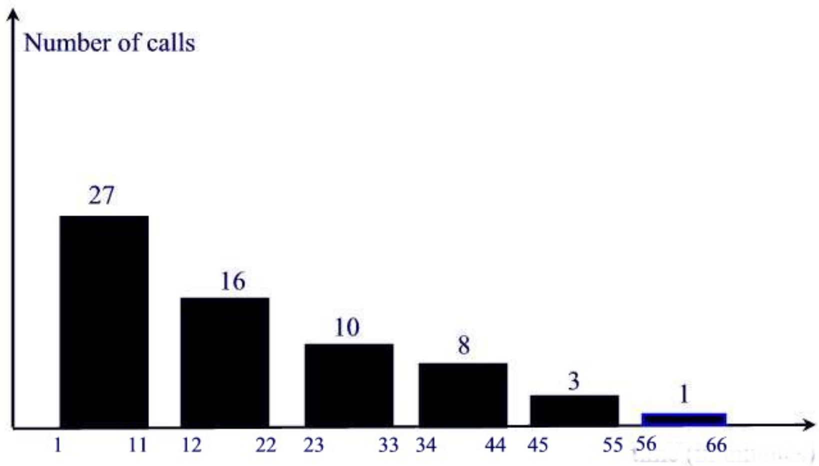
34	10	4	3	9	18	4
3	14	8	15	19	24	9
36	5	7	13	17	22	27
3	6	11	16	21	26	31
32	38	40	30	47	53	14
6	12	18	23	28	33	
3	4	62	24	35	54	
15	6	13	19	3	4	
4	20	5	4	5	5	
10	25	7	7	42	44	

Construct a frequency distribution and the corresponding histogram.

Solution.
We will use 6 class intervals so that $CW = \frac{62-3}{6} = 9.8 \approx 10$. Thus, a possible frequency distribution is the following.

Class	Frequency
1 - 11	27
12 - 22	16
23 - 33	10
34 - 44	8
45 - 55	3
56 - 66	1

The corresponding histogram is given by the figure below.



Problem 30.10
A nutritionist is interested in knowing the percent of calories from fat which Americans intake on a daily basis. To study this, the nutritionist randomly selects 25 Americans and evalustes the percent of calories from fat consumed in a typical day. The results of the study are as follows

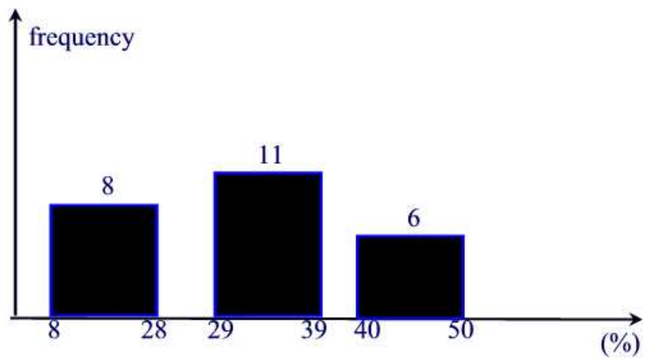
34%	18%	33%	25%	30%
42%	40%	33%	39%	40%
45%	35%	45%	25%	27%
23%	32%	33%	47%	23%
27%	32%	30%	28%	36%

Construct a frequency distribution and the corresponding histogram.

Solution.
Constructing 3 class intervals each of width $CW = \frac{47-18}{3} = 9.8 \approx 10$. The frequency distribution chart is

Class	Frequency
8 - 28	8
29 - 39	11
40 - 50	6

The corresponding histogram is given by the figure below.

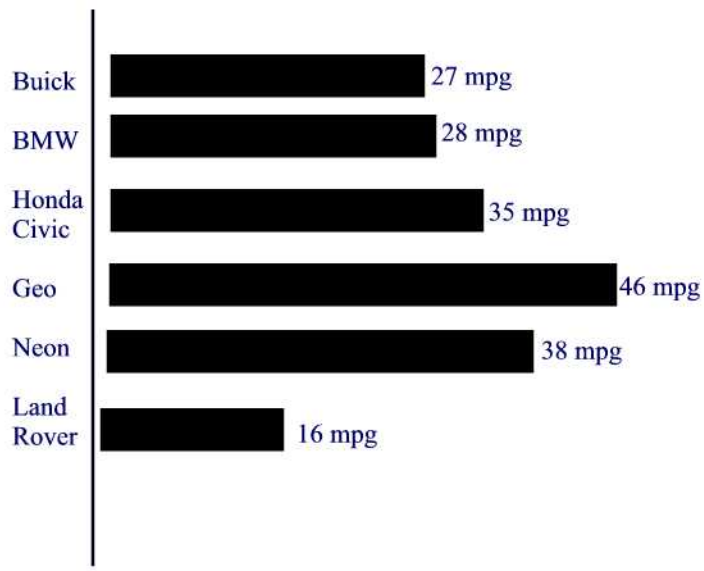


Problem 30.11
Given are several gasoline vehicles and their fuel consumption averages.

Buick	27 mpg
BMW	28 mpg
Honda Civic	35 mpg
Geo	46 mpg
Neon	38 mpg
Land Rover	16 mpg

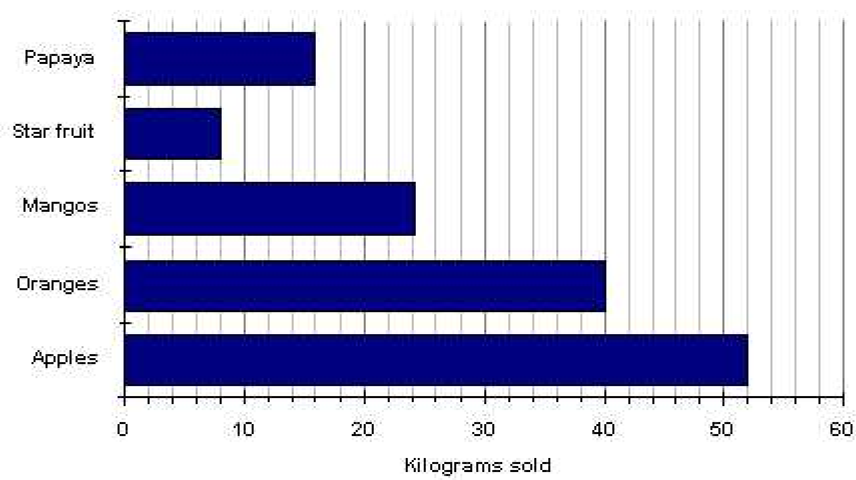
(a) Draw a bar graph to represent these data.
(b) Which model gets the least miles per gallon? the most?

Solution.
(a)The bar graph is given by the figure below.



(b) Land Rover gets the least mileage whereas Geo gets the most ■

Problem 30.12
The bar chart below shows the weight in kilograms of some fruit sold one day by a local market.



How many kg of apples were sold? How many kg of oranges were sold?

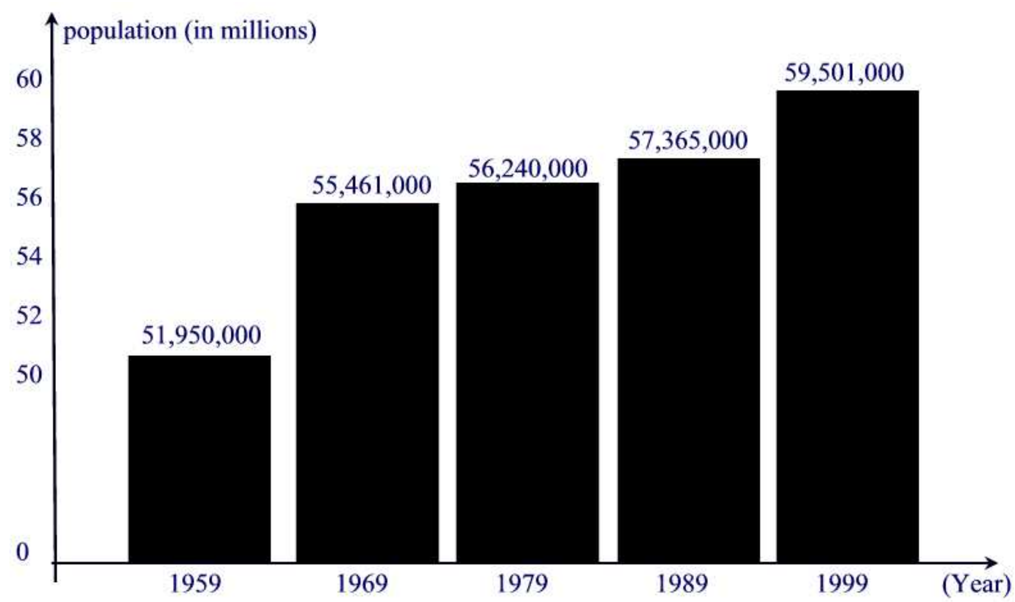
Solution.
52 kilograms of apples were sold and 40 kg of oranges were sold ■

Problem 30.13
The figures for total population at decade intervals since 1959 are given below:

Year	*Total UK Resident Population*
1959	51,956,000
1969	55,461,000
1979	56,240,000
1989	57,365,000
1999	59,501,000

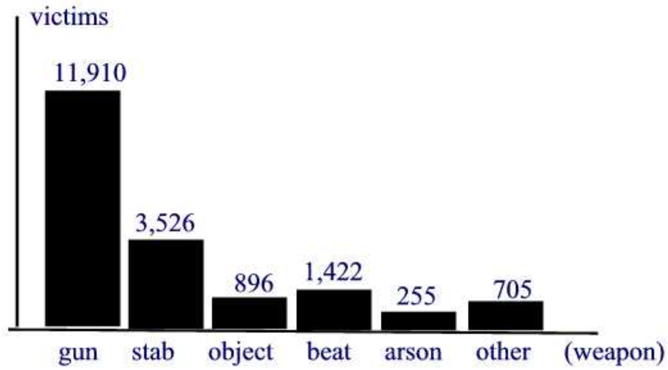
Construct a bar chart for this data.

Solution.



Problem 30.14
The following data gives the number of murder victims in the U.S in 1978 classified by the type of weapon used on them. Gun, 11,910; cutting/stabbing, 3,526; blunt object, 896; strangulation/beating, 1,422; arson, 255; all others 705. Construct a bar chart for this data. Use vertical bars.

Solution.



Problem 30.15
The table below shows the ingredients used to make a sausage and mushroom pizza.

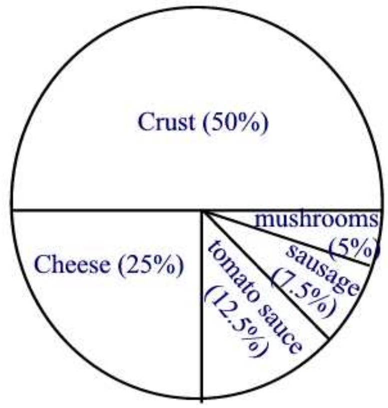
$Ingredient$	%
$Sausage$	7.5
$Cheese$	25
$Crust$	50
$TomatoSause$	12.5
$Mushroom$	5

Plot a pie chart for this data.

Solution.
First we find the measure of the central angle of each:

Sausage	$7.5 \times 3.6 = 27$
Cheese	$25 \times 3.6 = 90$
Crust	$50 \times 3.6 = 180$
Tomato Sauce	$12.5 \times 3.6 = 45$
Mushroom	$5 \times 3.6 = 18$

The pie chart is given below.


Crust (50%)
Cheese (25%)
tomato sauce (12.5%)
sausage (7.5%)
mushrooms (5%)

Problem 30.16
A newly qualified teacher was given the following information about the ethnic origins of the pupils in a class.

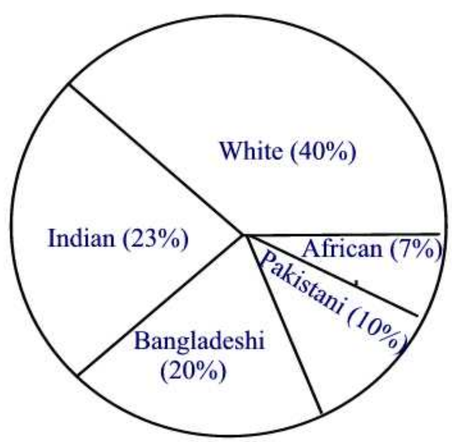
Ethnic origin	*No. of pupils*
White	12
Indian	7
BlackAfrican	2
Pakistani	3
Bangladeshi	6
TOTAL	30

Plot a pie chart representing the data.

Solution.
First we find the measure of the central angle of each:

White	$(12 \div 30) \times 360 = 144$
Indian	$(7 \div 30) \times 360 = 82.8$
Black African	$(2 \div 30) \times 360 = 25.2$
Pakistani	$(3 \div 30) \times 360 = 36$
Bangadeshi	$(6 \div 30) \times 360 = 72$

The pie chart is given below.



Problem 30.17
The following table represent a survey of people's favorite ice cream flavor

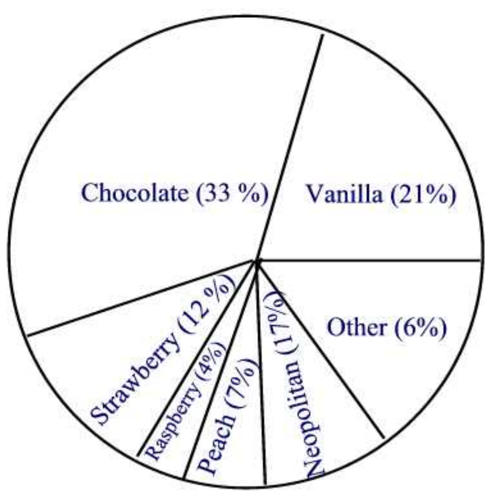
Flovor	*Number of people*
Vanilla	21.0%
Chocolate	33.0%
Strawberry	12.0%
Raspberry	4.0%
Peach	7.0%
Neopolitan	17.0%
Other	6.0%

Plot a pie chart to represent this data.

Solution.
First we find the measure of the central angle of each category:

Vanilla	$21.0 \times 3.6 = 75.6$
Chocolate	$33.0 \times 3.6 = 118.8$
Strawberry	$12.0 \times 3.6 = 43.2$
Raspberry	$4.0 \times 3.6 = 14.4$
Peach	$7.0 \times 3.6 = 25.2$
Neopolitan	$17.0 \times 3.6 = 61.2$
Other	$6.0 \times 3.6 = 21.6$

The pie chart is given below.

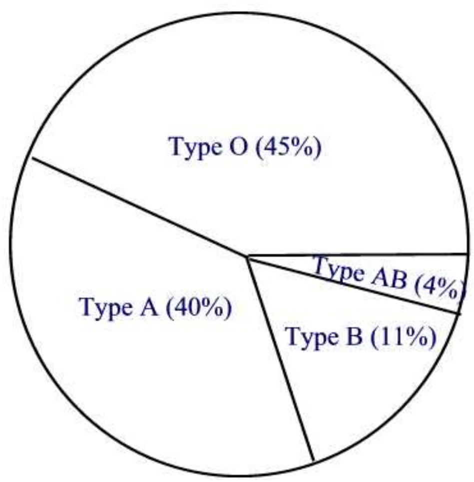


Problem 30.18
In the United States, approximately 45% of the population has blood type O; 40% type A; 11% type B; and 4% type AB. Illustrate this distribution of blood types with a pie chart.

Solution.
First we find the measure of the central angle of each category:

Blood type O	$45 \times 3.6 = 162$
Blood type A	$40 \times 3.6 = 144$
Blood type B	$11 \times 3.6 = 39.60$
Blood type AB	$4 \times 3.6 = 14.40$

The pie chart is given below.

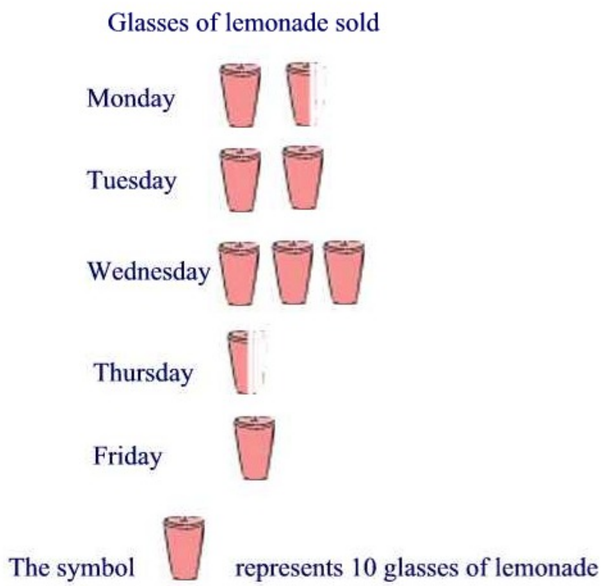


Problem 30.19

Make a pictograph to represent the data in the following table. Use  to represent 10 glasses of lemonade.

Day	Frequency
Monday	15
Tuesday	20
Wednesday	30
Thursday	5
Friday	10

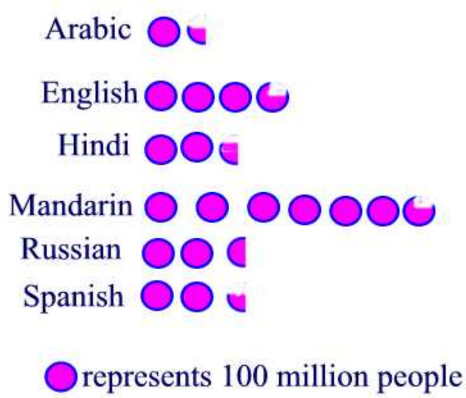
Solution.



Problem 30.20

The following pictograph shows the approximate number of people who speak the six common languages on earth.

(a) About how many people speak Spanish?
(b) About how many people speak English?
(c) About how many more people speak Mandarin than Arabic?



Solution.

(a) 225 million since there are 2 and a quarter symbols.
(b) 375 million people since there are $3\frac{3}{4}$ symbols.

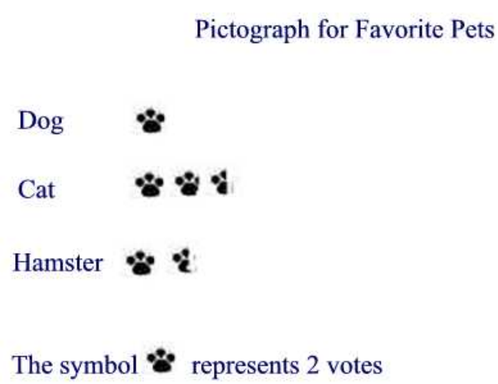
(c) 675 million people speak Mandarin and 125 million people speal Arabic. Thus, there are 550 million more people speaking Mandarin than Arabic■

Problem 30.21
Twenty people were surveyed about their favorite pets and the result is shown in the table below.

Pet	Frequency
Dog	2
Cat	5
Hamster	3

Make a pictograph for the following table of data. Let stand for 2 votes.

Solution.



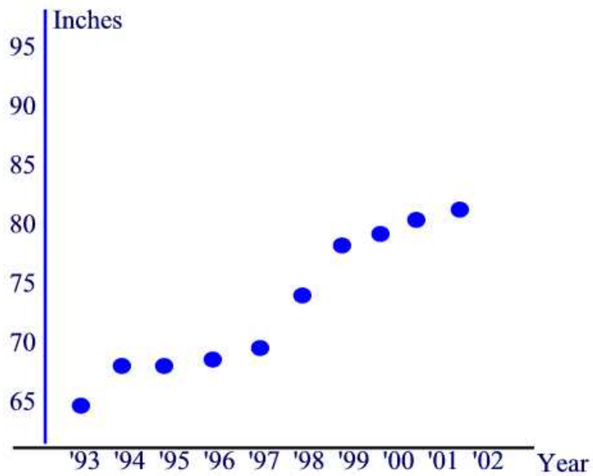
Problem 30.22
Coach Lewis kept track of the basketball team's jumping records for a 10-year period, as follows:

Year	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02
Record(nearest in)	65	67	67	68	70	74	77	78	80	81

(a) Draw a scatterplot for the data.
(b) What kind of corrolation is there for these data?

Solution.

(a)



(b) This scatterpplot represents a positive corrolation. This means that the jumping record increases with the years■

Problem 30.23

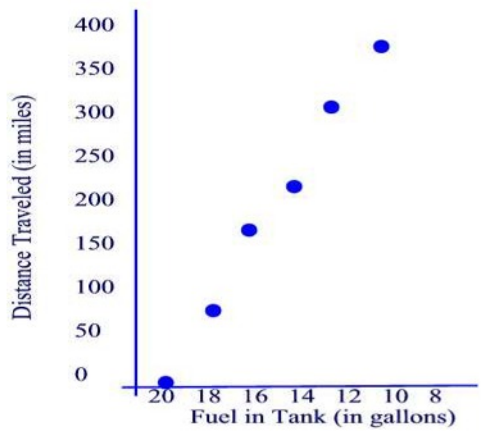
The gas tank of a National Motors Titan holds 20 gallons of gas. The following data are collected during a week.

Fuel in tank (gal.)	20	18	16	14	12	10
Dist. traveled (mi)	0	75	157	229	306	379

(a) Draw a scatterplot for the data.
(b) What kind of corrolation is there for these data?

Solution.

(a)

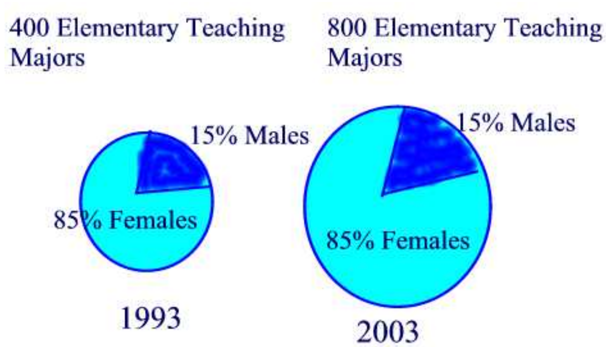


(b)The plot represents a positive corrolation. The more gas consumed the larger the distance traveled■

Problem 31.1
Jenny averaged 70 on her quizzes during the first part of the quarter and 80 on her quizzes during the second part of the quarter. When she found out that her final average for the quarter was not 75, she went to argue with her teacher. Give a possible explanation for Jenny's misunderstanding.

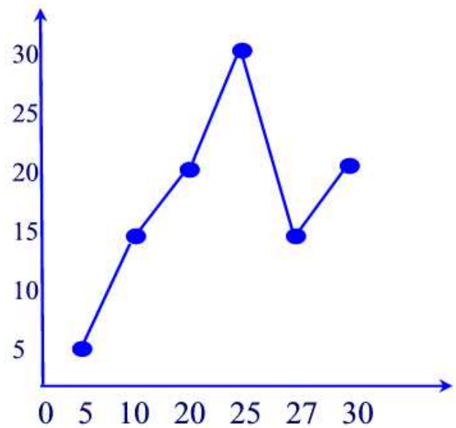
Solution.
She could have taken a different number of quizzes during the first part of the quarter than the second part■

Problem 31.2
Suppose the following circle graphs are used to illustrate the fact that the number of elementary teaching majors at teachers' colleges has doubled between 1993 and 2003, while the percent of male elementary teaching majors has stayed the same. What is misleading about the way the graphs are constructed?



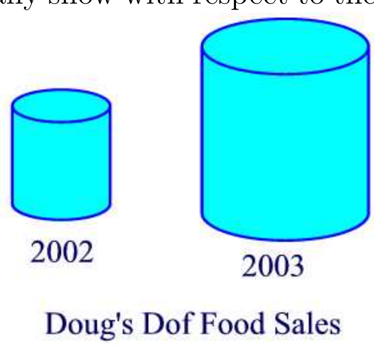
Solution.
When the radius of a circleiss doubled, the area is quadrupled, which is misleading since the population has only doubled. Recall that the area of a circle of radius r is $A = \pi r^2$ ■

Problem 31.3
What is wrong with the following line graph?



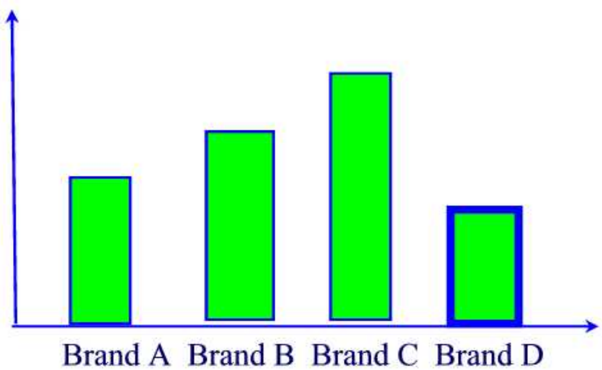
Solution.
The horizontal axis does not have uniformly sized intervals and both the horizontal axis and the graph are not labeled■

Problem 31.4
Doug's Dog Food Company wanted to impress the public with the magnitude of the company's growth. Sales of Doug's Dog Food had doubled from 2002 to 2003, so the company displayed the following graph, in which the radius of the base and the height of the 2003 can are double those of the 2002 can. What does the graph really show with respect to the growth of the company?



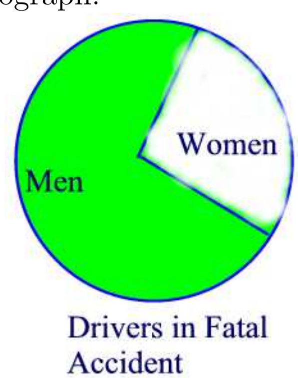
Solution.
The three-dimensional drawing distorts the graph. The result of doubling the radius and the height of the can is to increase the volume by a factor of 8. Recall that the volume of the can is $V = \pi r^2 h$ where r is the radius of the circle at the base and h is the height of the can■

Problem 31.5
What's wrong with the following graph?



Solution.
No label on the vertical axis, so we cannot compare actual sales. Also, there is no scale on the vertical axis■

Problem 31.6
Refer to the following pictograph:



Ms McNulty claims that on the basis of this information, we can conclude that men are worse drivers than women. Discuss whether you can reach that conclusion from the pictograph or you need more information. If more information is needed, what would you like to know?

Solution.
One would need more information; for example, is teh graph in percentages or actual numbers?$\blacksquare$

Problem 31.7
Larry and Marc took the same courses last quarter. Each bet that he would receive the better grades. Their courses and grades are as follows:

Course	Larry's Grades	Marc's Grades
Math(4 credits)	A	C
Chemistry(4 credits)	A	C
English(3 credits)	B	B
Psychology(3 credits)	C	A
Tennis(1 credit)	C	A

Marc claimed that the results constituted a tie, since both received 2 A's, 1 B, and 2 C's. Larry said that he won the bet because he had the higher GPA for the quarter. Who is correct?(Allow 4 points for A, 3 points for B, 2 points for C, 1 point for D, and 0 point for F.)

Solution.
Let's find the GPA of each.
Larry's GPA: $\frac{4\times4+4\times4+3\times3+3\times2+1\times2}{15} \approx 3.27$
Marc's GPA: $\frac{4\times2+4\times2+3\times3+3\times4+4\times1}{15} \approx 2.73$
Hence, Larry is right and Marc's is wrong$\blacksquare$

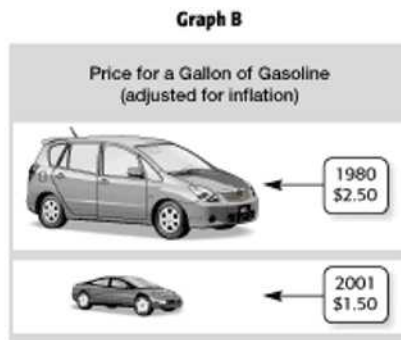
Problem 31.8
Oil prices went up 20% one year and 30% the next. Is it true that over the two years, prices went up 50%?

Solution.
Let x be the oil price two years ago. Then last year the price was $1.2x$ dollars and this year the new price is $1.2x + (0.3)(1.2x) = 1.56x$ so that the price went up 56% over the past two years$\blacksquare$

Problem 31.9
True or false? My rent went down 10% last year and then rose 20% this year. Over the two years my rent went up by 10%.

Solution.
Suppose that the rent was x dollars two years ago. Then the rent last year was $x - 0.1x = 0.9x$ dollars. The rent this year is $0.9x + 0.2(0.9x) = 1.08x$. Hence, the rent went up 8% over the two years■

Problem 31.10
Which graph could be used to indicate a greater decrease in the price of gasoline? Explain.



Solution.
Graph A. Graph B is misleading since in one year a van was used as a symbol and in the next a small compact car was used. The same symbol should be used in a good pictograph■

Problem 32.1
Find (a) the mean, (b) median, and (c) the mode for the following collection of data:

$$60 \quad 60 \quad 70 \quad 95 \quad 95 \quad 100$$

Solution.
(a) $\overline{x} = \frac{60+60+70+95+95+100}{6} = 80$
(b) The median rank is $MR = \frac{6+1}{2} = 3.5$ so that the median is $\frac{70+95}{2} = 82.5$
(c) The mode is 60 and 95■

Problem 32.2
Suppose a company employs 20 people. The president of the company earns \$200,000, the vice president earns \$75,000, and 18 employees earn \$10,000 each. Is the mean the best number to choose to represent the "average" salary of the company?

Solution.
The mean salary for this company is

$$\frac{\$200,000 + \$75,000 + 18(\$10,000)}{20} = \$22,750.$$

In this case, the mean salary of \$ 22,750 is not representative. Either the median or mode, both of which are \$ 10,000, would describe the typical salary better■

Problem 32.3
Suppose nine students make the following scores on a test:

$$30, 35, 40, 40, 92, 92, 93, 98, 99$$

Is the median the best "average" to represent the set of scores?

Solution.
The median rank is $MR = \frac{9+1}{2} = 5$ so that the median score is 92. From that score one might infer that individuals all scored very well, yet 92 is certainly not a typical score. In this case, the mean of approximately 69 might be more appropriate than the mode ■

Problem 32.4
Is the mode an appropriate "average" for the following test scores?

$$40, 42, 50, 62, 63, 65, 98, 98$$

Solution.
The mode is 98 because this score occurs most frequently. The score of 98 is not representative of the set of data because of the large spread of scores.■

Problem 32.5
The 20 meetings of a square dance club were attended by 26, 25, 28, 23, 25, 24, 24, 23, 26, 26, 28, 26, 24, 32, 25, 27, 24, 23, 24, and 22 of its members. Find the mode, median, and mean.

Solution.
The mode is 24. The sum of the given 20 numbers is 483 so that the mean is 24.15. To find the median we order the numbers for smallest to largest to obtain

22 23 23 23 24 24 24 24 24 25 25 25 26 26 26 26 27 28 28 32

The median rank is $MR = \frac{20+1}{2} = 10.5$ so that the median is 25 ■

Problem 32.6
If the mean annual salary paid to the top of three executives of a firm is \$96,000, can one of them receive an annual salary of \$300,000?

Solution.
If we denote the salaries of the three executives by x_1, x_2, and x_3 then $x_1 + x_2 + x_3 = 3(\$96,000) = \$288,000$. Thus, none can have a salary of \$ 300,000 ■

Problem 32.7
An instructor counts the final examination in a course four times as much as of the four one-hour examinations. What is the average grade of a student who received grades of $74, 80, 61,$ and 77 in the four one- hour examinations and 83 in the final examination?

Solution.
The average mean is

$$\frac{74 + 80 + 61 + 77 + 4(83)}{8} = 78 \blacksquare$$

Problem 32.8
In 1980 a college paid its 52 instructors a mean salary of \$13,200, its 96 assistant professors a mean salary of \$15,800, its 67 associate professors a mean salary of \$18,900, and its 35 full professors a mean salary of \$23,500. What was the mean salary paid to all the teaching staff of this college?

Solution.
The mean salary is

$$\frac{52(\$13,200) + 96(\$15,800) + 67(\$18,900) + 35(\$23,500)}{52 + 96 + 67 + 35} = \$17,168 \blacksquare$$

Problem 32.9
The following table gives the average costs of a single-lens reflex camera:

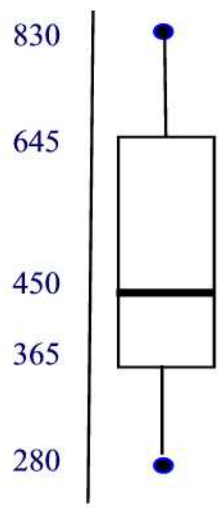
800	650	300	430	560	470	640	830
400	280	800	410	360	600	310	370

(a) Rank the data from smallest to largest.
(b) Find the quartiles Q_1, Q_2, and Q_3.
(c) Make a box-and-whisker plot.

Solution.
(a)

280	300	310	360	370	400	410	430
470	560	600	640	650	800	800	830

(b) The median rank of the whole set of data is $MR = \frac{16+1}{2} = 8.5$. So the second quartile is the average of the 8th and 9th numbers of the list, i.e., $Q_2 = \frac{430+470}{2} = 450$. This number split the set of data into two sets each with 8 numbers and with median rank $MR = \frac{8+1}{2} = 4.5$. Thus, $Q_1 = \frac{360+370}{2} = 365$ and $Q_3 = \frac{640+650}{2} = 645$
(c)



Problem 32.10
Mr. Eyha took a general aptitude test and scored in the 82nd percentile for aptitude in accounting. What percentage of the scores were at or below his score? What percentage were above?

Solution.
The 82nd percentile means that 82% of the scores were at or below his score and 18% were above his score■

Problem 32.11
At Center Hospital there is a concern about the high turnover of nurses. A survey was done to determine how long (in months) nurses had been in their current positions. The responses of 20 nurses were:

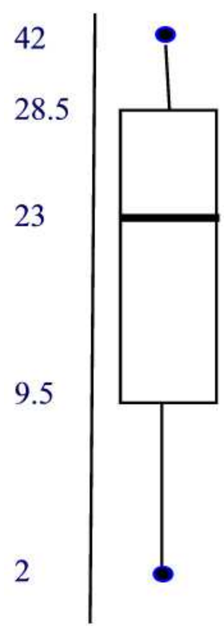
23	2	5	14	25	36	27	42	12	8
7	23	29	26	28	11	20	31	8	36

(a) Rank the data.
(b) Make a box-and-whisker plot of the data.
(c) What are your conclusions from the plot?

Solution.
(a)

2	5	7	8	8	11	12	14	20	23
23	25	26	27	28	29	31	36	36	42

(b) The three quartiles are $Q_1 = 9.5, Q_2 = 23$, and $Q_3 = 28.5$. The box-and-whisker plot is given next.



(c) Note that the median of the given set of data, i.e., $Q_2 = 23$, is above the center of the box. This shows that there are more nurses who have been in the job for more than 23 months compared to those who have been in the job for less than 23 months■

Problem 32.12
The following are the wind velocities reported at 6 P.M.on six consecutive days: 13,8,15,11,3 and 10. Find the range, sample mean, sample variance, and sample standard deviation.

Solution.
The range is $15-3=12$. The mean is $\frac{13+8+15+11+3+10}{6} = \frac{60}{6} = 10$. The sample variance is

$$s^2 = \frac{(13-10)^2 + (8-10)^2 + (15-10)^2 + (11-10)^2 + (10-10)^2}{5} = 17.6$$

and the sample standard deviation is $s = \sqrt{17.6} \approx 4.2$ ■

Problem 32.13
An airline's records show that the flights between two cities arrive on the average 4.6 minutes late with a standard deviation of 1.4 minutes. At least what percentage of its flights between these two cities arrive anywhere between 1.8 minutes late and 7.4 minutes late?

Solution.
1.8 and 7.4 are within two standard deviations of the mean so according to the emperical rule, 95% of the flights arrive anywhere between 1.8 minutes late and 7.4 minutes late ■

Problem 32.14
One patient's blood pressure, measured daily over several weeks, averaged 182 with a standard deviation of 5.3, while that of another patient averaged 124 with a standard deviation of 9.4. Which patient's blood pressure is relatively more variable?

Solution.
Since $\frac{5.3}{182} = 2.9\%$ and $\frac{9.4}{124} = 7.6\%$ then patient's 2 blood pressure is more variable■

Problem 32.15
By sampling different landscapes in a national park over a 2-year period, the number of deer per square kilometer was determined. The results were (deer per square kilometer)

30	20	5	29	58	7
20	18	4	29	22	9

Compute the range, sample mean, sample variance, and sample standard deviation.

Solution.
The range is $58 - 4 = 54$. The mean is the sum of the given numbers divided by 12, i.e. $\frac{251}{12} \approx 20.92$. Using a calculator one finds the variance to be about 225 and the standard deviation to be about 15 ■

Problem 32.16
A researcher wants to find the number of pets per household. The researcher conducts a survey of 35 households. Find the sample variance and standard deviation.

0	2	3	1	0
1	2	3	1	0
1	2	1	1	0
3	2	1	1	1
4	1	2	2	4
3	2	1	2	3
2	3	4	0	2

Solution.
The mean is $\frac{61}{35} \approx 1.74$. The variance is 1.38 and standard deviation is 1.17 ■

Problem 32.17
Suppose two machines produce nails which are on average 10 inches long. A sample of 11 nails is selected from each machine.

Machine A: 6, 8, 8, 10, 10, 10, 10, 10, 12, 12, 14.
Machine B: 6, 6, 6, 8, 8, 10, 12, 12, 14, 14, 14.

Which machine is better than the other?

Solution.
The standard deviation of Machine A is about 2.19 whereas that of Machine B is 3.35. Hence, Machine A is better than Machine B since more nails are close to the mean than the case of Machine B■

Problem 32.18
Find the missing age in the following set of four student ages.

Student	Age	Deviation from the Mean
A	19	-4
B	20	-3
C	?	1
D	29	6

Solution.
Note that each time the age increases by 1 the corresponding deviation increases by 1. Thus, the missing age 24 since the change in deviation is $1-(-4)=5$ so that $19+5=24$■

Problem 32.19
The maximum heart rates achieved while performing a particular aerobic exercise routine are measured (in beats per minute) for 9 randomly selected individuals.

145 155 130 185 170 165 150 160 125

(a) Calculate the range of the time until failure.
(b) Calculate the sample variance of the time until failure.
(c) Calculate the sample standard variation of the time until failure.

Solution.
(a) $185-125=60$
(b) The sample variance is about 361.11
(c) The sample standard variation is $\sqrt{361.11}\approx 19$ ■

Problem 32.20
The following data gives the number of home runs that Babe Ruth hit in each of his 15 years with the New York Yankees baseball team from 1920 to 1934:

$$54 \quad 59 \quad 35 \quad 41 \quad 46 \quad 25 \quad 47 \quad 60 \quad 54 \quad 46 \quad 49 \quad 46 \quad 41 \quad 34 \quad 22.$$

The following are the number of home runs that Roger Maris hit in each of the ten years he played in the major leagues from 1957 on:

$$8 \quad 13 \quad 14 \quad 16 \quad 23 \quad 26 \quad 28 \quad 33 \quad 39 \quad 61$$

Calculate the mean and standard deviation for each player's data and comment on the consistency of performance of each player.

Solution.
The mean of Babe Ruth is $\frac{659}{15} \approx 43.93$ and that of Roger Maris is $\frac{261}{10} = 26.1$ Ruth's standard deviation is 11.25 and that of Maris is 15.61. Since $\frac{11.25}{43.93} \approx$ 25.6% and $\frac{15.61}{26.1} = 59.8\%$ than Ruth is more consisten than Maris■

Problem 32.21
An office of Price Waterhouse Coopers LLP hired five accounting trainees this year. Their monthly starting salaries were: \$2536; \$2173; \$2448; \$2121; and \$2622.

(a) Compute the population mean.
(b) Compute the population variance.
(c) Compute the population standard deviation.

Solution.
(a) $\frac{2536+2173+2448+2121+2622}{5} = \$2380$
(b) \$ 49,363.50
(c) \$ 222.18■

Problem 32.22
On a final examination in Statistics, the mean was 72 and the standard deviation was 15. Assuming normal distribution, determine the z-score of students receiving the grades (a) 60, (b) 93, and (c) 72.

Solution.
(a) $z = \frac{60-72}{15} = -0.8$
(b) $z = \frac{93-72}{15} = 1.4$
(c) $z = \frac{72-72}{15} = 0$ ■

Problem 32.23
Referring to the previous exercise, find the grades corresponding to the z-score $z = 1.6$.

Solution.

$$x = 72 + 15(1.6) = 96 \blacksquare$$

Problem 32.24
If $z_1 = 0.8$, $z_2 = -0.4$ and the corresponding x-values are $x_1 = 88$ and $x_2 = 64$ then find the mean and the standard deviation, assuming we have a normal distribution.

Solution.
We have $88 = \mu + 0.8\sigma$ and $64 = \mu - 0.4\sigma$. Subtracting the second equation from the first we find $1.2\sigma = 24$ so that $\sigma = \frac{24}{1.2} = 20$. Replacing this value in the first equation and solving for μ we find $\mu = 72 \blacksquare$

Problem 32.25
A student has computed that it takes an average (mean) of 17 minutes with a standard deviation of 3 minutes to drive from home, park the car, and walk to an early morning class. Assuming normal distribution,

(a) One day it took the student 21 minutes to get to class. How many standard deviations from the average is that?
(b) Another day it took only 12 minutes for the student to get to class. What is this measurement in standard units?
(c) Another day it took him 17 minutes to be in class. What is the z-score?

Solution.
(a) since $21 - 17 = 4$ then 21 is $1\frac{1}{3}$ standard deviation from the mean.
(b) Since $17 - 12 = 5$ then 12 minutes is $1\frac{2}{3}$ standard deviation from the mean.
(c) $z = \frac{17-17}{3} = 0\blacksquare$

Problem 32.26
Mr. Eyha's z-score on a college exam is 1.3. If the x-scores have a mean of 480 and a standard deviation of 70 points, what is his x-score?

Solution.

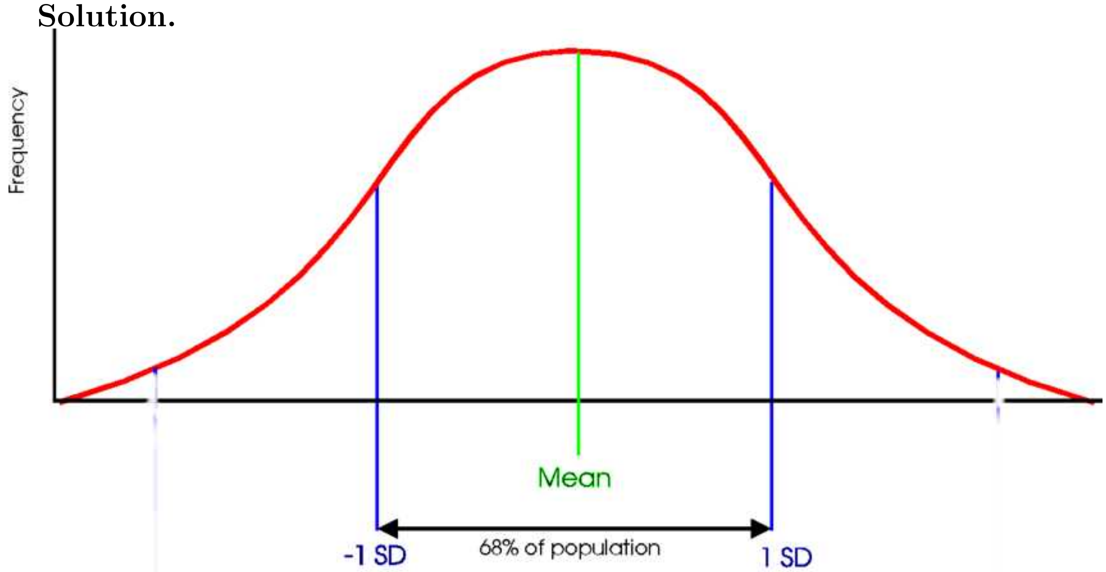
$$x = 480 + 1.3(70) = 571 \blacksquare$$

Problem 32.27
(a) If $\mu = 80, \sigma = 10$, what is the z-score for a person with a score of 92?
(b) If $\mu = 65, \sigma = 12$, what is the raw score for a z-score of -1.5?

Solution.
(a) $z = \frac{92-80}{10} = 1.2$
(b) $x = 65 - 1.5(12) = 47 \blacksquare$

Problem 32.28
Sketch a normal curve. Mark the axis corresponding to the parameter μ and the axis corresponding to $\mu + \sigma$ and $\mu - \sigma$.

Solution.



Problem 32.29
For the population of Canadian high school students, suppose that the number of hours of TV watched per week is normally distributed with a mean of 20 hours and a standard deviation of 4 hours. Approximately, what percentage of high school students watch

(a) between 16 and 24 hours per week?
(b) between 12 and 28 hours per week?
(c) between 8 and 32 hours per week?

Solution.
(a) Within one standard deviation: 68%
(b) Within two standard deviations: 95%
(c) Withoin three standard deviations: 99.7% ■

Problem 32.30
The length of human pregnancies from conception to birth varies according to a distribution that is approximately normal with mean 266 days and standard deviation 16 days. Use the emperical rule to answer the following questions.

(a) Between what values do the lengths of the middle 95% of all pregnancies fall?
(b) How short are the shortest 2.5% of all pregnancies?

Solution.
(a) Between $266 - 2(16) = 234$ days and $266 + 2(16) = 298$ days.
(b) less than or equal to 234 days■

Problem 33.1
An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.

Solution.

$$S = \{HH, HT, TH, TT\}\blacksquare$$

Problem 33.2
Three coins are thrown. List the outcomes which belong to each of the following events.

(a) exactly two tails (b) at least two tails (c) at most two tails.

Solution.
(a) $\{TTH, HTT, THT\}$
(b) $\{TTT, TTH, HTT, THT\}$
(c) $\{TTH, HTT, THT, THH, HTH, HHT, HHH\}$ ■

Problem 33.3
For each of the following events A, B, C, list and count the number of outcomes it contains and hence calculate the probability of A, B or C occurring.

(a) A = "throwing 3 or higher with one die",
(b) B = "throwing exactly two heads with three coins",
(c) C = "throwing a total score of 14 with two dice".

Solution.
(a) $A = \{3, 4, 5, 6\}$. $P(A) = \frac{4}{6} = \frac{2}{3}$
(b) $B = \{TTH, HTT, THT\}$. $P(B) = \frac{3}{8}$
(c) $C = \emptyset$. $P(C) = 0$ ■

Problem 33.4
An experiment consists of throwing two four-faced dice.

(a) Write down the sample space of this experiment.
(b) If E is the event 'total score is at least 4' list the outcomes belonging to E^c.
(c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?
(d) What is the probability that a double: (i.e. $\{(1,1),(2,2),(3,3),(4,4)\}$) will not be thrown?
(e) What is the probability that a double is not thrown nor is the score greater than 6?

Solution.
(a) $S = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),$
$(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$
(b) $E^c = \{(1,3),(1,4),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),$
$(4,2),(4,3),(4,4)\}$
(c) $P(\{(2,4),(3,3),(3,4),(4,2),(4,3),(4,4)\}) = \frac{6}{16} = \frac{3}{8}$. The probability that the total score is less than 6 is $1 - \frac{3}{8} = \frac{5}{8}$
(d) $\frac{12}{16} = \frac{3}{4}$
(e) $P(\{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}) = \frac{8}{16} = \frac{1}{2}$ ■

Problem 33.5
A lot consists of 10 good articles, 4 with minor defects and 2 with major

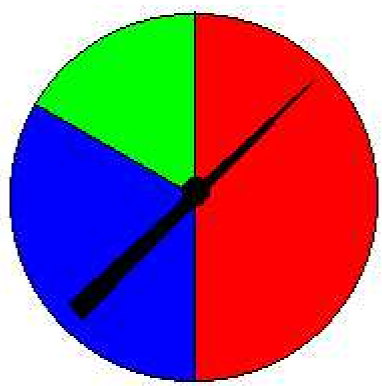
defects. One article is chosen at random. Find the probability that:

(a) it has no defects,
(b) it has no major defects,
(c) it is either good or has major defects.

Solution.
(a) $\frac{4}{10} = 40\%$
(b) $\frac{8}{10} = \frac{4}{5}$
(c) $\frac{6}{10} = \frac{3}{5}$ ■

Problem 33.6
Consider the experiment of spinning the pointer on the game spinner pictured below. There are three possible outcomes, that is, when the pointer stops it must point to one of the three colors. (We rule out the possibility of landing on the border between two colors.)

(a) What is the probability that the spinner is pointing to the red area?
(b) What is the probability that the spinner is pointing to the blue area?
(c) What is the probability that the spinner is pointing to the green area?



Solution.
(a) $\frac{1}{2}$
(b) $\frac{2}{3}$
(c) $\frac{1}{6}$ ■

Problem 33.7
Consider the experiment of flipping a coin three times. If we denote a

head by H and a tail by T, we can list the 8 possible ordered outcomes as $(H, H, H), (H, H, T), \cdots$ each of which occurs with probability of 1/8. Finish listing the remaining members of the sample space. Calculate the probability of the following events:

(a) All three flips are heads.
(b) Exactly two flips are heads.
(c) The first flip is tail.
(d) At least one flip is head.

Solution.
(a) $\frac{1}{8}$
(b) $\frac{3}{8}$
(c) $\frac{4}{8} = \frac{1}{2}$
(d) $\frac{7}{8}$ ■

Problem 33.8
Suppose an experimet consists of drawing one slip of paper from a jar containing 12 slips of paper, each with a different month of the year written on it. Find each of the following:

(a) The sample space S of the experiment.
(b) The event A consisting of the outcomes having a month begining with J.
(c) The event B consisting of outcomes having the name of a month that has exactly four letters.
(d) The event C consisting of outcomes having a month that begins with M or N.

Solution.
(a) The sample space consists of the 12 months of the year.
(b) $A = \{January, June, July\}$
(c) $B = \{June, July\}$
(d) $C = \{March, May, November\}$ ■

Problem 33.9
Let $S = \{1, 2, 3, \cdots, 25\}$. If a number is chosen at **random**, that is, with the same chance of being drawn as all other numbers in the set, calculate each of the following probabilities:

(a) The even A that an even number is drawn.
(b) The event B that a number less than 10 and greater than 20 is drawn.
(c) The event C that a number less than 26 is drawn.
(d) The event D that a prime number is drawn.
(e) The event E that a number both even and prime is drawn.

Solution.
(a) $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$ so that $P(A) = \frac{12}{25}$
(b) $B = \emptyset$ so that $P(B) = 0$
(c) $C = S$ so that $P(C) = 1$
(d) $D = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ so that $P(D) = \frac{9}{25}$
(e) $E = \{2\}$ so that $P(E) = \frac{1}{25}$ ■

Problem 33.10
Consider the experiment of drawing a single card from a standard deck of cards and determine which of the following are sample spaces with equally likely outcomes:
(a) {face card, not face card}
(b) {club, diamond, heart, spade}
(c) {black, red}
(d) {king, queen, jack, ace, even card, odd card}

Solution.
(a) Sincce $P(\{face\ card\}) = \frac{12}{52} = \frac{3}{13}$ and $P(\{not\ a\ face\ card\}) = \frac{40}{52} = \frac{10}{13}$ then the outcomes are not equally likely.
(b) Since $P(club) = P(diamond) = P(heart) = P(spade) = \frac{1}{4}$ then the outcomes are equally likely.
(c) Since $P(black) = P(red) = \frac{1}{2}$ then the outcomes are equally likely.
(d) Since $P(King) = \frac{4}{52} = \frac{1}{13}$ and $P(even\ card) = \frac{20}{52} = \frac{5}{13}$ then the outcomes are not equally likely ■

Problem 33.11
An experiment consists of selecting the last digit of a telephone number. Assume that each of the 10 digits is equally likely to appear as a last digit. List each of the following:
(a) The sample space
(b) The event consisting of outcomes that the digit is less than 5

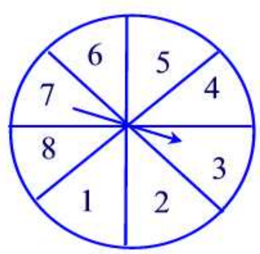
(c) The event consisting of outcomes that the digit is odd
(d) The event consisting of outcomes that the digit is not 2
(e) Find the probability of each of the events in (b) - (d)

Solution.
(a) $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
(b) $A = \{0, 1, 2, 3, 4\}$
(c) $B = \{1, 3, 5, 7, 9\}$
(d) $C = \{0, 1, 3, 4, 5, 6, 7, 8, 9\}$
(e) $P(A) = \frac{5}{10} = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{9}{10}$ ■

Problem 33.12
Each letter of the alphabet is written on a seperate piece of paper and placed in a box and then one piece is drawn at random.
(a) What is the probability that the selected piece of paper has a vowel written on it?
(b) What is the probability that it has a consonant written on it?

Solution.
(a) Since there are five vowels then the probability is $\frac{5}{26}$
(b) $\frac{21}{26}$■

Problem 33.13
The following spinner is spun:



Find the probabilities of obtaining each of the following:
(a) P(factor of 35)
(b) P(multiple of 3)
(c) P(even number)
(d) P(11)
(e) P(composite number)
(f) P(neither prime nor composite)

Solution.
(a) The factors of 35 in the figure are 1, 5, and 7 so that the probability is $\frac{3}{8}$
(b) The multiples of 3 in the figure are 3 and 6 so that the probability is $\frac{2}{8} = \frac{1}{4}$
(c) The outcomes of this event are 2,4,6,8 so that the probability is $\frac{4}{8} = \frac{1}{2}$
(d) $P(11) = 0$
(e) The composite numbers, i.e, not prime numbers, are 4,6,8 so that the probability is $\frac{3}{8}$
(f) The only number that is neither prime nor composite is 1 so that the probability is $\frac{1}{8}$ ■

Problem 33.14
An experiment consists of tossing four coins. List each of the following.

(a) The sample space
(b) The event of a head on the first coin
(c) The event of three heads

Solution.
(a) $S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$
(b) $E = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT\}$
(c) $F = \{HHHH, HHHT, HHTH, HTHH, THHH\}$ ■

Problem 33.15
Identify which of the following events are certain, impossible, or possible.
(a) You throw a 2 on a die
(b) A student in this class is less than 2 years old
(c) Next week has only 5 days

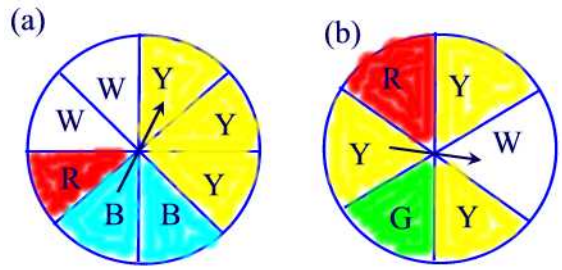
Solution.
(a) Possible
(b) impossible
(c) impossible ■

Problem 33.16
Two dice are thrown. If each face is equally likely to turn up, find the following probabilities.

(a) The sum is even
(b) The sum is not 10
(c) The sum is a prime
(d) The sum is less than 9
(e) The sum is not less than 9

Solution.
(a) $E = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}$ so $P(E) = \frac{18}{36} = \frac{1}{2}$
(b) If F is event that the sum is 10 then $F = \{(4,6),(6,4),(5,5)\}$ so that the probability that the sum is not 10 is $1 - \frac{3}{36} = 1 - \frac{1}{12} = \frac{11}{12}$
(c) $G = \{(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5)\}$ so that $P(G) = \frac{15}{36} = \frac{5}{12}$
(d) $H = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(6,1),(6,2)\}$
so that $P(H) = \frac{26}{36} = \frac{13}{18}$
(e) $1 - \frac{13}{18} = \frac{5}{18}$ ■

Problem 33.17
What is the probability of getting yellow on each of the following spinners?



Solution.
(a) $\frac{3}{8}$ (b) $\frac{3}{6} = \frac{1}{2}$ ■

Problem 33.18
A department store's records show that 782 of 920 women who entered the store on a Saturday afternoon made at least one purchase. Estimate the probability that a woman who enters the store on a Saturday afternoon will make at least one purchase.

Solution.
The estimated probability is $\frac{782}{920} = 85\%$ ■

Problem 33.19
Suppose that a set of 10 rock samples includes 3 that contain gold nuggets. If you were to pick up a sample at random, what is the probability that it includes a gold nugget?

Solution.
The probability is $\frac{3}{10} = 30\%$ ■

Problem 33.20
When do creative people get their best ideas? A magazine did a survey of 414 inventors (who hold U.S. patents) and obtained the following information:

Time of Day When Best Ideas Occur

Time	Number of Inventors
6 A.M. - 12 noon	46
12 noon - 6 P.M.	188
6 P.M. - 12 midnight	63
12 midnight - 6 A.M.	117

Assuming that the time interval includes the left limit and all the times up to but not including the right limit, estimate the probability (to two decimal places) that an inventor has a best idea during the time interval 6 A.M. - 12 noon.

Solution.
The estimated probability is $\frac{46}{414} \approx 11.1\%$ ■

Problem 33.21
Which of the following are mutually exclusive? Explain your answers.

(a) A driver getting a ticket for speeding and a ticket for going through a red light.
(b) Being foreign-born and being President of the United States.

Solution.
(a) Not mutually exclusive since a driver can get a ticket for speeding and going through a red light.
(b) Mutually exclusive since the President of the Unites States has to be born in the US■

Problem 33.22
If A and B are the events that a consumer testing service will rate a given stereo system very good or good, $P(A) = 0.22$, $P(B) = 0.35$. Find
(a) $P(A^c)$;
(b) $P(A \cup B)$;
(c) $P(A \cap B)$.

Solution.
(a) $P(A^c) = 1 - 0.22 = 0.78$
(b) $P(A \cup B) = P(A) + P(B) = 0.22 + 0.35 = 0.57$
(c) $P(A \cap B) = 0$ ■

Problem 33.23
If the probabilities are 0.20, 0.15, and 0.03 that a student will get a failing grade in Statistics, in English, or in both, what is the probability that the student will get a failing grade in at least one of these subjects?

Solution.

$$0.20 + 0.15 - 0.03 = 0.32 \blacksquare$$

Problem 33.24
If A is the event "drawing an ace" from a deck of cards and B is the event "drawing a spade". Are A and B mutually exclusive? Find $P(A \cup B)$.

Solution.
No since there is a card that is spade and ace. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$ ■

Problem 33.25
A bag contains 18 coloured marbles: 4 are coloured red, 8 are coloured yellow and 6 are coloured green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

Solution.

$$\frac{4}{18} + \frac{6}{18} = \frac{10}{18} = \frac{5}{9} \blacksquare$$

Problem 33.26
Show that for any events A and B, $P(A \cap B) \geq P(A) + P(B) - 1$.

Solution.
Since $P(A \cap B) \leq 1$ then $-P(A \cap B) \geq -1$. Add $P(A) + P(B)$ to both sides to obtain $P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1$. But the left hand side is just $P(A \cup B)$. ■

Problem 33.27
A golf bag contains 2 red tees, 4 blue tees, and 5 white tees.

(a) What is the probability of the event R that a tee drawn at random is red?
(b) What is the probability of the event "not R" that is, that a tee drawn at random is not red?
(c) What is the probability of the event that a tee drawn at random is either red or blue?

Solution.
(a) Because the bag contains a total of 11 tees and 2 tees are red then $P(R) = \frac{2}{11}$
(b) $P(not\ red) = \frac{9}{11}$
(c) Since the two events are mutually exclusive then $P(R \cup B) = P(R) + P(B) = \frac{2}{11} + \frac{4}{11} = \frac{6}{11}$ ■

Problem 33.28
A fair pair of dice is rolled. Let E be the event of rolling a sum that is an even number and P the event of rolling a sum that is a prime number. Find the probability of rolling a sum that is even or prime?

Solution.
Since $E = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}$ then $P(E) = \frac{18}{36} = \frac{1}{2}$. Since $P = \{(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),$
$(3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5)\}$ then $P(P) = \frac{15}{36} = \frac{5}{12}$. Also, $P(E \cap P) = \frac{1}{36}$ so that

$$P(E \cup P) = P(E) + P(P) - P(E \cap P) = \frac{18}{36} + \frac{15}{36} - \frac{1}{36} = \frac{32}{36} = \frac{8}{9} \blacksquare$$

Problem 33.29
If events A nd B are from the same sample space, and if P(A)=0.8 and P(B)=0.9, can events A and B be mutually exclusive?

Solution.
If A and B are mutually exclusive then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B) = 0.8 + 0.9 = 1.7$. But $P(A \cup B)$ is a probability and so it can not exceed 1. Hence, A and B can not be mutually exclusive■

Problem 34.1
If each of the 10 digits is chosen at random, how many ways can you choose the following numbers?

(a) A two-digit code number, repeated digits permitted.
(b) A three-digit identification card number, for which the first digit cannot be a 0.
(c) A four-digit bicycle lock number, where no digit can be used twice.
(d) A five-digit zip code number, with the first digit not zero.

Solution.
(a) This is a decision with two steps. For the first step we have 10 choices and for the second we have also 10 choices. By the Fundamental Principle of Counting there are $10 \cdot 10 = 100$ two-digit code numbers.
(b) This is a decision with three steps. For the first step we have 9 choices, for the second we have 10 choices, and for the third we have 10 choices. By the Fundamental Principle of Counting there are $9 \cdot 10 = 900$ three-digit identification card numbers.
(c) This is a decision with four steps. For the first step we have 10 choices, for the second 9 choices, for the third 8 choices, and for fourth 7 choices. By the Fundamental Principle of Counting there are $10 \cdot \cdot 9 \cdot 8 \cdot 7 = 5040$ four-digit code numbers.
(d) This is a decision with five steps. For the first step we have 9 choices, for the second 10 choices, for the third 10 choices, for the fourth 10 choices, and for the fifth 10 choices. By the Fundamental Principle of Counting there are $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90,000$ five-digit zip codes. ■

Problem 34.2
(a) If eight horses are entered in a race and three finishing places are considered, how many finishing orders can they finish?

(b) If the top three horses are Lucky one, Lucky Two, and Lucky Three, in how many possible orders can they finish?

Solution.
(a) This is a decision with three steps. For the first finisher we have 8 choices, 7 choices for the second and 6 choices for the third. By the Fundamental Principle of Counting there are $8 \cdot 7 \cdot 6 = 336$ different first three finishers.
(b) This is a decision with three steps. For the first finisher we have 3 choices, 2 choices for the second and 1 choice for the third. By the Fundamental Principle of Counting there are $3 \cdot 2 \cdot 1 = 6$ different arrangements.$\blacksquare$

Problem 34.3
You are taking 3 shirts(red, blue, yellow) and 2 pairs of pants (tan, gray) on a trip. How many different choices of outfits do you have?

Solution.
(a) This is a decision with two steps. For the choice of shirts there are 3 choices and for the choice of pants there are 2 choices. By the Fundamental Principle of Counting there are $3 \cdot 2 = 6$ different choices of outfits$\blacksquare$

Problem 34.4
The state of Maryland has automobile license plates consisting of 3 letters followed by three digits. How many possible license plates are there?

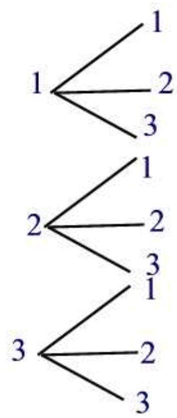
Solution.
This is a decision with 6 steps. For the first we have 26 choices, 26 choices for the second, 26 choices for the third and 10 for each of the fourth, fifth and sixth steps. By the Fundamental Principle of Counting we have: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$ different license plates$\blacksquare$

Problem 34.5
A club has 10 members. In how many ways can the club choose a president and vice-president if everyone is eligible?

Solution.
This is a decision with two steps. For the choice of president we have 10 possible choices, for the choice of the vice-president we have 9 choices. By the Fundamental Principle of Counting there are $10 \cdot 9 = 90$ different ways $\blacksquare$

Problem 34.6
A lottery allows you to select a two-digit number. Each digit may be either 1,2 or 3. Use a tree diagram to show the sample space and tell how many different numbers can be selected.

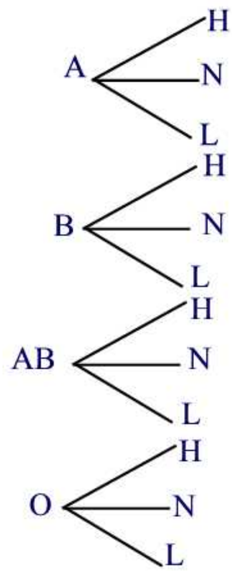
Solution.



The different numbers are $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ ■

Problem 34.7
In a medical study, patients are classified according to whether they have blood type A, B, AB, or O, and also according to whether their blood pressure is low, normal, or high. Use a tree diagram to represent the various outcomes that can occur.

Solution.



Problem 34.8
If a travel agency offers special weekend trips to 12 different cities, by air, rail, or bus, in how many different ways can such a trip be arranged?

Solution.
This is a decision with two steps. The first is to choose the destination and we have 12 choices. The second is to choose the way to travel and we have three choices. By the Fundamental Principle of Counting there are $12 \cdot 3 = 36$ ways for the trip ■

Problem 34.9
If twenty paintings are entered in art show, in how many different ways can the judges award a first prize and a second prize?

Solution.
For the first prize we have 20 choices and for the second we have 19 choices. By the Fundamental Principle of Counting we have $20 \cdot 19 = 380$ possible choices■

Problem 34.10
In how many ways can the 52 members of a labor union choose a president, a vice-president, a secretary, and a treasurer?

Solution.
For the president we have 52 choices, for the vice-president we have 51 choices, for the secretary we have 50 choices, and for the treasurer we have 49 choices. By the Fundamental Principle of Counting there are $52 \cdot 51 \cdot 50 \cdot 49 = 6,497,400$ possible choices$\blacksquare$

Problem 34.11
Find the number of ways in which four of ten new movies can be ranked first, second, third, and fourth according to their attendance figures for the first six months.

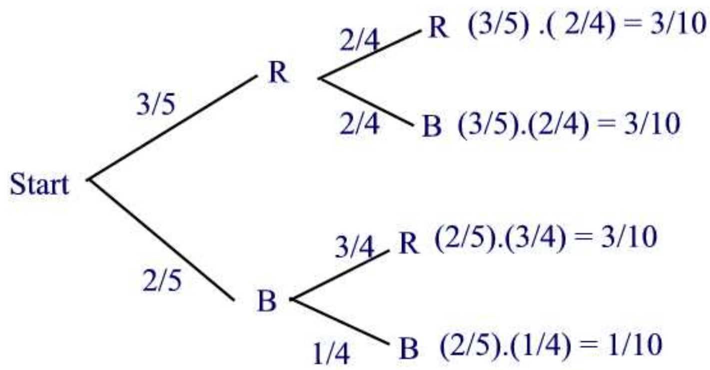
Solution.
For the first ranking we have 10 choices, for the second we have 9 choices, for the third 8 choices, and for the fourth 7 choices. By the Fundamental Principle of Counting there are $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ possible ways $\blacksquare$

Problem 34.12
To fill a number of vacancies, the personnel manager of a company has to choose three secretaries from among ten applicants and two bookkeepers from among five applicants. In how many different ways can the personnel manager fill the five vacancies?

Solution.
This is a decision with two steps. For the choice of secretaries there are $C(10,3)$ different ways and for the choice of the bookkeepers there are $C(5,2)$. By the Fundamental Principle of Counting there are $C(10,3) \cdot C(5,2) = 1200$ different ways. The notation $C(.,.)$ will be introduced and discussed in Section 35$\blacksquare$

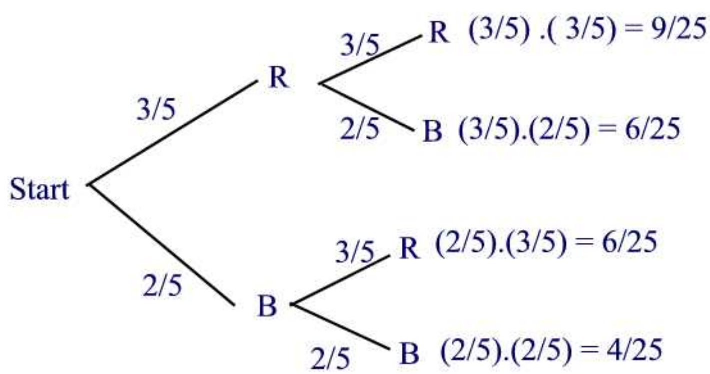
Problem 34.13
A box contains three red balls and two blue balls. Two balls are to be drawn without replacement. Use a tree diagram to represent the various outcomes that can occur. What is the probability of each outcome?

Solution.



Problem 34.14
Repeat the previous exercise but this time replace the first ball before drawing the second.

Solution.



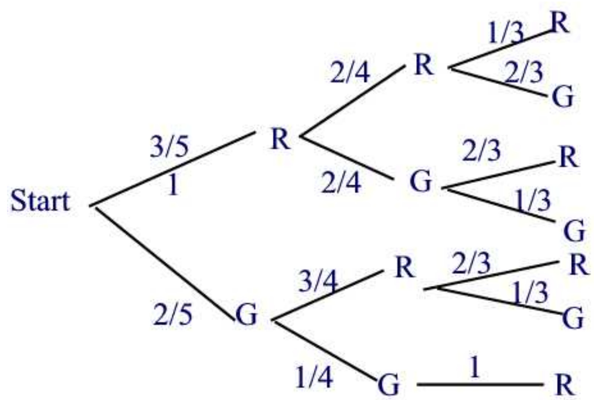
Problem 34.15
If a new-car buyer has the choice of four body styles, three engines, and ten colors, in how many different ways can s/he order one of these cars?

Solution.
This is a decision with three stages. For the choice of body styles we have 4 choices, for the engine three, and for the color ten. By the Fundamental Principle of Counting we have $4 \cdot 3 \cdot 10 = 120$ different choices■

Problem 34.16
A jar contains three red gumballs and two green gumballs. An experiment consists of drawing gumballs one at a time from the jar, without replacement, until a red one is obtained. Find the probability of the following events.

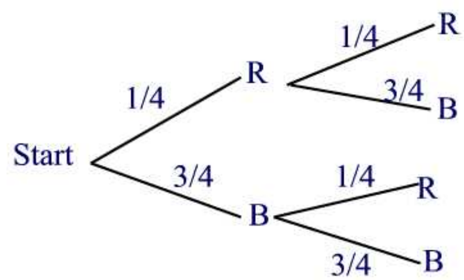
A: Only one draw is needed.
B: Exactly two draws are needed.
C: Exactly three draws are needed.

Solution.
(a) $P(A) = \frac{3}{5}$
(b) $P(B) = \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$
(c) $P(C) = \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{3}{3} = \frac{1}{10}$ ∎



Problem 34.17
Consider a jar with three black marbles and one red marble. For the experiment of drawing two marbles with replacement, what is the probability of drawing a black marble and then a red marble in that order?

Solution.
The probability is $\frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$



Problem 34.18
A jar contains three marbles, two black and one red. Two marbles are drawn with replacement. What is the probability that both marbles are black? Assume that the marbles are equally likely to be drawn.

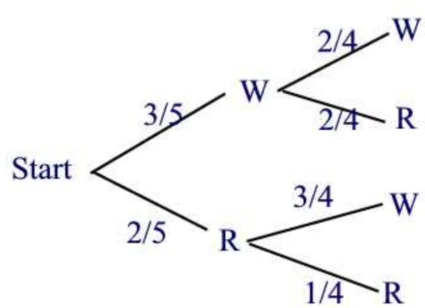
Solution.
The probability that both are black is $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$■

Problem 34.19
A jar contains four marbles-one red, one green, one yellow, and one white. If two marbles are drawn without replacement from the jar, what is the probability of getting a red marble and a white marble in that order?

Solution.
The probability is $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ ■

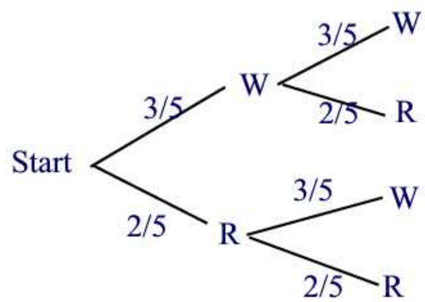
Problem 34.20
A jar contains 3 white balls and 2 red balls. A ball is drawn at random from the box and not replaced. Then a second ball is drawn from the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Solution.
The probability is $\frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}$■



Problem 34.21
Suppose that a ball is drawn from the box in the previous problem, its color recorded, and then it is put back in the box. Draw a tree diagram for this experiment and find the probability that the two balls are of different colors.

Solution.
The probability is $\frac{3}{5}\cdot\frac{2}{5}+\frac{2}{5}\cdot\frac{3}{5}=\frac{12}{25}$ ■



Problem 34.22
Suppose there are 19 balls in an urn. They are identical except in color. 16 of the balls are black and 3 are purple. You are instructed to draw out one ball, note its color, and set it aside. Then you are to draw out another ball and note its color. What is the probability of drawing out a black on the first draw and a purple on the second?

Solution.
The probability is $\frac{16}{19}\cdot\frac{3}{18}=\frac{8}{57}$ ■

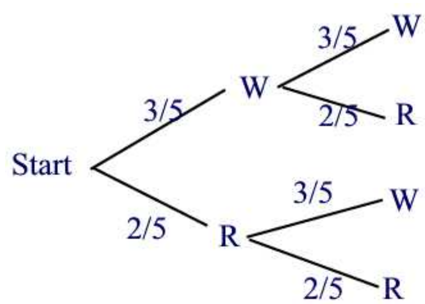
Problem 34.23
The row of Pascal's triangle that starts $1, 4, \cdots$ would be useful in finding probabilities for an experiment of tossing four coins.
(a) Interpret the meaning of each number.
(b) Find the probability of exactly one head and three tails.
(c) Find the probability of at least one tail turning up.

Solution.
(a) There are 1 four-head outcome and 4 three-head outcomes.
(b) From Pascals' trinagle in the notes we find that the probability of exactly one head and three tails is $\frac{4}{16}=\frac{1}{4}$
(c) $\frac{4}{16}(one\ tail)+\frac{6}{16}(two\ tails)+\frac{4}{16}(three\ tails)+\frac{1}{16}(four\ tails)=\frac{15}{16}$ ■

Problem 34.24
Four coins are tossed.

(a) Draw a tree diagram to represent the arrangements of heads (H) and tails (T).
(b) How many outcomes involve all heads? three heads, one tail? two heads, two tails? one head, three tails? no heads?
(c) How do these results relate to Pascal's triangle?

Solution.
(a)



(b) 1(4H0T), 4(3H1T), 6(2H2T), 4(1H3T), 1(0H4T)
(c) These numbers are the entries of the fourth row in Pascal's triangle■

Problem 34.25
A true-false problem has 6 questions.

(a) How many ways are there to answer the 6-question test?
(b) What is the probability of getting at least 5 right by guessing the answers at random?

Solution.
(a) This is a binary experiment. Each question is either true or false so by the Principle of Counting there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 6 = 2^6 = 64$ different ways.
(b) $\frac{1}{64}(all\ six\ are\ right) + \frac{6}{64}(five\ right\ and\ one\ wrong) = \frac{7}{64}$■

Problem 34.26
(a) Write the 7th row of Pascal's triangle.
(b) What is the probability of getting at least four heads when tossing seven coins?

Solution.
(a)1 7 21 35 35 21 7 1
(b) $\frac{1}{128} + \frac{7}{128} + \frac{21}{128} + \frac{35}{128} = \frac{64}{128} = \frac{1}{2}$■

Problem 34.27
Assume the probability is $\frac{1}{2}$ that a child born is a boy. What is the probability that if a family is going to have four children, they will all be boys?

Solution.
Using the fourth row of Pascal's triangle with boy = head and girl = tail we find $\frac{1}{16}$ ■

Problem 35.1
Compute each of the following expressions.

(a) $(2!)(3!)(4!)$
(b) $(4 \times 3)!$
(c) $4 \cdot 3!$
(d) $4! - 3!$
(e) $\frac{8!}{5!}$
(f) $\frac{8!}{0!}$

Solution.
(a) $(2!)(3!)(4!) = 2 \cdot 6 \cdot 24 = 288$
(b) $(4 \times 3)! = 12! = 479001600$
(c) $4 \cdot 3! = 4 \cdot 6 = 24$
(d) $4! - 3! = 24 - 6 = 18$
(e) $\frac{8!}{5!} = 336$
(f) $\frac{8!}{0!} = 8! = 40320$■

Problem 35.2
Compute each of the following.

(a) $P(7, 2)$ (b) $P(8, 8)$ (c) $P(25, 2)$

Solution.
(a) $P(7, 2) = \frac{7!}{(7-2)!} = 42$
(b) $P(8, 8) = \frac{8!}{(8-8)!} = 8! = 40320$
(c) $P(25, 2) = \frac{25!}{(25-2)!} = \frac{25!}{23!} = 25 \cdot 24 = 600$■

Problem 35.3
Find m and n so that $P(m,n) = \frac{9!}{6!}$

Solution.
Since $\frac{m!}{(m-n)!} = \frac{9!}{6!}$ then $m = 9$ and $n = 3$ ■

Problem 35.4
How many four-letter code words can be formed using a standard 26-letter alphabet
(a) if repetition is allowed?
(b) if repetition is not allowed?

Solution.
(a) $26 \cdot 26 \cdot 26 \cdot 26 = 456976$
(b) $P(26,4) = \frac{26!}{(26-4)!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358800$ ■

Problem 35.5
Certain automobile license plates consist of a sequence of three letters followed by three digits.

(a) If no repetitions of letters are permitted, how many possible license plates are there?
(b) If no letters and no digits are repeated, how many license plates are possible?

Solution.
(a) $P(26,3) \cdot 10 \cdot 10 \cdot 10 = 15600000$
(b) $P(26,3) \cdot P(10,3) = 15600 \cdot 720 = 11232000$ ■

Problem 35.6
A combination lock has 40 numbers on it.

(a) How many different three-number combinations can be made?
(b) How many different combinations are there if the numbers must be all different?

Solution.
(a) $40 \cdot 40 \cdot 40 = 64000$
(b) $P(40,3) = \frac{40!}{(40-3)!} = \frac{40!}{37!} = 59280$ ■

Problem 35.7
(a) Miss Murphy wants to seat 12 of her students in a row for a class picture. How many different seating arrangements are there?
(b) Seven of Miss Murphy's students are girls and 5 are boys. In how many different ways can she seat the 7 girls together on the left, and then the 5 boys together on the right?

Solution.
(a) $P(12, 12) = 12! = 479001600$
(b) $P(7, 7) \cdot P(5, 5) = 7! \cdot 5! = 5040 \cdot 120 = 604800$ ■

Problem 35.8
Using the digits 1, 3, 5, 7, and 9, with no repetitions of the digits, how many

(a) one-digit numbers can be made?
(b) two-digit numbers can be made?
(c) three-digit numbers can be made?
(d) four-digit numbers can be made?

Solution.
(a) 5
(b) $P(5, 2) = 20$
(c) $P(5, 3) = 60$
(d) $P(5, 4) = 120$ ■

Problem 35.9
There are five members of the Math Club. In how many ways can the positions of officers, a president and a treasurer, be chosen?

Solution.
There are $P(5, 3) = 20$ different ways ■

Problem 35.10
(a) A baseball team has nine players. Find the number of ways the manager can arrange the batting order.
(b) Find the number of ways of choosing three initials from the alphabet if none of the letters can be repeated.

Solution.
(a) $P(9,9) = 362880$
(b) $P(26,3) = 15600$ ■

Problem 35.11
Compute each of the following: (a) C(7,2) (b) C(8,8) (c) C(25,2)

Solution.
(a) $C(7,2) = \frac{7!}{2!(5-2)!} = \frac{7!}{2!3!} = 21$
(b) $C(8,8) = 1$
(c) $C(25,2) = 300$ ■

Problem 35.12
Find m and n so that $C(m,n) = 13$

Solution.
Since $C(m,n) = \frac{m!}{n!(m-n)!} = 13$ then we can choose $m = 13$ and $n = 1$ ■

Problem 35.13
The Library of Science Book Club offers three books from a list of 42. If you circle three choices from a list of 42 numbers on a postcard, how many possible choices are there?

Solution.
Since order is not of importance here then the number of different choices is $C(42,3) = 11480$■

Problem 35.14
At the beginning of the second quarter of a mathematics class for elementary school teachers, each of the class's 25 students shook hands with each of the other students exactly once. How many handshakes took place?

Solution.
Since the handshake between persons A and B is the same as that between B and A, this is a problem of choosing combinations of 25 people two at a time. There are $C(25,2) = 300$ different handshakes■

Problem 35.15
There are five members of the math club. In how many ways can the two-person Social Committee be chosen?

Solution.
Since order is irrelevant than the different two-person committe is $C(5,2) = 10$ ■

Problem 35.16
A consumer group plans to select 2 televisions from a shipment of 8 to check the picture quality. In how many ways can they choose 2 televisions?

Solution.
There are $C(8,2) = 28$ different ways ■

Problem 35.17
The Chess Club has six members. In how many ways
(a) can all six members line up for a picture?
(b) can they choose a president and a secretary?
(c) can they choose three members to attend a regional tournament with no regard to order?

Solution.
(a) $P(6,6) = 6! = 720$ different ways
(b) $P(6,2) = 30$ ways
(c) $C(6,3) = 20$ different ways ■

Problem 35.18
Find the smallest values m and n such that $C(m,n) = P(15,2)$

Solution.
We have $C(m,n) = \frac{m!}{n!(m-n)!} = \frac{15!}{13!}$. It follows that the smallest values of m and n are $m = 15$ and $n = 0$ ■

Problem 35.19
A school has 30 teachers. In how many ways can the school choose 3 people to attend a national meeting?

Solution.
There are $C(30,3) = 4060$ different ways ■

Problem 35.20
Which is usually greater the number of combinations of a set of objects or the number of permutations?

Solution.
Recall that $P(m,n) = \frac{m!}{(m-n)!}$ whereas $C(m,n) = \frac{m!}{n!(m-n)!}$. Since $n! \geq 1$ then $\frac{1}{n!} \leq 1$. Multiply both sides by $\frac{m!}{(m-n)!}$ to obtain $\frac{m!}{n!(m-n)!} \leq \frac{m!}{(m-n)!}$. Hence, $C(m,n) \leq P(m,n)$ ■

Problem 35.21
How many different 12-person juries can be chosen from a pool of 20 juries?

Solution.
There are $C(20,12) = 125970$ different ways■

Problem 35.22
John and Beth are hoping to be selected from their class of 30 as president and vice-president of the Social Committee. If the three-person committee (president, vice-president, and secretary) is selected at random, what is the probability that John and Beth would be president and vice president?

Solution.
There are 28 committees where John is the president and Beth is the vice-president. The number of groups of three out of 30 people is $C(30,3) = 4060$. Hence, the probability of John being selected as president and Beth as vice-president is $\frac{28}{4060} \approx 0.0069$ ■

Problem 35.23
There are 10 boys and 13 girls in Mr. Benson's fourth-grade class and 12 boys and 11 girls in Mr. Johnson fourth-grade class. A picnic committee of six people is selected at random from the total group of students in both classes.

(a) What is the probability that all the committee members are girls?
(b) What is the probability that the committee has three girls and three boys?

Solution.
(a) $P(committee\ has\ six\ girls) = \frac{C(24,6)}{C(46,6)} \approx 0.014$
(b) $P(three\ boys\ and\ three\ girls) = \frac{C(22,3)\cdot C(24,3)}{C(46,6)} \approx 0.333$■

Problem 35.24
A school dance committee of 4 people is selected at random from a group of 6 ninth graders, 11 eighth graders, and 10 seventh graders.

(a) What is the probability that the committee has all seventh graders?
(b) What is the probability that the committee has no seventh graders?

Solution.
(a) $P(four\ seventh\ graders) = \frac{C(10,4)}{C(27,4)} \approx 0.012$
(b) $1 - 0.012 = 0.988$ ■

Problem 35.25
In an effort to promote school spirit, Georgetown High School created ID numbers with just the letters G, H, and S. If each letter is used exactly three times,

(a) how many nine-letter ID numbers can be generated?
(b) what is the probability that a random ID number starts with GHS?

Solution.
(a) This is a decision with three stages. The first stage is to find the number of ways the three letters G can be arranged in 9 vacant positions. There are $C(9,3)$ ways. One these three positions were filled with three G, then we need to arrange the three letters H in 6 positions. There are $C(6,3)$ different ways. Finally, there are $C(3,3)$ to arrange the three letters S in the three remaining positions. By the Fundamental Principle of Counting there are $C(9,3) \cdot C(6,3) \cdot C(3,3) = 1680$ different ID numbers.
(b) The first three positions of the ID are occupied by GHS in order. There are 6 positions left to be used by the letters G, H and S repeated twice. There are $C(6,2) \cdot C(4,2) \cdot C(2,2) = 90$. Thus, the probability that a random ID number starts with GHS is $\frac{90}{1680} = \frac{3}{56}$ ■

Problem 35.26
The license plates in the state of Utah consist of three letters followed by three single-digit numbers.

(a) If Edward's initials are EAM, what is the probability that his license plate will have his initials on it (in any order)?
(b) What is the probability that his license plate will have his initials in the correct order?

Solution.
(a) The total number of license plates consisting of three letters followed by three digits is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$. The total number of plates with the letters E,A, M in any order is $6 \cdot 10 \cdot 10 \cdot 10 = 6000$. Thus, the probability that his license plate will have his initials on it (in any order) is

$$\frac{6000}{17576000}$$

(b) The probability in this case is $\frac{1000}{17576000}$ ■

Problem 36.1
If the probability of a boy's being born is $\frac{1}{2}$, and a family plans to have four children, what are the odds against having all boys?

Solution.
The probability of having all four boys is $\frac{1}{16}$. The odds against having all four boys is $15:1$ since

$$\frac{1-\frac{1}{16}}{\frac{1}{16}} = \frac{15}{1} \blacksquare$$

Problem 36.2
If the odds against Deborah's winning first prize in a chess tournament are 3 to 5, what is the probability that she will win first prize?

Solution.
The probability that she will win the first prize satisfies the equation

$$\frac{1-P(E)}{P(E)} = \frac{3}{5}$$

Solving this equation for $P(E)$ we find $P(E) = \frac{5}{8}$■

Problem 36.3
What are the odds in favor of getting at least two heads if a fair coin is tossed three times?

Solution.
The probability of getting at least two heads is $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$. The odds in favor is

$$\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \text{ } or \text{ } 1:1 \blacksquare$$

Problem 36.4
If the probability of rain for the day is 60%, what are the odds against its raining?

Solution.
The odds against rain is

$$\frac{1-0.6}{0.6}=\frac{4}{6}=\frac{2}{3}\ or\ 2:3\ qed$$

Problem 36.5
On a tote board at a race track, the odds for Gameylegs are listed as 26:1. Tote boards list the odds that the horse will lose the race. If this is the case, what is the probability of Gameylegs's winning the race?

Solution.
The odds against winning is 26:1. The probability of winning the race satisfies the equation

$$\frac{1-P(E)}{P(E)}=26$$

Solving for $P(E)$ we find $P(E)=\frac{1}{27}$ ■

Problem 36.6
If a die is tossed, what are the odds in favor of the following events?
(a) Getting a 4
(b) Getting a prime
(c) Getting a number greater than 0
(d) Getting a number greater than 6.

Solution.
(a) $\frac{\frac{1}{6}}{1-\frac{1}{6}}=\frac{1}{5}$ or 5 to 1

(b) There are three prime numbers: 2, 3, and 5. The probability of getting a prime number is $\frac{1}{2}$. The odds in favor is then

$$\frac{\frac{1}{2}}{1-\frac{1}{2}}=1\ or\ 1:1$$

(c) The probability of getting a number greater than 0 is 1. The odds in favor is $1:0$
(d) The probability of getting a number greater than 0 is 0 so that the odd in favor is $0:1$■

Problem 36.7
Find the odds against E if $P(E) = \frac{3}{4}$.

Solution.

$$\frac{1-P(E)}{P(E)} = \frac{1-\frac{3}{4}}{\frac{3}{4}} = \frac{1}{3} \textit{ or } 1:3 \blacksquare$$

Problem 36.8
Find P(E) in each case.

(a) The odds in favor of E are 3:4
(b) The odds against E are 7:3

Solution.
(a) In this case $P(E)$ satisfies $\frac{P(E)}{1-P(E)} = \frac{3}{4}$. Solving for $P(E)$ we find $P(E) = \frac{3}{7}$
(b) Solving the equation $\frac{1-P(E)}{P(E)} = \frac{7}{3}$ for $P(E)$ we find $P(E) = \frac{3}{10} \blacksquare$

Problem 36.9
Compute the expected value of the score when rolling two dice.

Solution.

Score	2	3	4	5	6	7	8	9	10	11	12
# of outcome(s)	1	2	3	4	5	6	5	4	3	2	1

The expected value is

$$\begin{aligned} E &= 2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36} \\ &+ 8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}=7 \blacksquare \end{aligned}$$

Problem 36.10
A game consists of rolling two dice. You win the amounts shown for rolling the score shown.

Score	2	3	4	5	6	7	8	9	10	11	12
$ won	4	6	8	10	20	40	20	10	8	6	4

Compute the expected value of the game.

Solution.

$$\begin{aligned} E &= 4\times\tfrac{1}{36}+6\times\tfrac{2}{36}+8\times\tfrac{3}{36}+10\times\tfrac{4}{36}+20\times\tfrac{5}{36} \\ &+ 40\times\tfrac{6}{36}+20\times\tfrac{5}{36}+10\times\tfrac{4}{36}+8\times\tfrac{3}{36}+6\times\tfrac{2}{36}+4\times\tfrac{1}{36}=\tfrac{50}{3}\approx \$16.67 \blacksquare\end{aligned}$$

Problem 36.11
Consider the spinner in Figure 36.1, with the payoff in each sector of the circle. Should the owner of this spinner expect to make money over an extended period of time if the charge is $2.00 per spin?

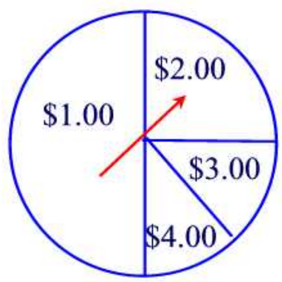


Figure 36.1

Solution.

$$\begin{aligned} E &= -1\times\tfrac{1}{2}+0\times\tfrac{1}{4}+1\times\tfrac{1}{8}+2\times\tfrac{1}{8} \\ &= -\tfrac{1}{8}=-\$0.125\end{aligned}$$

So the owner will make on average 12.5 cents per spin ■

Problem 36.12
You play a game in which two dice are rolled. If a sum of 7 appears, you win $10; otherwise, you lose $2.00. If you intend to play this game for a long time, should you expect to make money, lose money, or come out about even? Explain.

Solution.

$$E = 10\times\frac{1}{6}-2\times\frac{5}{6}=0$$

Therefore, you should come out about even if you play for a long time■

Problem 36.13
Suppose it costs $8 to roll a pair of dice. You get paid the sum of the numbers in dollars that appear on the dice. What is the expected value of this game?

Solution.

$$\begin{array}{rcl} E & = & -6\times\frac{1}{36}-5\times\frac{2}{36}-4\times\frac{3}{36}-3\times\frac{4}{36}-2\times\frac{5}{36} \\ & - & 1\times\frac{6}{36}+0\times\frac{5}{36}+1\times\frac{4}{36}+2\times\frac{3}{36}+3\times\frac{2}{36}+4\times\frac{1}{36} \\ & = & \$-1 \blacksquare \end{array}$$

Problem 36.14
An insurance company will insure your dorm room against theft for a semester. Suppose the value of your possessions is \$800. The probability of your being robbed of \$400 worth of goods during a semester is $\frac{1}{100}$, and the probability of your being robbed of \$800 worth of goods is $\frac{1}{400}$. Assume that these are the only possible kinds of robberies. How much should the insurance company charge people like you to cover the money they pay out and to make an additional \$20 profit per person on the average?

Solution.
The expected value is

$$E = (400)\times\frac{1}{100} + (800)\times\frac{1}{400} = \$6$$

They should charge \$ 6 + \$ 20 = \$ 26 for the policy to make an average gain of \$ 20 per policy$\blacksquare$

Problem 36.15
Consider a lottery game in which 7 out of 10 people lose, 1 out of 10 wins \$50, and 2 out of 10 wins \$35. If you played 10 times, about how much would you expect to win?

Solution.

$$E = -1\times\frac{7}{10} + 50\times\frac{1}{10} + 35\times\frac{2}{10} = \frac{113}{10}$$

So if you play this game 10 times you expect to win on average \$ 113 $\blacksquare$

Problem 36.16
Suppose a lottery game allows you to select a 2-digit number. Each digit may be either 1, 2, 3, 4, or 5. If you pick the winning number, you win \$10. Otherwise, you win nothing. What is the expected payoff?

Solution.
There are $5 \times 5 = 25$ 2-digit number. The expected value is

$$E = -1 \times \frac{24}{25} + 10 \times \frac{1}{25} = -\frac{14}{25}$$

So if you play this game 25 times you expect to lose \$ 14 ■

Problem 36.17
Suppose that A is the event of rolling a sum of 7 with two fair dice. Make up an event B so that
(a) A and B are independent.
(b) A and B are dependent.

Solution.
(a) Let B be the event " It rains tomorrow". Then A and B are independent.
(b) Let B be the event "of rolling an even sum". Then A and B are dependent■

Problem 36.18
When tossing three fair coins, what is the probability of getting two tails given that the first coin came up heads?

Solution.
Let A be the event of getting two tails and B the event that the first coing came up head. Then $A = \{TTT, TTH, THT, HTT\}, B = \{HHH, HTT, HHT, HTH\}, A \cap B = \{HTT\}$. Thus, $P(A) = P(B) = \frac{1}{2}$ and $P(A \cap B)\frac{1}{8}$. Finally, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$ ■

Problem 36.19
Suppose a 20-sided die has the following numerals on its face:1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 7, ,8 9, 10, 11, 12, 13, 14, 15, 16. The die is rolled once and the number on the top face is recorded. Let A be the event the number is prime, and B be the event the number is odd. Find $P(A|B)$ and $P(B|A)$.

Solution.
$A = \{2, 2, 2, 3, 3, 5, 7, 11, 13\}$ and $B = \{1, 3, 3, 5, 7, 9, 11, 13, 15\}$. Thus, $A \cap B = \{3, 3, 5, 7, 11, 13\}$. Hence, $P(A) = \frac{9}{20}, P(B) = \frac{9}{20} <$ and $P(A \cap B) = \frac{6}{20}$. So

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{6}{20}}{\frac{9}{20}} = \frac{2}{3}$$

and

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{6}{20}}{\frac{9}{20}} = \frac{2}{3} \blacksquare$$

Problem 36.20
What is the probability of rolling a 6 on a fair die if you know that the roll is an even number?

Solution.
Let A be the event of rolling a 6 so that $P(A) = \frac{1}{6}$. Let $B = \{2, 4, 6\}$. Then $P(B) = \frac{1}{2}$. Also, $A \cap B = \{6\}$ so that $P(A \cap B) = \frac{1}{6}$. Hence,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \blacksquare$$

Problem 36.21
A red die and a green die are rolled. What is the probability of obtaining an even number on the red die and a multiple of 3 on the green die?

Solution.
Let A be the event of obtaining an even number on the red die and B the event of obtaining a multiple of 3 on the green die. The two events are indepndent so that $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ $\blacksquare$

Problem 36.22
Two coins are tossed. What is the probability of obtaining a head on the first coin and a tail on the second coin?

Solution.
Let A be the outcome of getting a head on the first toss and B be the event of getting a tail on the second toss. Then A and B are independent. Thus, $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\blacksquare$

Problem 36.23
Consider two boxes: Box 1 contains 2 white and 2 black balls, and box 2 contains 2 white balls and three black balls. What is the probability of drawing a black ball from each box?

Solution.
Let A be the event of drawing a black ball from Box 1 and B that of Box 2. Then A and B are independent so that $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$
■

Problem 36.24
A container holds three red balls and five blue balls. One ball is drawn and discarded. Then a second ball is drawn.
(a) What is the probability that the second ball drawn is red if you drew a red ball the first time?
(b) What is the probability of drawing a blue ball second if the first ball was red?
(c) What is the probability of drawing a blue ball second if the first ball was blue?

Solution.
(a) $\frac{2}{7}$
(b) $\frac{5}{7}$
(c) $\frac{4}{7}$ ■

Problem 36.25
Consider the following events.

A: rain tomorrow
B: You carry an umbrella
C: coin flipped tomorrow lands on heads

Which of two events are dependent and which are independent?

Solution.
A and B are depedent. A and C, B and C are independent■

Problem 36.26
You roll a regular red die and a regular green die. Consider the following events.

A: a 4 on the red die
B: a 3 on the green die
C: a sum of 9 on the two dice

Tell whether each pair of events is independent or dependent.
(a) A and B (b) B and C

Solution.
(a) Independent
(b) Depedent since a 3 on the green die makes it more likely $\frac{1}{6}$ that you will get a sum of 9■

Problem 37.1
Find three objects in your classroom with surfaces that suggests common geometric figures.

Solution.
Answer may vary. A door (rectangle), a ceiling (a square), a wall clock (a circle) ■

Problem 37.2
A fifth grader says a square is not a rectangle because a square has four congruent sides and rectangles don't have that. A second fifth grader says a square is a type of rectangle because it is a parallelogram and it has four right angles.

(a) Which child is right?
(b) How can you use the definitions to help the other child understand?

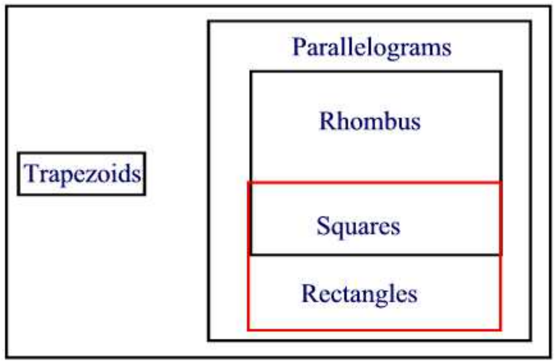
Solution.
(a) The second child
(b) Write the definition of a rectangle and show the child that a square satisfies it■

Problem 37.3
Suppose P={parallelograms}, Rh={rhombus}, S={squares}, Re={rectangles}, T={trapezoids}, and Q={quadrilaterals}. Find
(a) $Rh \cap Re$ (b) $T \cap P$

Solution.
(a) $Rh \cap Re = S$
(b) $T \cap P = \emptyset$■

Problem 37.4
Organize the sets P, Rh, S, Re, T, and Q using Venn diagram.

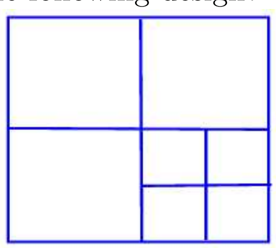
Solution.



Problem 37.5
(a) True or false? No scalene triangle is isosceles.
(b) What shape is the diamond in a deck of cards?

Solution.
(a) True. A scalene triangle is a triangle with the three sides of different lengths. An isosceless triangle has two sides of equal length.
(b) Rhombus■

Problem 37.6
How many squares are in the following design?



Solution.
Nine square■

Problem 37.7
Tell whether each of the following shapes must, can, or cannot have at least

one right angle.

(a) Rhombus
(b) Square
(c) Trapezoid
(d) Rectangle
(e) Parallelogram

Solution.
(a) Cannot
(b) must
(c) can
(d) must
(e) can■

Problem 37.8
In which of the following shapes are both pairs of opposite sides parallel?

(a) Rhombus
(b) Square
(c) Trapezoid
(d) Rectangle
(e) Parallelogram

Solution.
(a), (b), (d), and (e) ■

Problem 37.9
A square is also which of the following?
(a) Quadrilateral
(b) Parallelogram
(c) Rhombus
(d) Rectangle

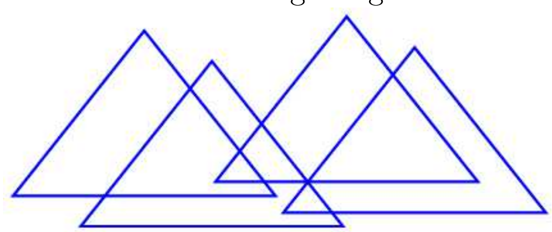
Solution.
(a), (b), (c), and (d) ■

Problem 37.10
Fill in the blank with "All", "Some", or "No"

(a) ______ rectangles are squares.
(b) ______ parallelograms are trapezoids.
(c) ______ rhombuses are quadrilaterals.

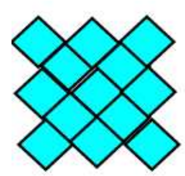
Solution.
(a) Some rectangles are squares.
(b) No parallelograms are trapezoids.
(c) All rhombuses are quadrilaterals.■

Problem 37.11
How many triangles are in the following design?



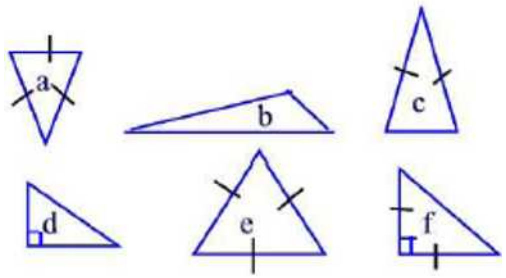
Solution.
Nine triangles■

Problem 37.12
How many squares are found in the following figure?



Solution.
Eighteen squares ■

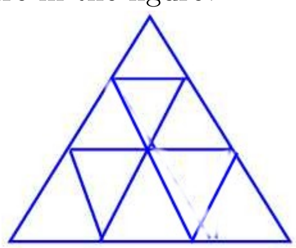
Problem 37.13
Given are a variety of triangles. Sides with the same length are indicated. Right angles are indicated.



(a) Name the triangles that are scalene.
(b) Name the triangles that are isosceles.
(c) Name the triangles that are equilateral.
(d) Name the triangles that contain a right angle.

Solution.
(a) (b) and (d)
(b) (a), (c), (e), and (f)
(c) (a), (e)
(d) (d) and (f) ■

Problem 37.14
(a) How many triangles are in the figure?
(b) How many parallelograms are in the figure?
(c) How many trapezoids are in the figure?

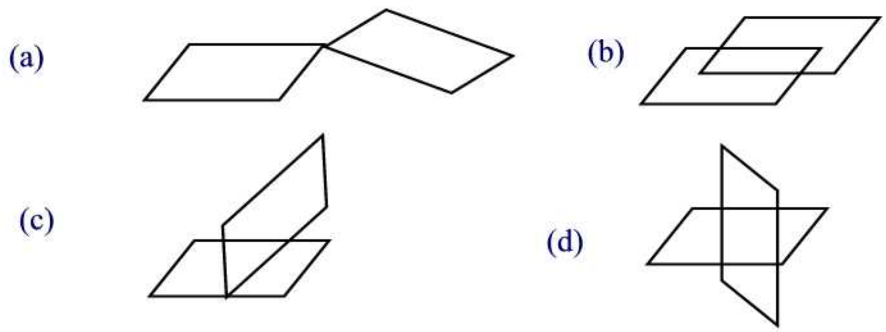


Solution.
(a) Thirteen triangles
(b) Ten parallelograms
(c) Nine Trapezoids■

Problem 37.15
If possible, sketch two parallelograms that intersect at exactly

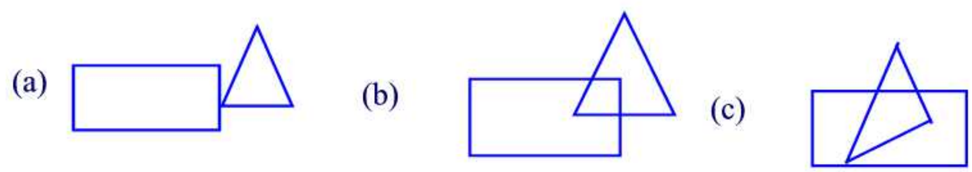
(a) one point
(b) two points
(c) three points
(d) four points.

Solution.



Problem 37.16
If possible, draw a triangle and a quadrilateral that intersect at exactly
(a) one point
(b) two points
(c) three points.

Solution.



Problem 37.17
Suppose P={parallelograms}, S={squares}, T={trapezoids}, and Q={quadrilaterals}. Find

(a) $P \cap S$
(b) $P \cup Q$

Solution.
(a) $P \cap S = S$
(b) $P \cup Q = Q$ ■

Problem 37.18
A fifth grader does not think that a rectangle is a type of parallelogram. Tell why it is.

Solution.
A rectangle is a parallelogram since the opposite sides are parallel■

Problem 37.19
Tell whether each definition has sufficient information. If it is not sufficient, tell what information is missing.

(a) A rhombus is a quadrilateral with both pairs of opposite sides parallel.
(b) A square is a quadrilateral with four congruent sides.
(c) A rhombus is a quadrilateral that has four congruent sides.

Solution.
(a) All four sides are congruent
(b) All four angles are right angles
(c) No additional information is needed■

Problem 37.20
Name properties that a square, parallelogram, and rhombus have in common.

Solution.
At this point the only common property we can state is that the opposite sides are parallel■

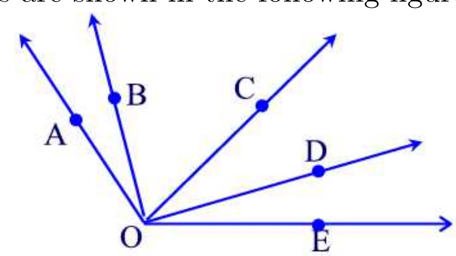
Problem 37.21
How many different line segments are contained in the following portion of a line?

Solution.
Sixteen different line segments■

Problem 38.1
Using Figure 38.5 show that $m(\angle 1) + m(\angle 5) = 180°$.

Solution.
Clearly $m(\angle 6) + m(\angle 5) = 180°$. But the angles $\angle 1$ and $\angle 6$ are alternate interior angles so that $m(\angle 6) = m(\angle 1)$. Thus, $m(\angle 1) + m(\angle 5) = 180°$ ■

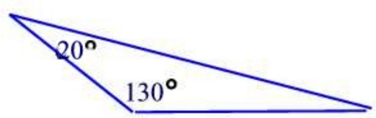
Problem 38.2
(a) How many angles are shown in the following figure?



(b) How many are obtuse?
(c) How many are acute?

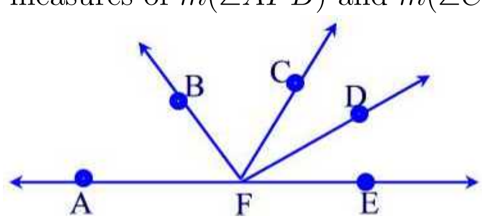
Solution.
(a) Ten angles
(b) $\angle BOE, \angle AOE, \angle AOD$
(c) $\angle DOE, \angle COD, \angle BOC, \angle AOB, \angle COE$ ■

Problem 38.3
Find the missing angle in the following triangle.



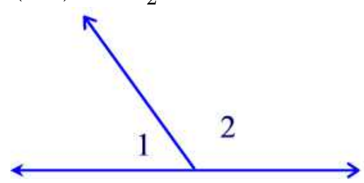
Solution.
The measure of the missing angle is 30° ■

Problem 38.4
In the figure below, $m(\angle BFC) = 55°, m(\angle AFD) = 150°$, $m(\angle BFE) = 120°$. Determine the measures of $m(\angle AFB)$ and $m(\angle CFD)$.



Solution.
The angles $\angle AFB$ and $\angle BFE$ are complementary so that $m(\angle AFB) = 180° - m(\angle BFE) = 180° - 120° = 60°$. On the other hand, $m(\angle CFD) = 150° - m(\angle AFB) - m(\angle BFC) = 150° - 60° - 55° = 35°$ ■

Problem 38.5
In the following figure $m(\angle 1) = \frac{m(\angle 2)}{2} - 9°$. Determine $m(\angle 1)$ and $m(\angle 2)$.



Solution.
Since $m(\angle 1) + m(\angle 2) = 180°$ and $m(\angle 1) = \frac{m(\angle 2)}{2} - 9°$ then $\frac{m(\angle 2)}{2} - 9° + m(\angle 2) = 180°$. Solving for $3m(\angle 2) = 2 \times 189° = 378°$ or $m(\angle 2) = \frac{378°}{3} = 126°$. Finally, $m(\angle 1) = \frac{m(\angle 2)}{2} - 9° = 63° - 9° = 54°$ ■

Problem 38.6
Angles 3 and 8; 2 and 7 in Figure 38.5 are called **alternate exterior angles**. Show that $m(\angle 3) = m(\angle 8)$.

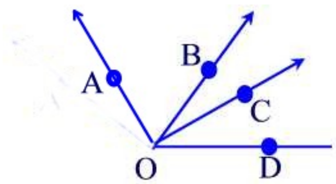
Solution.
We have $m(\angle 3) = m(\angle 6)$ (corresponding angles) and $m(\angle 6) = m(\angle 8)$ (vertical angles). Hence, $m(\angle 3) = m(\angle 8)$■

Problem 38.7
Following are the measures of $\angle A, \angle B, \angle C$. Can a triangle ΔABC be made that has the given angles? Explain.

(a) $m(\angle A) = 36°, m(\angle B) = 78°, m(\angle C) = 66°$.
(b) $m(\angle A) = 124°, m(\angle B) = 56°, m(\angle C) = 20°$.
(c) $m(\angle A) = 90°, m(\angle B) = 74°, m(\angle C) = 18°$.

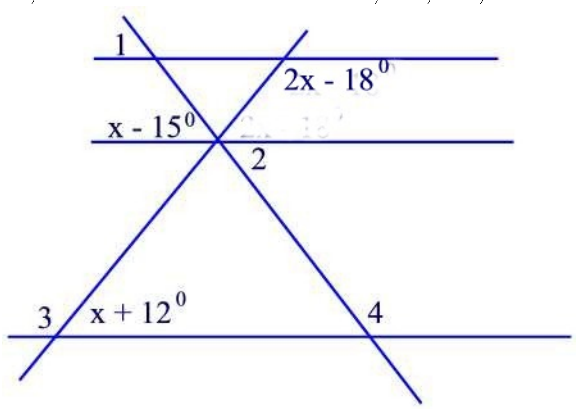
Solution.
(a) Since $m(\angle A) + m(\angle B) + m(\angle C) = 180°$ then ΔABC can exist.
(b) Since $m(\angle A) + m(\angle B) + m(\angle C) = 200° > 180°$ then ΔABC can not exist.
(c) Since Since $m(\angle A) + m(\angle B) + m(\angle C) = 182° > 180°$ then ΔABC can not exist ■

Problem 38.8
In the following figure $\overline{AO}$ is perpendicular to $\overline{CO}$. If $m(\angle AOD) = 165°$ and $m(\angle BOD) = 82°$, determine the measures of $\angle AOB$ and $\angle BOC$.



Solution.
We have $m(\angle AOB) = 165° - 82° = 83°$. Also, $m(\angle BOC) = 90° - m(\angle AOB) = 90° - 83° = 7°$ ■

Problem 38.9
In the figure below, find the measures of $\angle 1, \angle 2, \angle 3$, and $\angle 4$.

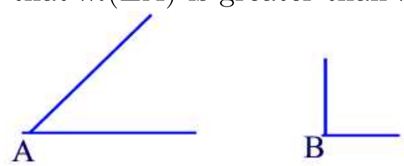


Solution.
We have: $2x - 18° + x + 12° = 180°$ (See Problem 38.1). Solving for x we find $x = \frac{186}{3} = 62°$. Now, $m(\angle 1) = m(\angle 2) = 62° - 15° = 47°$. Also, $m(\angle 3) = 180° - (62° + 12°) = 106°$. Finally, $m(\angle 4) = 180° - m(\angle 2) = 180° - 47° = 133°$ ■

Problem 38.10
(a) How is a line segment different from a line?
(b) What is the vertex of the angle $\angle PAT$?
(c) How is $\overline{AB}$ different from AB?

Solution.
(a) A line segment has two endpoints; a line has none.
(b) A
(c) $\overline{AB}$ is the notation for the line segment with endpoints A and B whereas AB is its length■

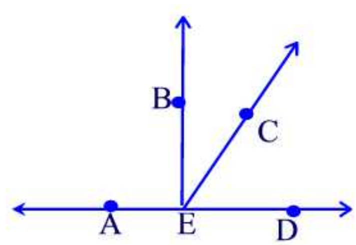
Problem 38.11
A fourth grader thinks that $m(\angle A)$ is greater than $m(\angle B)$.



(a) Why might the child think this?
(b) How could you put the angles together to show that $m(\angle A) < m(\angle B)$?

Solution.
(a) $\angle A$ appears to have longer sides.
(b) Put the vertex of $\angle B$ on top of the vertex of $\angle A$ then you see that the opening of $\angle A$ is smaller than that of $\angle B$ ■

Problem 38.12
In the figure below



(a) name two supplementary angles
(b) name two complementary angles.

Solution.
(a) $\angle CED$ and $\angle BEC$
(b) $\angle CED$ and $\angle AEC$ ■

Problem 38.13
An angle measures 20°. What is the measure of
(a) its supplement? (b) its complement?

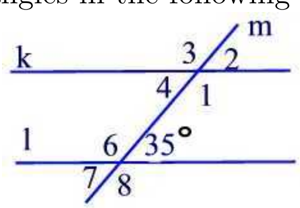
Solution.
(a) $160°$
(b) $70°$ ■

Problem 38.14
True or false? If false give an example.

(a) All right angles are congruent.
(b) Two complementary angles are congruent.
(c) Two supplementary angles are congruent.

Solution.
(a) True
(b) False. For example, $20°$ and $70°$ are complementary angles but are not congruent.
(c) False. For example, $20°$ and $160°$ are supplementary angles but are not congruent■

Problem 38.15
Find the measures of the angles in the following figure.



Solution.
$m(\angle 4) = 35°$ (corresponding angles).
$m(\angle 6) = 180° - 35° = 145°$
$m(\angle 2) = m(\angle 4) = 35°$ (vertical angles)
$m(\angle 8) = m(\angle 6) = 145°$ (vertical angles)
$m(\angle 7) = 35°$ (vertical angles)
$m(\angle 3) = m(\angle 1) = 180° - m(\angle 4) = 180° - 35° = 145°$ ■

Problem 38.16
How many degrees does the minute hand of a clock turn through

(a) in sixty minutes?
(b) in five minutes?
(c) in one minute?

Solution.
(a) $360°$
(b) $\frac{360°}{12} = 30°$
(c) $6°$■

Problem 38.17
How many degrees does the hour hand of a clock turn through

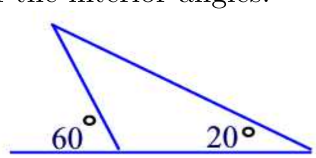
(a) in sixty minutes?
(b) in five minutes?

Solution.
(a) $\frac{360°}{12} = 30°$
(b) $\frac{5\times30°}{60} = 2.5°$■

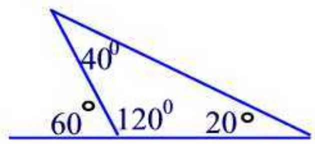
Problem 38.18
Find the angle formed by the minute and hour hands of a clock at these times.
(a) 3:00   (b) 6:00   (c) 4:30   (d) 10:20

Solution.
(a) $90°$
(b) $180°$
(c) The minute hand is on 6 and the hour hand is midway between 4 and 5. So the angle between the two is $30° + 15° = 45°$
(d) The minute hand is on 4 and the hour hand is $\frac{1}{3}$ away from 10 so $\frac{2}{3}$ away from 11. Hence, the angle is $10° + 10° + 4 \times 30° = 140°$■

Problem 38.19
Determine the measures of the interior angles.



Solution.

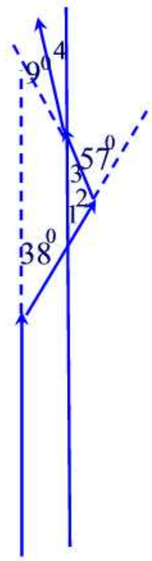


Problem 38.20
(a) Can a triangle have two obtuse angles? Why?
(b) Can a triangle have two right angles? Why?
(c) Can a triangle have two acute angles?

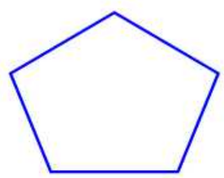
Solution.
(a) No, because an obtuse angle has measure greater than 90° and adding two such measures would exceed 180° which is the sum of all three interior angles for any triangle.
(b) No, same reaon as in (a).
(c) Yes■

Problem 38.21
A hiker started heading due north, then turned to the right 38°, then turned to the left 57°, and next turned right 9°. To resume heading north, what turn must be made?

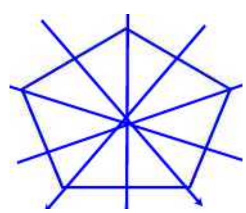
Solution.
From the figure below we see that $m(\angle 4) + 9° = m(\angle 3)$ (vertical angles). Also, we have $m(\angle 1) = 38°$ (alternate interior angles) and $m(\angle 2) = 180° - 57° = 123°$. Hence, $m(\angle 3) = 180° - 38° - 123° = 19°$. It follows that $m(\angle 4) = 10°$. So the hiker needs to make a turn to the right by 10° ■



Problem 39.1
Draw all lines of symmetry for the figure below.

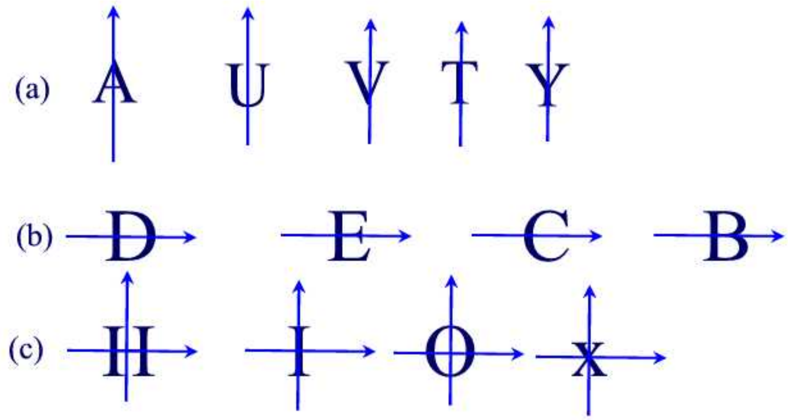


Solution.

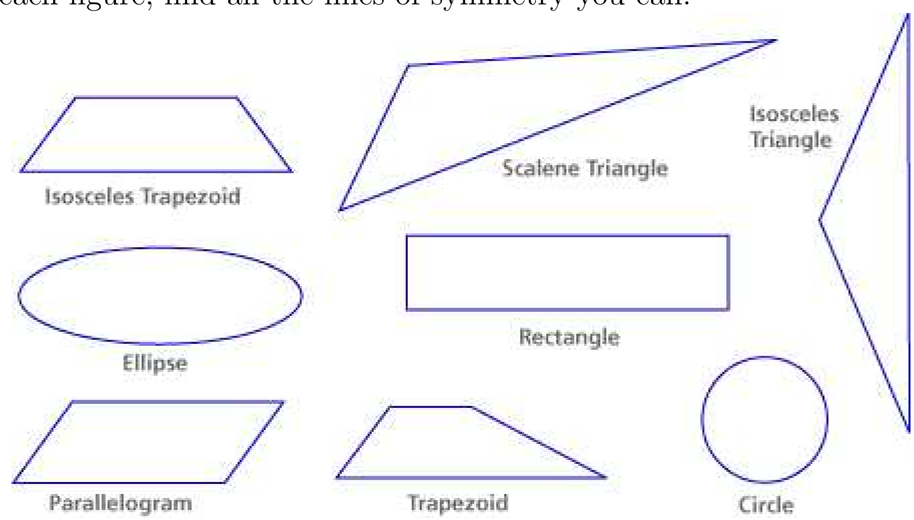


Problem 39.2
(a) Find the vertical line(s) of symmetry of the letters A, U, V, T, Y.
(b) Find the horizontal line(s) of symmetry of the letters D, E, C, B.
(c) Find the vertical and horizontal line(s) of symmetry of the letters H, I, O, X.

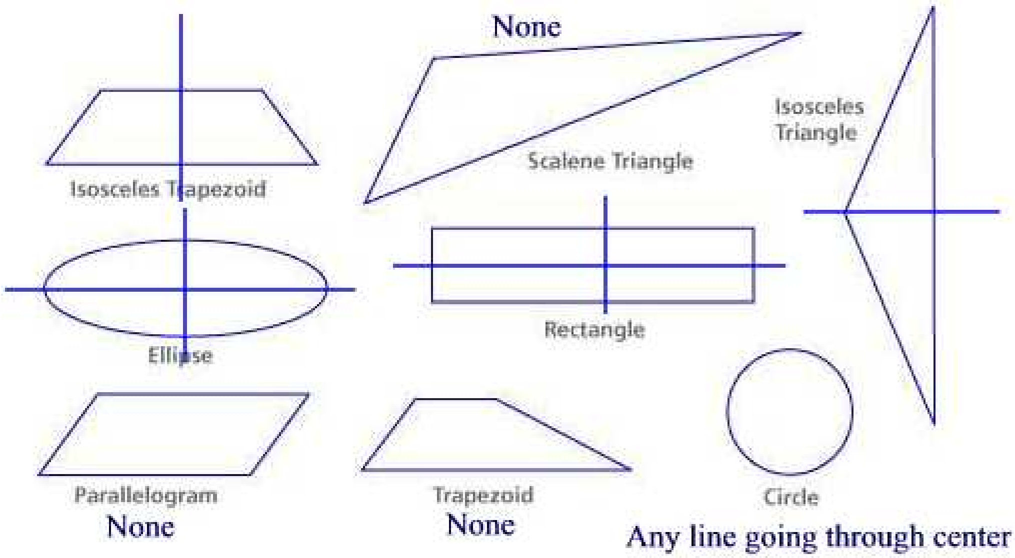
Solution.



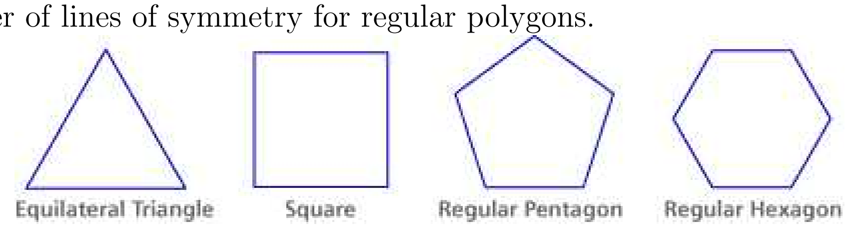
Problem 39.3
For each figure, find all the lines of symmetry you can.



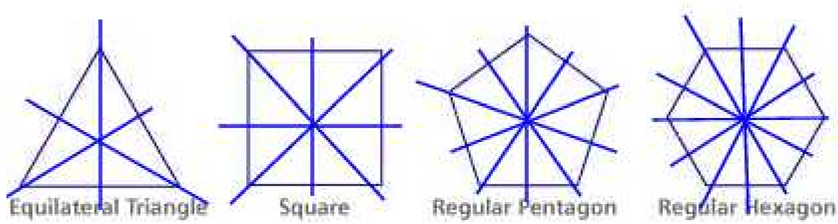
Solution.



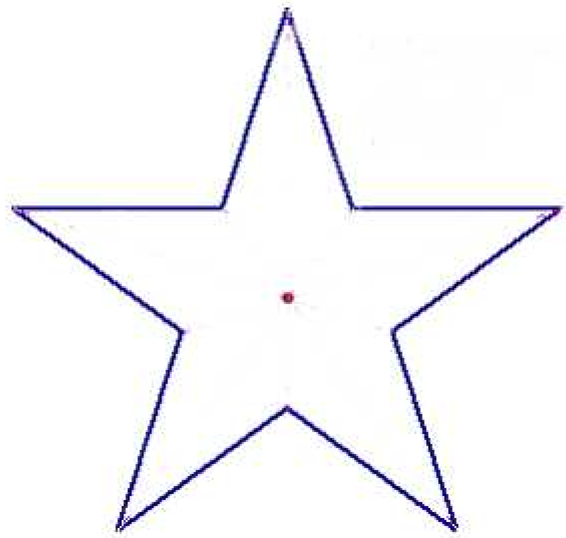
Problem 39.4
A regular polygon is a closed figure with all sides congruent. Find all the lines of symmetry for these regular polygons. Generalize a rule about the number of lines of symmetry for regular polygons.



Solution.

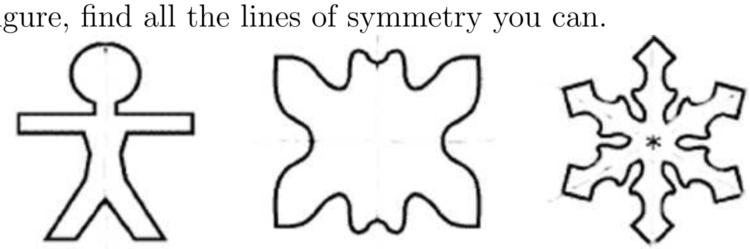


Problem 39.5
Find the number of rotations of the following geometric shape.

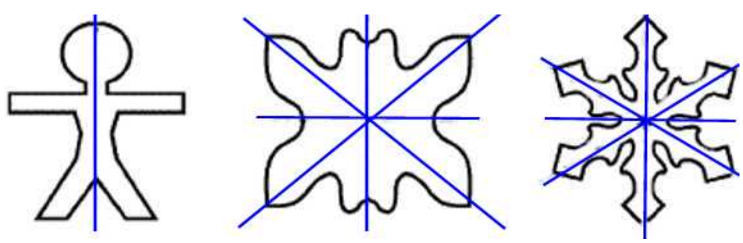


Solution.
The rotational symmetries are: $72°, 144°, 216°, 288°$ and $360°$ ■

Problem 39.6
For each figure, find all the lines of symmetry you can.

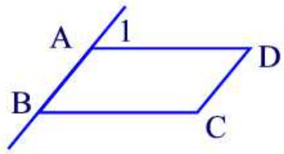


Solution.

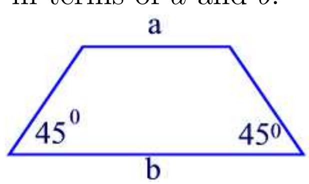


Problem 39.7
Let ABCD be a parallelogram.
(a) Prove that $\angle A$ and $\angle B$ are supplementary.
(b) Prove that angles $\angle A$ and $\angle C$ are congruent.

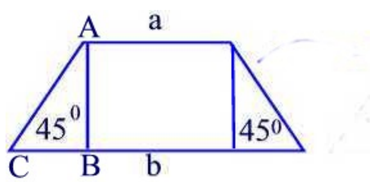
Solution.
(a) According to the figure below we have that $m(\angle 1) = m(\angle B)$ (corresponding angles). But $m(\angle 1) + m(\angle A) = 180°$. Thus, $m(\angle A) + m(\angle B) = 180°$
(b) From (a) we deduce that $m(\angle A) + m(\angle B) = 180°$ and $m(\angle C) + m(\angle B) = 180°$. Thus, $m(\angle A) + m(\angle B) = m(\angle C) + m(\angle B)$. Subtracting $m(\angle B)$ from both sides we find $m(\angle A) = m(\angle C)$ ■



Problem 39.8
Using the figure below find the height of the trapezoid(i.e., the distance between the parallel sides) in terms of a and b.



Solution.
The triangle ABC is isosceles with $BC = \frac{b-a}{2}$. Thus, $AB = \frac{b-a}{2}$ ■



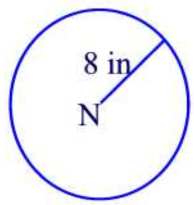
Problem 39.9
"The diagonals of a rectangle are congruent." Why does this statement imply that the diagonals of a square must also be congruent?

Solution.
Becuase is square is at the same time a rectangle ■

Problem 39.10
You learn the theorem that the diagonals of a parallelogram bisect each other. What other quadrilaterals must have this property?

Solution.
Since a rhombus, a rectangle, and a square are all quadrilaterals so they share all the properties of a parallelogram ■

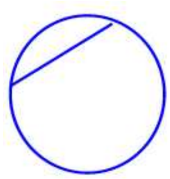
Problem 39.11
Fill in the blank to describe the following circle with center N. Circle N is the set of _____ in a plane that are _____ from ____.



Solution.
Circle N is the set of *points* in a plane that are *8 in.* from N ■

Problem 39.12
A **chord** is a line segment with endpoints on a circle. True or false? Every chord is also a diameter of the circle.

Solution.
False. The figure below shows a chord that is not a diameter

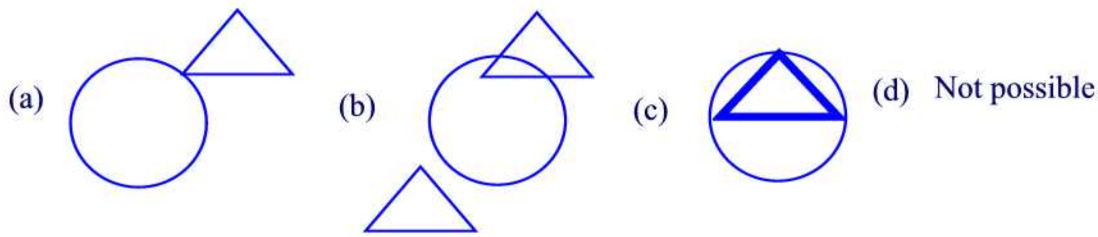


Problem 39.13
The diameter of a circle divides its interior into two congruent regions. How can someone use this property in dividing a circular pizza or pie in half?

Solution.
By cutting along a straight line that passes through the center of the pizaa you divide it in half ■

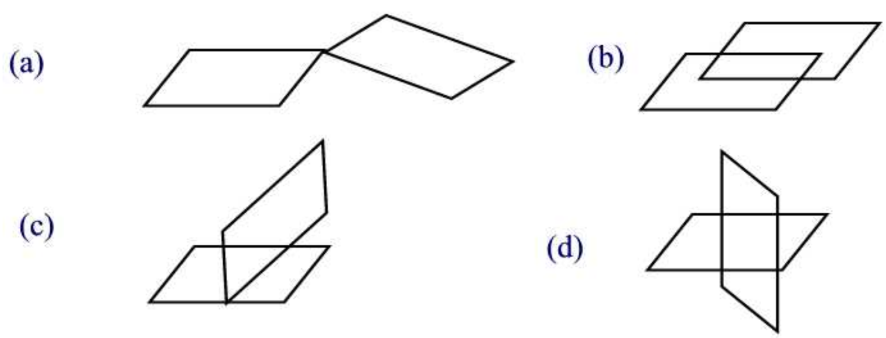
Problem 39.14
If possible draw a triangle and a circle that intersect at exactly
(a) one point (b) two points (c) three points (d) four points

Solution.



Problem 39.15
If possible draw two parallelograms that intersect at exactly
(a) one point (b) two points (c) three points (d) four points

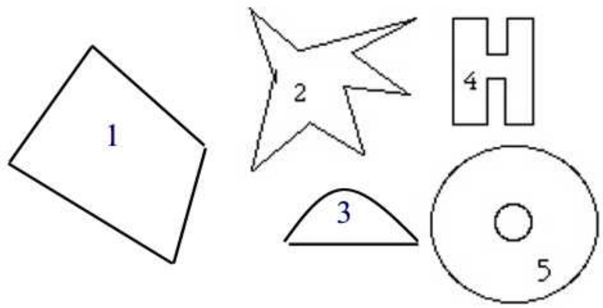
Solution.



Problem 39.16
A quadrilateral has two right angles. What can you deduce about the measures of the other two angles?

Solution.
The remaining the angles are supplementary since the some of the interior angles of a quadrilateral is always 360° ■

Problem 40.1
List the numerical values of the shapes that are convex .



Solution.
The only convex shapes are 1 and 3. The rest are concave since one can find two points with line segment not entirely contained in the shape$\blacksquare$

Problem 40.2
Determine how many diagonals each of the following has:
(a) 20-gon (b) 100-gon (c) n-gon

Solution.
We look for a pattern. Let D denote the number of diagonals and n the number of sides. We have the following table

n	4	5	6	7
D	2	5	9	14

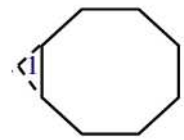
One finds that the formula for D is $D = \frac{n(n-3)}{2}$.
(a) For $n = 20, D = \frac{20(20-3)}{2} = 170$
(b) $D = \frac{100(100-3)}{2} = 4650$
(v) $D = \frac{n(n-3)}{2}$ $\blacksquare$

Problem 40.3
In a regular polygon, the measure of each interior angle is 162°. How many sides does the polygon have?

Solution.
Since $180° - \frac{360°}{n} = 162°$ then

$$\begin{aligned} 180° - \frac{360°}{n} - 180° &= 162° - -180° \\ -\frac{360°}{n} &= -18° \\ n &= \frac{360°}{18°} = 20 \ sides \ \blacksquare \end{aligned}$$

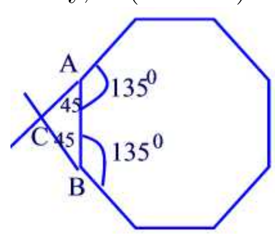
Problem 40.4
Two sides of a regular octagon are extended as shown in the following figure. Find the measure of $\angle 1$.



Solution.
Consider the triangle ABC shown in the figure below. The exterior angle at A of the triangle is just an interior angle of the regular octagon and has measure

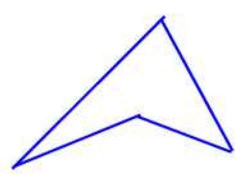
$$180^\circ - \frac{360^\circ}{n} = 180^\circ - \frac{360^\circ}{8} = 135^\circ$$

Thus, $m(\angle CAB) = 45^\circ$. Similarly, $m(\angle ABC) = 45^\circ$. Hence, $m(\angle 1) = 90^\circ$ ∎



Problem 40.5
Draw a quadrilateral that is not convex.

Solution.



Problem 40.6
What is the sum of the interior angle measures of a 40-gon?

Solution.
For an $n-$gone we have seen in the notes that the sum of all interior angles is given by the formula

$$(n-2) \times 180^\circ$$

For $n = 40$ we find

$$(40 - 2) \times 180° = 6840° \blacksquare$$

Problem 40.7
A Canadian nickel has the shape of a regular dodecagon (12 sides). How many degrees are in each interior angle?

Solution.
Each interior angle measures

$$180° - \frac{360°}{12} = 150° \blacksquare$$

Problem 40.8
Is a rectangle a regular polygon? Why or why not?

Solution.
A rectangle is polygon that is not regular since the sides are not all congruent $\blacksquare$

Problem 40.9
Find the measures of the interior, exterior, and central angles of a 12-gon.

Solution.
The measure of the interior angle is

$$180° - \frac{360°}{12} = 150°$$

The measure of the exterior angle which is also the measure of the central angle is

$$\frac{360°}{12} = 30° \blacksquare$$

Problem 40.10
Suppose that the measure of the interior angle of a regular polygon is 176°. What is the measure of the central angle?

Solution.
Since the measure of the interior angle is given by the formula

$$180^\circ - \frac{360^\circ}{n}$$

and that of the central angle is $\frac{360^\circ}{n}$ then from the previous equation we find

$$\frac{360^\circ}{n} = 180^\circ - 176^\circ = 4^\circ \blacksquare$$

Problem 40.11
The measure of the exterior angle of a regular polygon is 10°. How many sides does this polygon have?

Solution.
Since the measure of the exterior angle is equal to the measure of the interior angle then we have

$$\frac{360^\circ}{n} = 10^\circ$$

Solving this equation for n we find $n = \frac{360^\circ}{10^\circ} = 36$ *sides* $\blacksquare$

Problem 40.12
The measure of the central angle of a regular polygon is 12°. How many sides does this polygon have?

Solution.
We have

$$\frac{360^\circ}{n} = 12^\circ$$

Solving this equation for n we find $n = \frac{360^\circ}{12^\circ} = 30$ *sides* $\blacksquare$

Problem 40.13
The sum of the measures of the interior angles of a regular polygon is 2880°. How many sides does the polygon have?

Solution.
The sum of the measures of the interior angles of a regular polygon is given by the formula

$$(n-2) \cdot 180^\circ$$

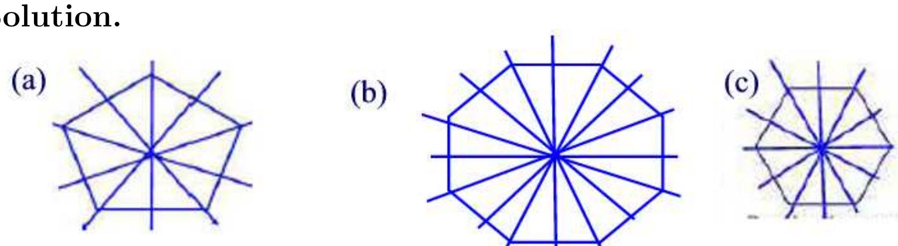
Thus,

$$\begin{array}{rcl}(n-2)\cdot 180^\circ & = & 2880^\circ \\ n-2 & = & \frac{2880^\circ}{180^\circ} = 16 \\ n-2+2 & = & 16+2 \\ n & = & 18\ sides\ \blacksquare\end{array}$$

Problem 40.14
How many lines of symmetry does each of the following have?

(a) a regular pentagon
(b) a regular octagon
(c) a regular hexagon.

Solution.



Five lines of symmetries Eight lines of symmetries Six lines of symmetries

Problem 40.15
How many rotational symmetry does a pentagon have?

Solution.
Since each central angle of a pentagon measures

$$\frac{360^\circ}{5} = 72^\circ$$

then the rotational symmetries are: $72^\circ, 144^\circ, 216^\circ, 288^\circ$ and 360° $\blacksquare$

Problem 41.1
Determine the measures of all dihedral angles of a right prism whose bases are regular octagons.

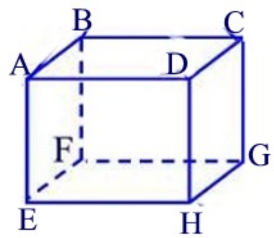
Solution.
Since the prism is a right prism, the dihedral angles made by either base to

any lateral face measure 90°. Now, the measure of the interior angles of the regular octagon is

$$180^\circ - \frac{360^\circ}{8} = 135^\circ$$

then the lateral faces meet at 135°. The dihedral angle between a lateral face and a base is 90° ■

Problem 41.2
Consider the cube given in the figure below.



(a) How many planes are determined by the faces of the cube?
(b) Which edges of the cube are parallel to edge $\overline{AB}$?
(c) Which edges of the cube are contained in lines that are skew to the line going through A and B?

Solution.
(a) Six planes: ABCD, EFGH, ABFE, CDHG, BCGF, ADHE
(b) CD, GH, EF
(c) DH, CG, FG, EH ■

Problem 41.3
If a pyramid has an octagon for a base, how many lateral faces does it have?

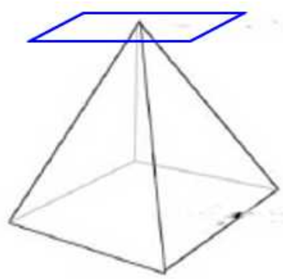
Solution.
Eight lateral faces since a lateral face is determined by the apex and a side of the octagon ■

Problem 41.4
Determine the number of faces, vertices, and edges for a hexagonal pyramid.

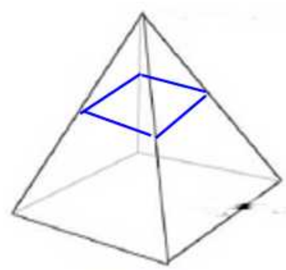
Solution.
There are seven faces, seven vertices, and twelve edges ■

Problem 41.5
Sketch drawings to illustrate different possible intersections of a square pyramid and a plane.

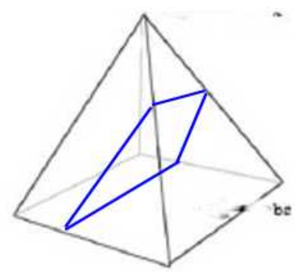
Solution.
Some of the cross sections that can be obtained: A plane can intersect the pyramid at exactly one point. If the plane is perpendicular to the line connecting the apex and the center of the square then a cross section is a square. If the plane crosses the base then the cross section is a rectangle. A cross-section can consists of a line segement, an edge of the pyramid. Finally, a cross-section can be a triangle ■



a point



a aquare



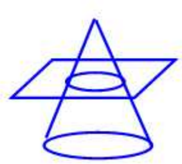
a rectangle

Problem 41.6
Sketch each of the following.

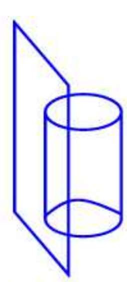
(a) A plane and a cone that intersect in a circle.
(b) A plane and a cylinder that intersect in a segment.
(c) Two pyramids that intersect in a triangle.

Solution.

(a)

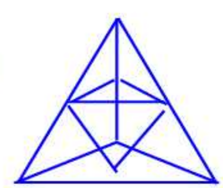


(b)

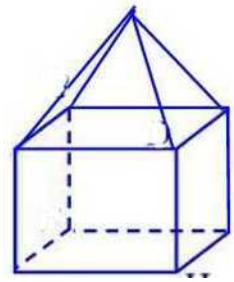


tangent plane

(c)



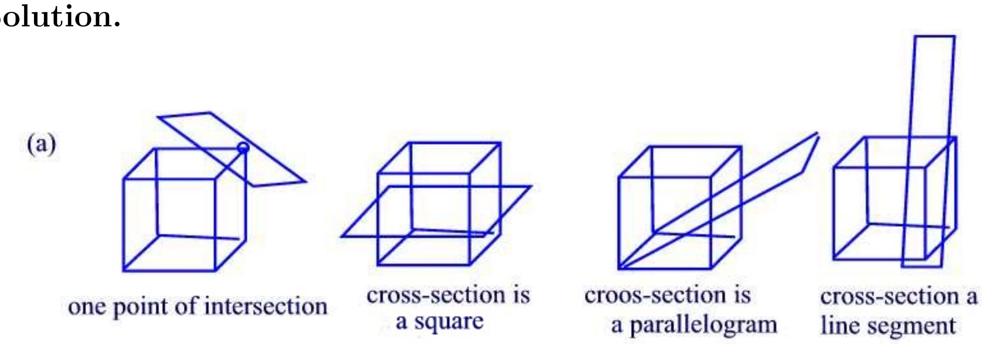
Problem 41.7
For the following figure, find the number of faces, edges, and vertices.



Solution.
Nine faces, sixteen edges, Nine vertices ■

Problem 41.8
For each of the following figures, draw all possible intersections of a plane with
(a) Cube (b) Cylinder

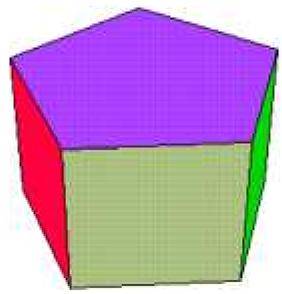
Solution.



(b) A cross-section can be a point, a circle, a line segement, or an oval ■

Problem 41.9
A diagonal of a prism is any segment determined by two vertices that do not lie in the same face. How many diagonals does a pentagonal prism have?

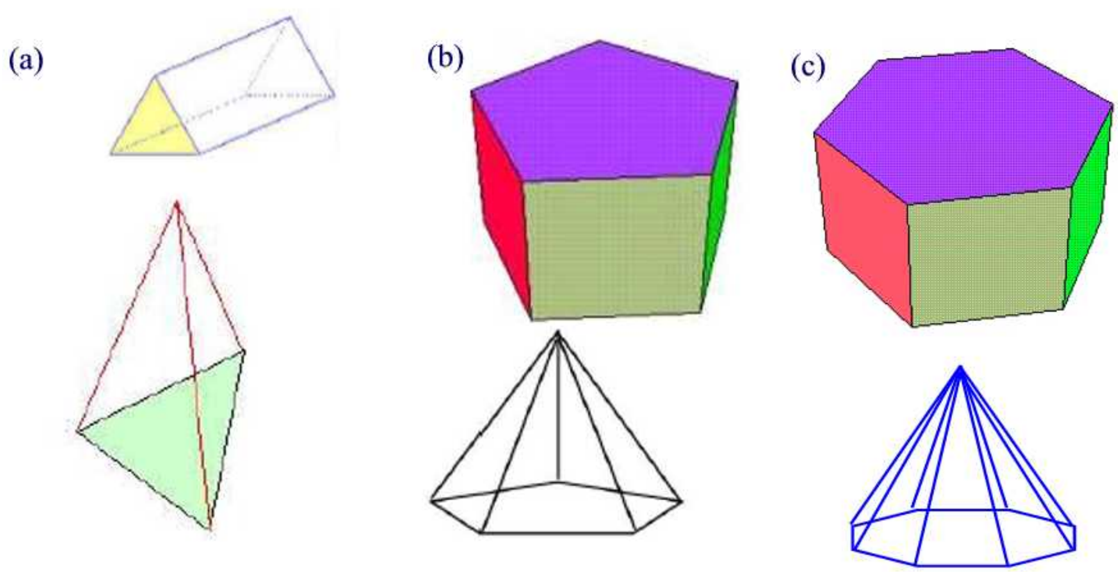
Solution.
$2 \times 5 = 10$ diagonals ■



Problem 41.10
For each of the following, draw a prism and a pyramid that have the given region as a base:

(a) Triangle
(b) Pentagon
(c) Regular hexagon

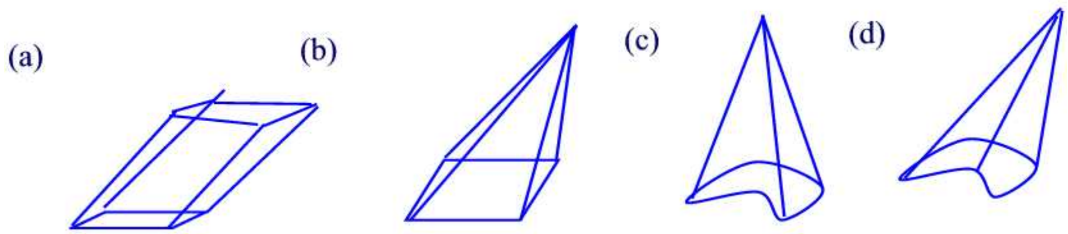
Solution.



Problem 41.11
If possible, sketch each of the following:
(a) An oblique square prism
(b) An oblique square pyramid

(c) A noncircular right cone
(d) A noncircular cone that is not right

Solution.



Problem 41.12
A simple relationship among the number of faces(F), the number of edges(E), and the number of vertices(V) of any convex polyhedron exists and is known as **Euler's formula.** Use the following table to find this relationship.

Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Hexahedron	6	12	8
Icosahedron	20	30	12
Dodecahedron	12	30	20

Solution.

$$V + F = E + 2$$

where V=vertex, F=face, E=edge ■

Problem 41.13
What is the sum of the angles at the each vertex of a
(a) Tetrahedron
(b) Octahedron
(c) Cube
(d) Icosahedron
(e) Dodecahedron

Solution.
(a) The faces consist of equilateral triangles so that each interior angle measures $60°$. Thus, the sum of the angles at each vertex is $180°$
(b) There are 8 faces consisting of equilateral triangles. Thus, the sum of angles at each vertex is $240°$
(c) The sum of angles at each vertex is $270°$
(d) There are five equilateral triangles at each vertex so that the sum of the angles at each vertex is $300°$
(e) The faces consist of pentagons so the measure of each interior angle is $108°$. The number of pentagons at each vertex is three so that the sum of the angles at each vertex is $324°$ ■

Problem 41.14
Determine for each of the following the minimum number of faces possible:
(a) Prism (b) Pyramid (c) Hexahedron

Solution.
(a) Five (triangular base) (b) Four (triangular base) (c) Six ■

Problem 41.15
True or false?
(a) Through a given point not on plane P, there is exactly one line parallel to P.
(b) Every set of four points is contained in one plane.
(c) If a line is perpendicular to one of two parallel planes then it is perpendicular to the other.

Solution.
(a) False. Any line through the point not crossing P is parallel to P
(b) False. Three distinct points determine a unique plane. Take the fourth point oustide that plane.
(c) True ■

Problem 41.16
Tell whether each of the following suggests a polygon or a polygonal region.
(a) A picture frame.
(b) A page in this book.
(c) A stop sign.

Solution.
(a) Polygon
(b) Polygonal region
(c) Polygonal region ■

Problem 41.17
A certain prism has 20 vertices. How many faces and edges does it have?

Solution.
Decagonal prism has 20 vertices, 12 faces (12 regular pentagons) and 30 edges ■

Problem 41.18
Why are there no such things as skew planes?

Solution.
Two distinct planes must be parallel or intersecting ■

Problem 41.19
A prism has a base with n sides.

(a) How many faces does it have?
(b) How many vertices does it have?
(c) How many edges does it have?

Solution.
(a) $n+2$
(b) $2n$
(c) $3n$ ■

Problem 41.20
A pyramid has a base with n sides.

(a) How many faces does it have?
(b) How many vertices does it have?
(c) How many edges does it have?

Solution.
(a) $n+1$ faces
(b) $n+1$ vertices
(c) $2n$ ■

Problem 42.1
A small bottle of Perrier sparking water contains 33 cL. What is the volume in mL?

Solution.
We have: 1 cL = 10 mL so that 33 cL = 330 mL ■

Problem 42.2
Fill in the blanks.
(a) 58728 g = ____ kg
(b) 632 mg = ____ g
(c) 0.23 kg = ____ g

Solution.
(a) 58728 g = $58728\ g \times \frac{1\ kg}{1000\ g} = 58.728\ kg$
(b) 632 mg = $632\ mg \times \frac{1\ g}{1000\ mg} = 0.632\ g$
(c) 0.23 kg = $0.23\ kg \times \frac{1000\ g}{1\ kg} = 230\ g$ ■

Problem 42.3
Convert each of the following.
(a) 100 in = ____ yd
(b) 400 yd = ____ in
(c) 300 ft = ____ yd
(d) 372 in = ____ ft

Solution.
(a) 100 in = $100\ in \times \frac{1\ ft}{12\ in} \times \frac{1\ yd}{3\ ft} = \frac{100}{36} = \frac{25}{9}\ yd$
(b) 400 yd = $400\ yd \times \frac{3\ ft}{1\ yd} \times \frac{12\ in}{1\ ft} = 14400\ in$
(c) 300 ft = $300\ ft \times \frac{1\ yd}{3\ ft} = 100\ yd$
(d) 372 in = $372\ in \times \frac{1\ ft}{12\ in} = 31\ ft$ ■

Problem 42.4
Complete the following table.

Item	m	cm	mm
Length of a piece of paper		35	
Height of a woman	1.63		
Width of a filmstrip			35
Length of a cigarette			10

Solution.

Item	m	cm	mm
Length of a piece of paper	0.35	35	350
Height of a woman	1.63	163	1630
Width of a filmstrip	0.035	3.5	35
Length of a cigarette	0.1	10	100

Problem 42.5
List the following in decreasing order: 8 cm, 5218 mm, 245 cm, 91 mm, 6 m, 700 mm.

Solution.
We have: 8 cm = 80 mm, 245 cm = 2450 mm, 6 m =6000 mm. Thus

$$8\ cm < 91\ mm < 700\ mm < 2450\ mm < 5218\ mm < 6\ m \blacksquare$$

Problem 42.6
Complete each of the following.
(a) 10 mm = ____ cm
(b) 3 km = ____ m
(c) 35 m = ____ cm
(d) 647 mm = ____ cm

Solution.
(a) 10 mm = $10\ mm \times \frac{1\ cm}{10\ mm} = 1\ cm$
(b) 3 km = $3\ km \times \frac{1000\ m}{1\ km} = 3000\ m$
(c) 35 m = $35\ m \times \frac{100\ cm}{1\ m} = 3500\ cm$
(d) 647 mm = $647\ mm \times \frac{1\ cm}{10\ mm} = 64.7\ cm \blacksquare$

Problem 42.7
Complete the following conversions.
(a) 3 feet = ____ inches
(b) 2 miles = ____ feet
(c) 5 feet = ____ yards

Solution.
(a) 3 ft = $3\ ft \times \frac{12\ in}{1\ ft} = 36\ in$
(b) 2 mi = $2\ mi \times \frac{5280\ ft}{1\ mi} = 10560\ ft$
(c) 5 ft = $5\ ft \times \frac{1\ yd}{3\ ft} = \frac{5}{3}\ ft \blacksquare$

Problem 42.8
Complete the following conversions.
(a) 7 yards = ____ feet
(b) 9 inches = ____ feet
(c) 500 yards = ____ miles

Solution.
(a) 7 yd = $7\ yd \times \frac{3\ ft}{1\ yd} = 21\ ft$
(b) 9 in = $9\ in \times \frac{1\ ft}{12\ in} = 0.75\ ft$
(c) 500 yd = $500\ yd \times \frac{1\ mi}{5280\ ft} \times \frac{3\ ft}{1\ yd} = \frac{25}{88}\ mi$ ∎

Problem 42.9
Complete the following conversions.
(a) 9.4 L = ____ mL
(b) 37 mg = ____ g
(c) 346 mL = ____ L

Solution.
(a) 9.4 L = $9.4\ L \times \frac{1000\ mL}{1\ L} = 9400\ mL$
(b) 37 mg = $37\ mg \times \frac{1\ g}{1000\ mg} = 0.037\ g$
(c) 346 mL = $346\ mL \times \frac{1\ L}{1000\ mL} = 0.346\ L$ ∎

Problem 42.10
A nurse wants to give a patient 0.3 mg of a certain drug. The drug comes in a solution containing 0.5 mg per 2 mL. How many milliters should be used?

Solution.
The nurse must use
$$\frac{0.3 \times 2}{0.5} = 1.2\ mL \blacksquare$$

Problem 42.11
A nurse wants to give a patient 3 gm of sulfisoxable. It comes in 500 mg tablets. How many tablets should be used?

Solution.
Since 3 g = 3000 mg and $3000 \div 500 = 6$ then the nurse must use 6 tablets ∎

Problem 42.12
Complete the following conversions.
(a) 3 gallons = ____ quarts
(b) 5 cups = ____ pints
(c) 7 pints = ____ quarts
(d) 12 cups = ____ gallons

Solution.
(a) 3 gallons = $3\ gallons \times \frac{4\ quarts}{1\ gallon} = 12\ quarts$
(b) 5 cups = $5\ cups \times \frac{1\ pint}{2\ cups} = 2.5\ pints$
(c) 7 pints = $7\ pints \times \frac{1\ quart}{2\ pints} = 3.5\ quarts$
(d) 12 cups = $12\ cups \times \frac{1\ pint}{2\ cups} \times \frac{1\ quart}{2\ pints} \times \frac{1\ gallon}{4\ quarts} = 0.75\ gallons$ ■

Problem 42.13
True or false? Expain.
(a) 1 mm is longer than 1 in.
(b) 1 m is longer than 1 km.
(c) 1 g is heavier than 1 lb.
(d) 1 gallon is more than 1 L.

Solution.
(a) Since 1 in = 2.54 cm = 25.4 mm then the statement is false
(b) Since 1 km = 1000 m then the statement is false
(c) Since 1 lb = 16 oz = 16×29 g = 464 g then the statement is false
(d) Since $1\ gal \approx 3.79\ L$ then the statement is true ■

Problem 42.14
Derive a conversion formula for degrees Celsius to degrees Fahrenheit.

Solution.
F is a linear function of C. That is, $F = mC + b$. But $F = 32$ when $C = 0$ so that $b = 32$. Also, $F = 212$ when $C = 100$ so that $m = \frac{212-32}{100} = \frac{9}{5}$. Thus,

$$F = \frac{9}{5}C + 32 \blacksquare$$

Problem 42.15
A temperature of $-10°C$ is about
(a) $-20°F$ (b) $10°F$ (c) $40°F$ (d) $70°F$

Solution.
$F = -10 \times \frac{9}{5} + 32 = 14°F$ so that (b) is the closest choice ■

Problem 42.16
Convert the following to the nearest degree
(a) Moderate oven ($350°F$) to degrees Celsius.
(b) $20°C$ to degrees Fahrenheit.
(c) $-5°C$ to degrees Fahrenheit.

Solution.
(a) $350 = \frac{9}{5}C + 32$. Solving for C we find $C \approx 177°C$
(b) $F = \frac{9}{5} \times 20 + 32 = 68°F$
(c) $F = \frac{9}{5} \times (-5) + 32 = 23°F$ ■

Problem 42.17
Complete the following conversions.
(a) $1\ cm^2 =$ ____ mm^2
(b) $610\ dam^2 =$ ____ hm^2
(c) $564\ m^2 =$ ____ km^2
(d) $0.382\ km^2 =$ ____ m^2

Solution.
(a) Since 1 cm = 10 mm then $1\ cm^2 = 100\ mm^2$
(b) Since 1 dam = 0.1 hm then $1\ dam^2 = 0.01\ hm^2$ and therefore $610\ dam^2 = 6.1\ hm^2$
(c) Since 1 m = 0.001 km then $1\ m^2 = 0.000001\ km^2$ so $564\ m^2 = 0.000564\ km^2$
(d) Since 1 km = 1000 m then $1\ km^2 = 10^6\ m^2$ so that $0.382\ km^2 = 382000\ m^2$ ■

Problem 42.18
Suppose that a bullet train is traveling 200 mph. How many feet per seconds is it traveling?

Solution.
We have

$$\frac{200\ mi}{1\ hr} = \frac{200\ mi}{1\ hr} \times \frac{5280\ ft}{1\ mi} \times \frac{1\ hr}{3600\ sec} \approx 203.3 ft/sec$$

Problem 42.19
A pole vaulter vaulted 19ft$4\frac{1}{2}$in. Find the height in meters.

Solution.
Converting to inches we find: 19ft$4\frac{1}{2}$in $= 19 \times 12 + \frac{9}{2} = 232.5\ in$. But 1 in $=$ 2.54 cm so that $232.5\ in = 232.5 \times 2.54 = 590.55\ cm = 5.9055\ m$ ■

Problem 42.20
The area of a rectangular lot is 25375 ft^2. What is the area of the lot in acres? Use the fact that 640 acres = 1 square mile.

Solution.
since 1 mi = 5280 ft then 1 $mi^2 = 27878400\ ft^2$. Thus, $25375\ ft^2 = \frac{25375}{27878400}\ mi^2 = \frac{25375}{27878400} \times 640 \approx 0.58\ acres\ qed$

Problem 42.21
A vase holds 4286 grams of water. What is the capacity in liters? Recall that the density of water is 1 g/cm^3.

Solution.
Since density $= \frac{mass}{volume}$ then the capacity of 4286 g in cm^3 is 4286 cm^3. But 1 cm^3 = 1 mL = 0.001 L. Thus, the capacity is liters is $4286 \times 0.001 = 4.286\ L$ ■

Problem 42.22
By using dimensional analysis, make the following conversions.
(a) 3.6 lb to oz
(b) 55 mi/hr to ft/min
(c) 35 mi/hr to in/sec
(d) $575 per day to dollars per minute.

Solution.
(a) 3.6 lb $= 3.6\ lb \times \frac{16\ oz}{1\ lb} = 57.7\ oz$
(b) 55 mi/hr $= \frac{55\ mi}{1\ hr} \times \frac{5280\ ft}{1\ mi} \times \frac{1\ hr}{60\ min} = 4840\ ft/min$
(c) 35 mi/hr $= \frac{35\ mi}{1\ hr} \times \frac{5280\ ft}{1\ mi} \times \frac{12\ in}{1\ ft} \times \frac{1\ hr}{3600\ sec} = 616\ in/sec$
(d) \$ 575 per day $= \frac{\$575}{1\ day} \times \frac{1\ day}{1440\ min} \approx 0.40\ dollars\ per\ minute$ ■

Problem 42.23
The density of a substance is the ratio of its mass to its volume. A chunk of oak firewood weighs 2.85 kg and has a volume of 4100 cm^3. Determine the density of oak in g/cm^3, rounding to the nearest thousandth.

Solution.
We have
$$\frac{2.85\ kg}{4100\ cm^3} = \frac{2.85\ kg}{4100\ cm^3} \times \frac{1000\ g}{1\ kg} \approx 0.695\ g/cm^3 \blacksquare$$

Problem 42.24
The speed of sound is 1100 ft/sec at sea level. Express the speed of sound in mi/hr.

Solution.
Using dimensional analysis we find

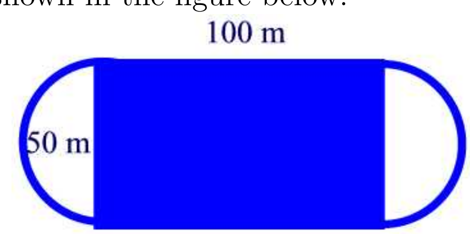
$$1100\ ft/sec = \frac{1100\ ft}{1\ sec} \times \frac{1\ mi}{5280\ ft} \times \frac{3600\ sec}{1\ hr} = 750\ mi/hr \blacksquare$$

Problem 42.25
What temperature is numerically the same in degrees Celsius and degrees Fahrenheit?

Solution.
We are asked to solve the equation $C = \frac{9}{5}C + 32$.

$$\begin{array}{rcl} C & = & \frac{9}{5}C + 32 \\ -32 & = & \frac{9}{5}C - C \\ \frac{4}{5}C & = & -32 \\ C & = & -40 \blacksquare \end{array}$$

Problem 43.1
An oval track is made up by erecting semicircles on each end of a 50-m by 100-m rectangle as shown in the figure below.



What is the perimeter of the track?

Solution.
The perimter of the rectangular part is $2 \times 100 + 2 \times 50 = 300\ m$. Since the width of the rectangle is 50 m then the radius of each semi-circle is 25 m. When the two half circles are joing they form a circle with perimeter $2\pi(25) = 50\pi\ m$. Hence, the perimeter of the oval track is $p = 300 + 50\pi - 100 = 200 + 50\pi\ m$ ■

Problem 43.2
Find each of the following:
(a) The circumference of a circle if the radius is 2 m.
(b) The radius of a circle if the circumference is 15π m.

Solution.
(a) $C = 2\pi r = 4\pi\ m$
(b) $r = \frac{C}{2\pi} = \frac{15\pi}{2\pi} = 7.5\ m$ ■

Problem 43.3
Draw a triangle ABC. Measure the length of each side. For each of the following, tell which is greater?
(a) AB + BC or AC
(b) BC + CA or AB
(c) AB + CA or BC

Solution.
(a) $AB + BC > AC$
(b) $BC + CA > AB$
(c) $AB + CA > BC$ ■

Problem 43.4
Can the following be the lengths of the sides of a triangle? Why or Why not?
(a) 23 cm, 50 cm, 60 cm
(b) 10 cm, 40 cm, 50 cm
(c) 410 mm, 260 mm, 14 cm

Solution.
According to the previous problem, the length of one side of a triangle is less than the sum of the lenghts of the remaining two sides.
(a) The given numbers can be the lengths of the sides of a triangle.
(b) Since 50 cm = 10 cm + 40 cm then the given numbers can not be the

lengths of the sides of a triangle.
(c) Since $410 > 260 + 14$ then the given numbers can not be the lengths of the sides of a triangle ■

Problem 43.5
Find the circumference of a circle with diameter 6π cm.

Solution.
We have $C = 2\pi r = \pi d = \pi(6\pi) = 6\pi^2\ cm$ ■

Problem 43.6
What happens to the circumference of a circle if the radius is doubled?

Solution.
If the radius is doubles then the new circumference is $4\pi r$. That is, the new perimeter is also doubled ■

Problem 43.7
Find the length of the side of a square that has the same perimeter as a rectangle that is 66 cm by 32 cm.

Solution. Let a be the length of the side of the square. Then $4a = 2(66 + 32) = 4 \times 49$. Divide both sides by 4 to obtain $a = 49\ cm$ ■

Problem 43.8
A bicycle wheel has a diameter of 26 in. How far a rider travel in one full revolution of the tire? Use 3.14 for π.

Solution.
We are asked to find the perimeter of the wheel. Since the wheel is a circle with diameter 26 in then its circumference is $C = \pi d = 26\pi = 26 \times 3.14 = 81.64\ in$ ■

Problem 43.9
A car has wheels with radii of 40 cm. How many revolutions per minute must a wheel turn so that the car travels 50 km/h?

Solution.
Using dimensional analysis we find

$$\frac{50\ km}{1\ hr} = \frac{50\ km}{1\ hr} \times \frac{1\ hr}{60\ min} \times \frac{1\ rev}{0.0008\pi\ km} \approx 332\ rev/min \ ■$$

Problem 43.10
A lot is 21 ft by 30 ft. To support a fence, an architect wants an upright post at each corner and an upright post every 3 ft in between. How many of these posts are needed?

Solution.
Draw a picture. There are 22 posts to cover the lengths of the rectangle and 12 posts to cover the widths a total of 34 posts ■

Problem 43.11
Convert each of the following:
(a) 1 $cm^2 =$ ____ mm^2
(b) 124,000,000 $m^2 =$ ____ km^2

Solution.
(a) Since 1 cm = 10 mm then 1 $cm^2 = 100$ mm^2
(b) Since 1 m = 0.001 km then 124000000 m^2 = 124000000 × 0.000001 = 124 km^2 ■

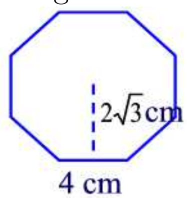
Problem 43.12
Find the cost of carpeting a 6.5 m × 4.5 m rectangular room if one meter square of carpet costs \$ 13.85.

Solution.
The total area is $6.5 \times 4.5 = 29.25$ m^2. Each square meter costs \$ 13.85 so the total cost is $29.25 \times 13.85 \approx \$405.11$ ■

Problem 43.13
A rectangular plot of land is to be seeded with grass. If the plot is 22 m × 28 m and 1-kg bag of seed is needed for 85 m^2 of land, how many bags of seed are needed?

Solution.
The area of the plot is $22 \times 28 = 616$ m^2 so that $\frac{616}{85} =\approx 7.2$ then we need 8 bags of seed ■

Problem 43.14
Find the area of the following octagon.



Solution.
A hexagon consists of 8 equilateral triangle. According to the figure, each side of the triangle is 4 cm and the height is $2\sqrt{3}$ cm. The area of a triangle is $\frac{4\times 2\sqrt{3}}{2} = 4\sqrt{3}$. Hence, the total area of the hexagon is $8 \times 4\sqrt{3} = 32\sqrt{3}$ cm^2 ■

Problem 43.15
(a) If a circle has a circumference of 8π cm, what is its area?
(b) If a circle of radius r and a square with a side of length s have equal areas, express r in terms of s.

Solution.
(a) The radius of the circle is $r = \frac{C}{2\pi} = \frac{8\pi}{2\pi} = 4$ cm. Thus, its area is $A = \pi r^2 = 16\pi$ cm^2
(b) We have $\pi r^2 = s^2$. Solving for r we find $r = \frac{s\sqrt{\pi}}{\pi}$ ■

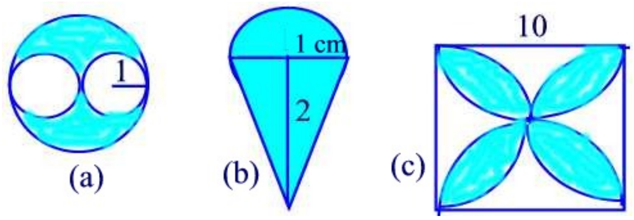
Problem 43.16
A circular flower bed is 6 m in diameter and has a circular sidewalk around it 1 m wide. Find the area of the sidewalk.

Solution.
The radius of the flower bed is 3 m. The radius of the circle consisting of the flower bed and the sidewalk is 4 m. Thus, the area of the sidewalk is $16\pi - 9\pi = 7\pi$ m^2 ■

Problem 43.17
(a) If the area of a square is 144 cm^2, what is its perimeter?
(b) If the perimeter of a square is 32 cm, what is its area?

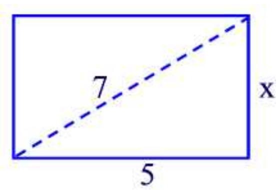
Solution.
(a) The side of the square is $s = \sqrt{144} = 12$ cm. The perimeter of the square is $p = 4s = 48$ cm
(b) The side of the square is $s = \frac{32}{4} = 8$ cm. Its area is $A = s^2 = 64$ cm^2 ■

Problem 43.18
Find the area of each of the following shaded parts. Assume all arcs are circular. The unit is cm.



Solution.
(a) The area of the larger circle is $4\pi\ cm^2$. The area of each of the smaller circles is $\pi\ cm^2$. Thus, the shaded area is $4\pi - 2\pi = 2\pi\ cm^2$
(b) The area of the half circle is $0.5\pi\ cm^2$ and the area of the triangle is $\frac{2\times 2}{2} = 2\ cm^2$. Hence, the shaded area is $2 + 0.5\pi\ cm^2$
(c) The area is twice the area of the circle of radius 5 cm minus the area of the square of side 10 cm. That is, $50\pi - 100\ cm^2$ ■

Problem 43.19
For the drawing below find the value of x.



Solution.
By the Pythagorean formula we have: $7^2 = x^2 + 5^2$ or $x^2 + 25 = 49$. Thus, $x^2 = 24$ and $x = 2\sqrt{6}$ ■

Problem 43.20
The size of a rectangular television screen is given as the length of the diagonal of the screen. If the length of the screen is 24 cm and the width is 18 cm, what is the diagonal length?

Solution.
By the Pythagorean formula, the length of the diagonal is $\sqrt{18^2 + 24^2} = 30\ cm$ ■

Problem 43.21
If the hypotenuse of a right triangle is 30 cm long and one leg is twice as long as the other, how long are the legs of the triangle?

Solution.
Let x be the length of one side. Then by the Pythagorean formula we have

$$x^2 + (2x)^2 = 30^2 = 900$$

Sovling for x we find $5x^2 = 900$ or $x^2 = 180$. Hence, $x = 6\sqrt{5}\ cm$ ■

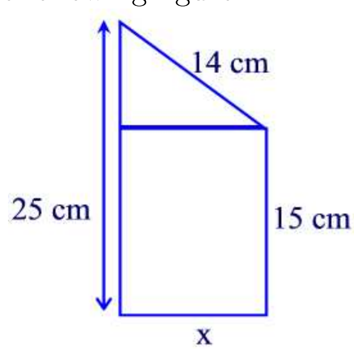
Problem 43.22
A 15-ft ladder is leaning against a wall. The base of the ladder is 3 ft from the wall. How high above the ground is the top of the ladder?

Solution.
Let x be the distance from the top of the ladder to the ground. Then by the Pythagorean formula we have

$$x^2 + 3^2 = 15^2$$

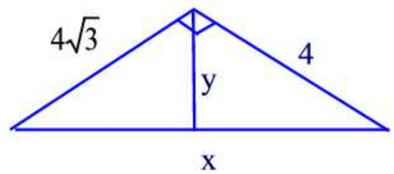
Thus, $x^2 + 9 = 225$ or $x^2 = 216$. Hence, $x = \sqrt{216} = 6\sqrt{6}\ ft$ ■

Problem 43.23
Find the value of x in the following figure.



Solution.
Let's look at the triangle. The hypotenuse is 14 cm, one leg is x cm and the other leg is 25 - 15 = 10 cm. Using the Pythagorean formula we can write $x^2 + 10^2 = 14^2$ or $x^2 + 100 = 196$. Thus, $x^2 = 96$ or $x = 4\sqrt{6}\ cm$ ■

Problem 43.24
Find x and y in the following figure.



Solution.
Using the Pythagorean formula we can write $x^2 = 4^2 + (4\sqrt{3})^2 = 64$. Thus, $x = 8$. On the other hand, $\frac{y\times 8}{2} = \frac{4\sqrt{3}\times 4}{2}$. Solving for y we find $y = 2\sqrt{3}$ ■

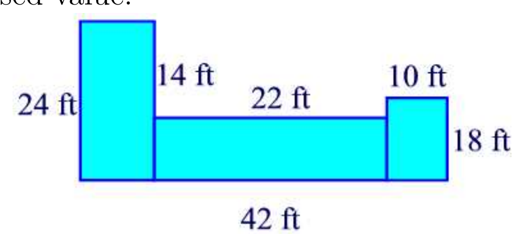
Problem 43.25
Complete the following table which concerns circles:

Radius	Diameter	Circumference	Area	
5 cm				
	24 cm			
			$17\pi\ m^2$	
		20πcm		

Solution.

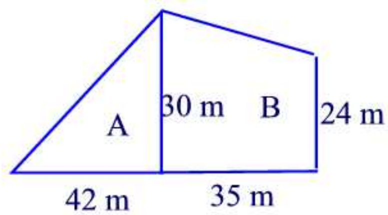
Radius	Diameter	Circumference	Area
5 cm	10 cm	10π cm	$25\pi\ cm^2$
12 cm	24 cm	24π cm	$144\pi\ cm^2$
$\sqrt{7}$ cm	$2\sqrt{7}$ cm	$2\sqrt{7}\pi$ cm	$17\pi\ cm^2$
10 cm	20 cm	20πcm	$100\pi\ cm^2$

Problem 43.26
Suppose the housing authority has valued the house shown here at \$220 per ft^2. Find the assessed value.



Solution.
The total area of the house is: $24 \times 10 + 22 \times 10 + 18 \times 10 = 640\ ft^2$. The assessed value of the house is $640 \times 220 = \$140800$ ∎

Problem 43.27
Two adjacent lots are for sale. Lot A cost \$ 20,000 and lot B costs \$27,000. Which lot has the lower cost per square meter?

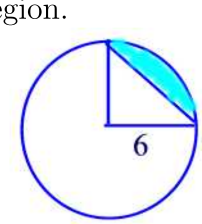


Solution.
The area of lot A is $\frac{42\times 30}{2} = 630\ m^2$ whereas that of lot B is $\frac{(24+30)\times 35}{2} = 945\ m^2$. The cost of lot A per square meter is $\frac{20000}{630} \approx \$31.75$. The cost of lot B per square meter is $\frac{27000}{945} \approx \$28.57$. Thus, lot B has the lower cost per square meter ∎

Problem 43.28
If the radius of a circle increases by 30% then by what percent does the area of the circle increase?

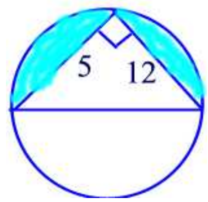
Solution.
If $A = \pi r^2$ and r increases by 30% then the new area is $\pi(1.3r)^2 = 1.69\pi r^2$ so the area increased by 69% ∎

Problem 43.29
Find the area of the shaded region.



Solution.
The area of a quarter of the circle is 9π. The area of the triangle is 18. Thus the shaded area is $9\pi - 18$ ∎

Problem 43.30
Find the area of the shaded region.



Solutio.
The diameter of the circle is: $d^2 = 5^2 + 12^2 = 25 + 144 = 169$. Thus, $d = 13$. The area of half of the circle is $\frac{169}{2}\pi$. The area of the triangle is $\frac{5\times 12}{2} = 30$. Thus the shaded area is $\frac{169}{2}\pi - 30$ ■

Problem 44.1
A small can of frozen orange juice is about 9.5 cm tall and has a diameter of about 5.5 cm. The circular ends are metal and the rest of the can is cardboard. How much metal and how much cardboard are needed to make a juice can?

Solution.
If we cut the lateral face and make it flat we will get a rectangle of height 9.5 cm and length 5.5π. Thus, the area of the cardboard is $(9.5)\cdot(5.5\pi) = 52.25\pi\ cm^2$.Tne metal ends each is a circle of radius 2.75 cm and therefore with area $\pi(2.75)^2\ cm^2$. So twice that is the area of the two metal, i.e., $15.125\ cm^2$ ■

Problem 44.2
A pyramid has a square base 10 cm on a side. The edges that meet the apex have length 13 cm. Find the slant height of the pyramid, and then calculate the total surface area of the pyramid.

Solution.
The base of the pyramid has area $B = 100\ cm^2$ and perimeter $P = 40$ cm.The slant height can be computed with the Pythagorean formula which gives $l = \sqrt{13^2 - 5^2} = 12$ cm. Thus, the surface area of the pyramid is

$$SA = 100 + \frac{1}{2}\cdot 40\cdot 12 = 340\ cm^2 \blacksquare$$

Problem 44.3
An ice cream cone has a diameter of 2.5 in and a slant height of 6 in. What is the lateral surface area of the cone?

Solution.
The lateral surface area is $\pi r l$ where r is the radius of the base circle and l is the slant height. Thus, the lateral surface area is $\pi \cdot \left(\frac{2.5}{2}\right) \cdot 6 = 7.5\pi\ in^2$ ■

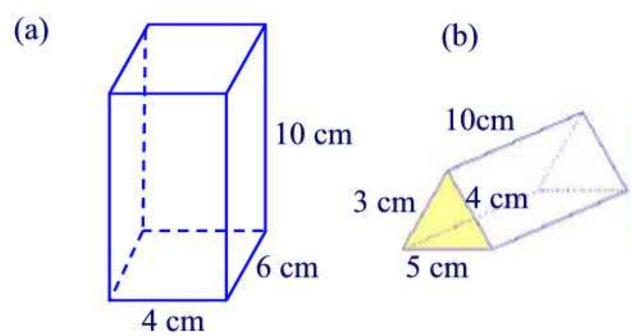
Problem 44.4
The diameter of Jupiter is about 11 times larger than the diameter of the planet Earth. How many times greater is the surface area of Jupiter?

Solution.
Let S_J be the surface area of Jupiter and S_E that of the Earth. Then

$$\frac{S_J}{S_E} = \frac{4\pi r_J^2}{4\pi r_E^2} = \frac{\left(\frac{d_J}{2}\right)^2}{\left(\frac{d_E}{2}\right)^2} = \frac{d_J^2}{d_E^2} = \frac{121 d_E^2}{d_E^2} = 121$$

Thus, the surface area of Jupiter is 121 times larger than the surface area of the Earth ■

Problem 44.5
Find the surface area of each of the right prisms below.



Solution.
(a) $SA = 2 \cdot 10 \cdot 6 + 2 \cdot 10 \cdot 4 + 2 \cdot 4 \cdot 6 = 248\ cm^2$
(b) Note that the base is a right triangle with hypotenuse of length 5 cm. The area of the base is then $\frac{3\times 4}{2} = 6\ cm^2$. Thus,

$$SA = 2(6) + (3 + 4 + 5) \cdot 10 = 132\ cm^2 \ ■$$

Problem 44.6
The Great Pyramid of Cheops is a right square pyramid with a height of 148 m and a square base with perimeter of 930 m. The slant height is 188 m. The basic shape of the Transamerica Building in San Fransisco is a right square pyramid that has a height of 260 m and a square base with a perimeter of 140 m. The altitude of slant height is 261 m. How do the lateral surface areas of the two structures compare?

Solution.
The length of one side of the square base of the Great Pyramid is $\frac{930}{4} = 232.5\ m$. The length of one side of the square base of the Transamerica Building is 35 m. The lateral surface area of each is then

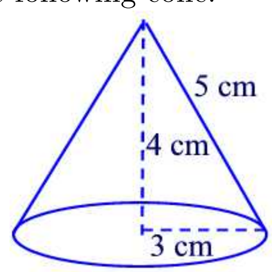
$$LSA\ of\ Great\ Pyramid = 4 \cdot \frac{1}{2} \cdot (232.5) \cdot 188 = 87420\ m^2$$

and

$$LSA\ of\ Transamerica = 4 \cdot \frac{1}{2} \cdot 35 \cdot 261 = 18270\ m^2$$

Thus, the lateral surface area of the Great Pyramid is about 4.8 times larger than that of the Transamerica Building ■

Problem 44.7
Find the surface area of the following cone.



Solution.
The base of the cone is a circle with radius 3 cm and area $9\pi\ cm^2$. The lateral surface area is $\pi \cdot 3 \cdot 5 = 15\pi\ cm^2$. Thus, the total surface area is $SA = 9\pi + 15\pi = 24\pi\ cm^2$ ■

Problem 44.8
The Earth has a spherical shape of radius 6370 km. What is its surface area?

Solution.
The surface area of the Earth is $SA = 4\pi r = 4\pi(6370) = 25480\pi\ km^2$ ■

Problem 44.9
Suppose one right circular cylinder has radius 2 m and height 6 m and another has radius 6 m and height 2 m.

(a) Which cylinder has the greater lateral surface area?
(b) Which cylinder has the greater total surface area?

Solution.
(a) The lateral surface area of the first cylinder is $2\pi \cdot 2 \cdot 6 = 24\pi\ m^2$ and that of the second cylinder is $2\pi \cdot 6 \cdot 2 = 24\pi\ m^2$. Thus, the two cylinder have the same lateral surface area.
(b) The surface area of the first cylinder is $2\pi \cdot 2 \cdot (2+6) = 32\pi\ m^2$ and that of the second cylinder is $2\pi \cdot 6 \cdot (6+2) = 96\pi\ m^2$. Thus, the total surface area of the first cylinder smaller than the total surface are of the second cylinder ■

Problem 44.10
The base of a right pyramid is a regular hexagon with sides of length 12 m. The height of the pyramid is 9 m. Find the total surface area of the pyramid.

Soluion.
The base is a hexagon with consists of 6 equilateral triangles each with sides of length 12 m. The height in any such a triangle is found by the Pythagorean formula to be $6\sqrt{3}$ m. Thus, the area of each triangle is $\frac{6\sqrt{3}\cdot 12}{2} = 36\sqrt{3}\ m^2$. Hence, the area of base is $6 \cdot 36\sqrt{3} = 216\pi\ m^2$. A face of the pyramid is a rectangle of length 12 m and width of 9 m so that the area of a face is 108 m^2. Hence, the total surface area of the pyramid is

$$SA = 6 \cdot 108 + 2 \cdot 216\pi = 648 + 432\pi\ m^2 \ ■$$

Problem 44.11
A square piece of paper 10 cm on a side is rolled to form the lateral surface area of a right circular cylinder and then a top and bottom are added. What is the surface area of the cylinder?

Solution.
The height of the cylinder is 10 cm. The radius of the base is found is follows: $2\pi r = 10$ or $r = \frac{5}{\pi}$. Thus, the total surface area of the cylinder is

$$SA = 2\pi \cdot \frac{5}{\pi} \cdot \left(\frac{5}{\pi} + 10\right) = \frac{50}{\pi} + 100\ cm^2 \blacksquare$$

Problem 44.12
The top of a rectangular box has an area of 88 cm^2. The sides have areas 32 cm^2 and 44 cm^2. What are the dimensions of the box?

Solution.
Let L be the length, W the width, and h the height. We have $LW = 88$, $Lh = 32$, and $Wh = 44$. From the last two equations we find $L = \frac{8}{11}W$. Thus, using the first equation we find $\frac{8}{11}W^2 = 88$. Solving for W we find $W = 11\ cm$. Thus, $L = \frac{8}{11} \cdot 11 = 8\ cm$ and $h = \frac{44}{11} = 4\ cm$ $\blacksquare$

Problem 44.13
What happens to the surface area of a sphere if the radius is tripled?

Solution.
The surface area of a sphere is $4\pi r^2$. If the radius is tripled then the surface area becomes $4\pi(3r)^2 = 36\pi r^2$. This shows that the surface area is 9 times larger $\blacksquare$

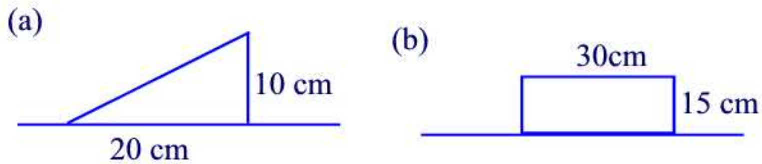
Problem 44.14
Find the surface area of a square pyramid if the area of the base is 100 cm^2 and the height is 20 cm.

Solution.
Since the area of the base is 100 then the side of the square is 10 cm. Using the Pythagorean formula we find that the slant height is $l = \sqrt{20^2 + 5^2} = \sqrt{425} = 5\sqrt{17}$. Thus, the surface area of the pyramid is

$$SA = 100 + 20 \cdot 5\sqrt{17} = 100(1 + sqrt17)\ cm^2 \blacksquare$$

Problem 44.15
Each region in the following figure revolves about the horizontal axis. For each case, sketch the three dimensional figure obtained and find its surface

area.



Solution.
(a) The three dimensional shape is a cone with radius of base 10 cm and height 20 cm. Its surface area is

$$SA = \pi \cdot 10(10 + \sqrt{100 + 400}) = 10\pi(10 + 10\sqrt{5}) = 100\pi(1 + \sqrt{5})\ cm^2$$

(b) The three dimensional shape is a cylinder of height 30 cm and radius of base 15 cm. Its surface area is

$$SA = 2\pi \cdot 15(15 + 30) = 1350\pi\ cm^2 \blacksquare$$

Problem 44.16
The total surface area of a cube is 10,648 cm^2. Find the length of a diagonal that is not a diagonal of a face.

Solution.
The surface area of a cube of side s is $6s^2$. Thus, $6s^2 = 10648$ or $s^2 = \frac{5324}{2}$. Hence, $s = \sqrt{\frac{5324}{2}} \approx 42\ cm$. The length of a diagonal that is not a diagonal of a face is found by using the Pythagorean formula:

$$d = \sqrt{2 \cdot 42^2 + 42^2} = 42\sqrt{3} \approx 73\ cm \blacksquare$$

Problem 44.17
If the length, width, and height of a rectangular prism is tripled, how does the surface area change?

Solution.
The surface area of a rectangular prism is given by the formula

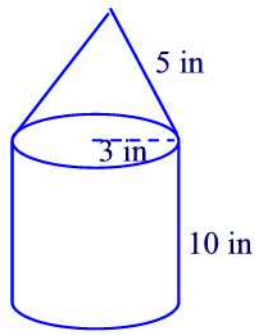
$$SA = 2lw + 2h(l + w)$$

If we triple the length, width and height then the new SA is

$$SA = 2(3l)(3w) + 2(3h)(3l + 3w) = 9(2lw + 2h(l + w))$$

Thus the new surface area is 9 times larger then the previous one $\blacksquare$

Problem 44.18
Find the surface area of the following figure.



Solution.
The surface area of the cylinder is

$$SA = 2\pi \cdot 3(3 + 10) = 78\pi \ cm^2$$

The lateral surface are of the cone is

$$LSA = \pi \cdot 3\sqrt{9 + 100} = 3\pi\sqrt{109} \ cm^2$$

Thus, the total surface area is

$$78\pi + 3\pi\sqrt{109} \ cm^2 \ \blacksquare$$

Problem 44.19
A room measures 4 meters by 7 meters and the ceiling is 3 meters high. A liter of paint covers 40 square meters. How many liters of paint will it take to paint all but the floor of the room?

Solution.
The total surface area to be painted is

$$SA = 4 \times 7 + 2 \times 3 \times 4 + 2 \times 3 \times 7 = 94 \ m^2$$

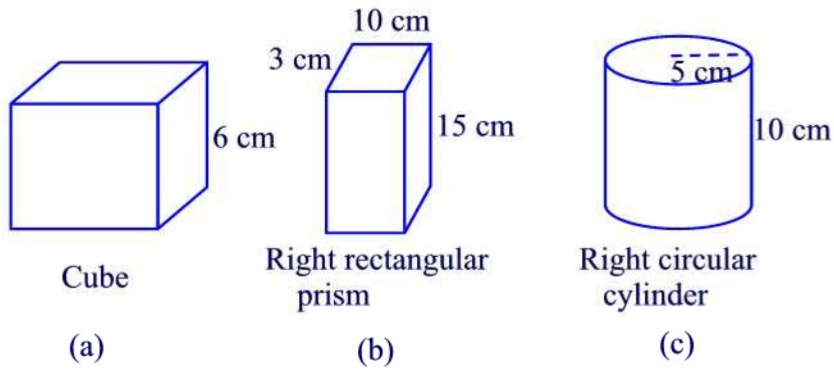
Since $\frac{94}{40} = 2.35$ then 3 liters of paint should be purchased $\blacksquare$

Problem 44.20
Given a sphere with diameter 10 , find the surface area of the smallest cylinder containing the sphere.

Solution.
The smallest cylinder containing the sphere has height 10 and radius of base 5. Thus, the surface area of the cylinder is

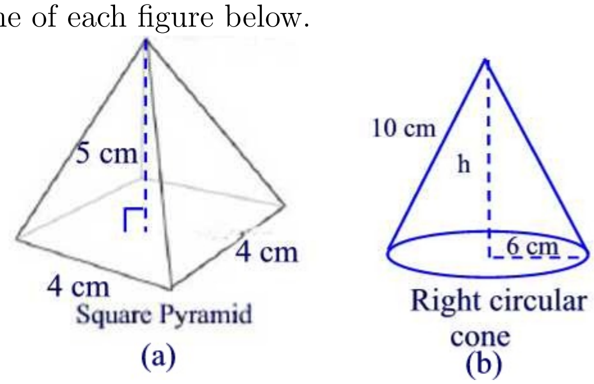
$$SA = 2\pi \cdot 5(5 + 10) = 150\pi \blacksquare$$

Problem 45.1
Find the volume of each figure below.



Solution.
(a) $V = 6^3 = 216\ cm^3$
(b) $V = 3 \cdot 10 \cdot 15 = 450\ cm^3$
(c) $V = h\pi r^2 = 10 \cdot \pi \cdot 25 = 250\pi\ cm^3$ $\blacksquare$

Problem 45.2
Find the volume of each figure below.



Solution.
(a) $V = \frac{1}{3} \cdot 4^2 \cdot 5 = \frac{80}{3}\pi\ cm^3$
(b) Using the Pythagorean formula we find $h = \sqrt{100 - 36} = 8$. Thus, $V = \frac{1}{3}\pi 6^2 \times 8 = \frac{288}{3}\pi$ $\blacksquare$

Problem 45.3
Maggie is planning to build a new one-story house with floor area of 2000 ft^2. She is thinking of putting in a 9-ft ceiling. If she does this, how many cubic feet of space will she have to heat or cool?

Solution.
The house has the shape of rectangular prism with volume equals to $2000 \times 9 = 18,000\ ft^3$ ■

Problem 45.4
Two cubes have sides lengths 4 cm and 6 cm, respectively. What is the ratio of their volumes?

Solution.
The volume of the cube with side of length 4 cm is $V_4 = \frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi$. The volume of the cube with side of length 6 cm is $V_6 = \frac{4}{3}\pi \cdot 6^3 = \frac{864}{3}\pi$. Thus,

$$\frac{V_4}{V_6} = \frac{\frac{256}{3}\pi}{\frac{864}{3}\pi} = \frac{256}{864} = \frac{8}{27}\ \blacksquare$$

Problem 45.5
What happens to the volume of a sphere if its radius is doubled?

Solution.
The volume of s a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. If we double the radius then the volume of the new sphere is $V' = \frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3 = 8V$. That is the new volume is three times larger than the old volume ■

Problem 45.6
An olympic-sized pool in the shape of a right rectangular prism is 50 m × 25 m. If it is 2 m deep throughout, how many liters of water does it hold? Recall that 1 m^3 = 1000 L.

Solution.
The volume of the prism is $V = 50 \times 25 \times 2 = 2500\ m^3 = 2500 \times 10^3 = 2500000\ L$ ■

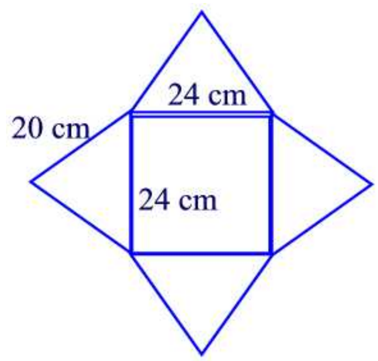
Problem 45.7
A standard straw is 25 cm long and 4 mm in diameter. How much liquid can be held in the straw at one time?

Solution.
The volume is given by $V = 25\pi \cdot 0.2^2 = \pi\ cm^3 = \pi\ mL$ ■

Problem 45.8
The pyramid of Khufu is 147 m high and its square base is 231 m on each side. What is the volume of the pyramid?

Solution.
The volume of the pyramid is $V = \frac{1}{3} \cdot 231^2 \cdot 147 = 2614689\ m^3$ ■

Problem 45.9
A square right regular pyramid is formed by cutting, folding, and gluing the following pattern.



(a) What is the slant height of the pyramid?
(b) What is the lateral surface area of the pyramid?
(c) Use the Pythagorean formula to find the height of the pyramid.
(d) What is the volume of the pyramid?

Solution.
(a) Using the Pythagorean formula we find that the slant height is $l = \sqrt{20^2 - 12^2} = 16\ cm$
(b) $LSA = 2 \cdot 24 \cdot 16 = 768\ cm^2$
(c) By the Pythagorean formula the height is $h = \sqrt{256 - 144} = \sqrt{112} = 4\sqrt{7}\ cm$
(d) $V = \frac{1}{3} \cdot 24^2 \times 4\sqrt{7} = \frac{2304}{3}\sqrt{7}\ cm^3$ ■

Problem 45.10
A cube 10 cm on a side holds 1 liter. How many liters does a cube 20 cm on a side hold?

Solution.
Since

$$V = 20^3 = 8000\ cm^3$$

then a cube 20 cm on a side hold 8 liters ■

Problem 45.11
A right circular cone has height r and a circular base of radius $2r$. Compare the volume of the cone to that of a sphere of radius r.

Solution.
The volume of the cone is $V_C = \frac{1}{3}r \cdot \pi \cdot (2r)^2 = \frac{4}{3}\pi r^3$. The volume of the sphere of radius r is $V_S = \frac{4}{3}\pi r^3$. Thus, $V_C = V_S$ ■

Problem 45.12
A store sell two types of freezers. Freezer A costs \$350 and measures 2 ft by 2 ft by 4.5 ft. Freezer B cots \$480 and measures 3 ft by 3 ft by 3.5 ft. Which freezer is the better buy?

Solution.
The volume of freezer A is $V_A = 2 \cdot 2 \cdot 4.5 = 18\ ft^3$ and that of freezer B is $V_B = 3 \cdot 3 \cdot 3.5 = 31.5\ ft^3$. The cost per cubic foot of freezer A is $\frac{350}{18} \approx \$19.44$. The cost per cubic foot of freezer B is $\frac{480}{31.5} \approx \$15.24$. Thus, Freezer B is the better buy ■

Problem 45.13
Write a sentence that tells the difference between the surface area and volume of a prism.

Solution.
The surface area measures the total area of all its surfaces or faces, while the volume measures the total amount of space in its interior ■

Problem 45.14
A cylindrical water tank has a radius of 6.0 m. About how high must be filled to hold 400.0 m^3?

Solution.
The volume of the cylinder is $V = h \cdot \pi \cdot r^2$. Thus, $h \cdot \pi \cdot 36.0 = 400.0$. Solving for h we find $h = \frac{400.0}{36.0\pi} \approx 3.5\ m$ ■

Problem 45.15
Roll an 8.5 by 11 in sheet of paper into a cylindrical tube. What is the diameter?

Solution.
The cylinder is of height 11 and circumference of base equals to 8.5. Thus, $\pi d = 8.5$ and $d = \frac{8.5}{\pi} \approx 2.7$ ■

Problem 45.16
A cylindrical pipe has an inner radius r, an outer radius R, and length l. Find its volume.

Solution.
The volume of the pipe is $\pi R^2 l - \pi r^2 l$ ■

Problem 45.17
A basketball has a diameter of 10 in. What is its volume?

Solution.
The volume is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi 5^3 = \frac{500}{3}\pi\ in^3$ ■

Problem 45.18
A standard tennis can is a cylinder that holds three tennis ball.

(a) Which is greater the circumference of the can or its height?
(b) Find the volume of the can?

Solution.
(a) Suppose that the tennis balls touch the sides, the top, and the bottom of the can. Let r be the radius of a tennis ball. Then the circumference of the can is $2\pi r$ and the height is $6r$. Since $2\pi > 6$ then $2\pi r > 6r$ and therefore the circumference of the can is larger than its height.
(b) The volume of the can is $V = (6r)(\pi r^2) = 6\pi r^3$ ■

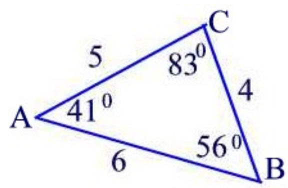
Problem 45.19
A cylindrical aquarium has a circular base with diameter 2 ft and height 3 ft. How much water does it hold, in cubic feet?

Solution.
The volume of the aquarium is $V = 3 \cdot \pi \cdot 4 = 12\pi\ ft^3$ ■

Problem 45.20
The circumference of a beach ball is 73 inches. How many cubic inches of air does the ball hold? Round your answer to the nearest cubic inch.

Solution.
The radius of the ball is $r = \frac{73}{2\pi}$. The volume of the ball is $V = \frac{4}{3}\pi\left(\frac{73}{2\pi}\right)^3 \approx 6569\ in^3$ ■

Problem 46.1
Suppose $\Delta JKL \cong \Delta ABC$, where ΔABC is shown below.



Find the following
(a) KL (b) LJ (c) $m(\angle L)$ (d) $m(\angle J)$

Solution.
(a) $KL = BC = 4$
(b) $LJ = CA = 5$
(c) $m(\angle L) = m(\angle C) = 83°$
(d) $m(\angle J) = m(\angle A) = 41°$ ■

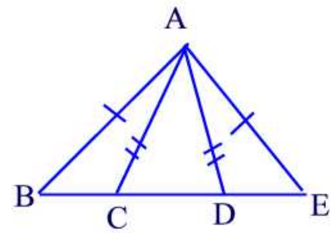
Problem 46.2
Using congruence of triangles show that equilateral triangles are equiangular.

Solution.
Let ABC be an equilateral triangles. Then $\Delta ABC \cong \Delta BAC$ by SSS property. This shows that $m(\angle A) = m(\angle B)$. Similarly, $\Delta ABC \cong \Delta ACB$ by SSS. Hence, $m(\angle B) = m(\angle C)$ ■

Problem 46.3
Let the diagonals of a parallelogram ABCD intersect at a point M.
(a) Show that $\Delta ABM \cong \Delta CDM$.
(b) Use part (a) to explain why M is the midpoint of both diagonals of the parallelogram.

Solution.
(a) We have $AB = CD, m(\angle ABM) = m(\angle CDM), m(\angle BAM) = m(\angle DCM)$ (alternate interior angles). Hence, by ASA, $\Delta ABM \cong \Delta CDM$.
(b) By part (a), $MA = MC$ so that M is the midpoint of $\overline{AC}$. Similarly, $MB = MD$ so that M is the midpoint of $\overline{BD}$ ■

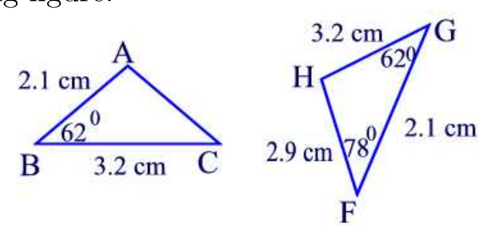
Problem 46.4
In the figure below, AB = AE and AC=AD.



(a) Show that $m(\angle B) = m(\angle E)$
(b) Show that $m(\angle ACD) = m(\angle ADC)$
(c) Show that $\Delta ABC \cong \Delta AED$
(d) Show that $BC = DE$.

Solution.
(a) Since $AB = AE$ then ΔABE is isosceles so that $m(\angle B) = m(\angle E)$
(b) Since $AC = AD$ then ΔACD is isosceles so that $m(\angle ACD) = m(\angle ADC)$
(c) By (a) and (b) we have $m(\angle ACB) = m(\angle ADE)$ and $m(\angle BAC) = m(\angle DAE)$. Hence, by SAS property we have $\Delta ABC \cong \Delta AED$
(d) By (c) we have $BC = DE$ ■

Problem 46.5
Consider the following figure.



(a) Find AC
(b) Find $m(\angle H), m(\angle A)$, and $m(\angle C)$.

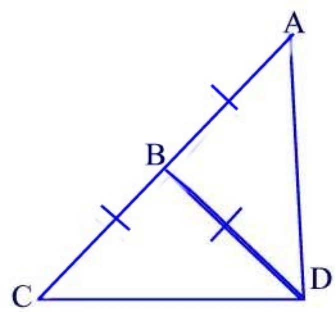
Solution.
(a) By SAS property we have $\Delta ABC \cong \Delta FGH$. Hence, $AC = FH = 2.9\ cm$

(b) $m(\angle H) = 180^\circ - (62^\circ + 78^\circ) = 40^\circ$, $m(\angle A) = m(\angle F) = 78^\circ, m(\angle C) = m(\angle H) = 40^\circ$ ■

Problem 46.6
Show that if $\Delta ABC \cong \Delta A'B'C'$ and $\Delta A'B'C' \cong \Delta A''B''C''$ then $\Delta ABC \cong \Delta A''B''C''$.

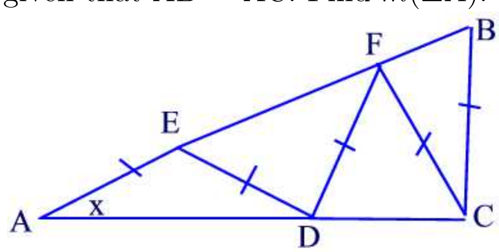
Solution.
Since $\Delta ABC \cong \Delta A'B'C'$ then $AB = A'B', AC = A'C'$, and $BC = B'C'$. Similarly, since $\Delta A'B'C' \cong \Delta A"B"C"$ then $A'B' = A"B", A'C' = A"C"$, and $B'C' = B"C"$. Thus, $AB = A"B", AC = A"C"$, and $BC = B"C"$ so that by SSS property $\Delta ABC \cong \Delta A"B"C"$ ■

Problem 46.7
In the figure below, given that AB=BC=BD. Find $m(\angle ADC)$.

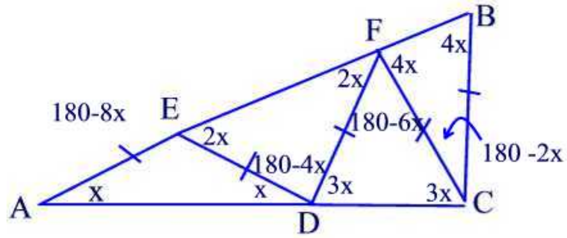


Solution.
Triangle CBD is a right isosceles triangle with $m(\angle CBD) = 90^\circ$. Hence, $m(\angle CDB) = 45^{circ}$. Similar argument shows that $m(\angle BDA) = 45^\circ$. Therefore, $m(\angle ADC) = m(\angle CDB) + m(\angle BDA) = 90^\circ$ ■

Problem 46.8
In the figure below given that AB = AC. Find $m(\angle A)$.

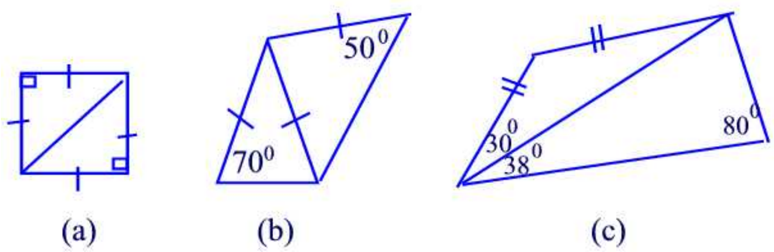


Solution.
Various angles are shown in the figure below.

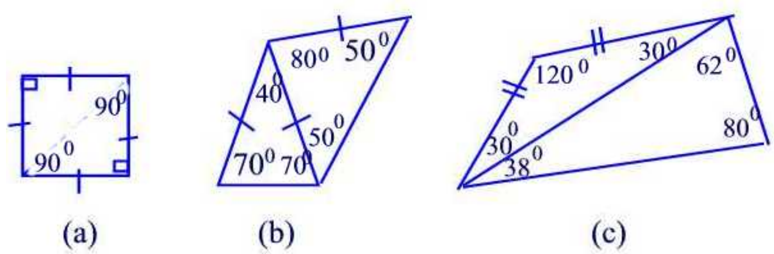


Since $AB = AC$ then triangle ABC is isosceles so that $m(angleACB) = m(\angle ABC)$. Hence, $180° - 8x + 3x = 4x$. Solving this equation for x we find $x = 18°$ ■

Problem 46.9
Find all missing angle measures in each figure.

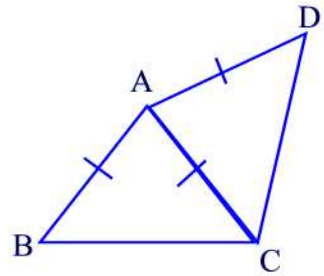


Solution.



Problem 46.10
An eighth grader says that AB=AC=AD , as shown in the figure below, then $m(\angle B) = m(\angle C) = m(\angle D)$. Is this right? If not, what would you tell the

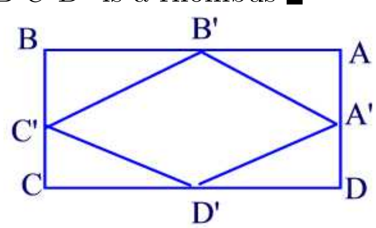
child?



Solution.
The conclusion is false since $m(\angle B) = m(\angle BCA) < m(\angle C)$ ■

Problem 46.11
What type of figure is formed by joining the midpoints of a rectangle?

Solution.
From the figure below we see that $BB' = CD', CC' = BC'$ and $m(\angle B) = m(\angle C) = 90°$. Thus, by the SAS property we conclude that $\Delta B'BC' \cong D'CC'$ so that $B'C' = C'D'$. Similar arguments show that $B'C' = C'D' = A'B' = A'D'$ so that $A'B'C'D'$ is a rhombus ■



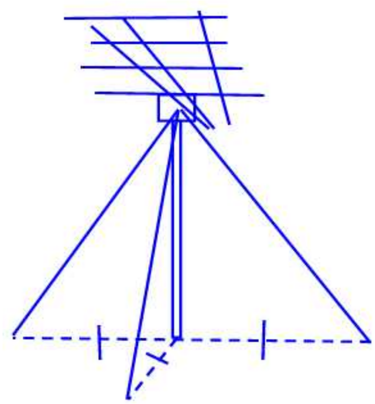
Problem 46.12
If two triangles are congruent what can be said about their perimeters? areas?

Solution.
Suppose that $\Delta ABC \cong \Delta A'B'C'$. Then $AB = A'B', AC = A'C', BC = B'C'$. Thus the two triangles have equal perimeters ■

Problem 46.13
In a pair of right triangles, suppose two legs of one are congruent respectively to two legs of the other. Explain whether the triangles are congruent and why.

Solution.
The two triangles are congruent by the SAS property ■

Problem 46.14
A rural homeowner had his television antenna held in place by three guy wires, as shown in the following figure. If the distance to each of the stakes from the base of the antenna are the same, what is true about the lengths of the wires? Why?



Solution.
The lengths must be the same because they are corresponding sides of congruent triangles ■

Problem 46.15
For each of the following, determine whether the given conditions are sufficient to prove that $\Delta PQR \cong \Delta MNO$. Justify your answer.
(a) PQ = MN, PR = MO, $m(\angle P) = m(\angle M)$
(b) PQ = MN, PR = MO, QR = NO
(c) PQ = MN, PR = MO, $m(\angle Q) = m(\angle N)$

Solution.
(a) SAS property guarantees that the two triangles are congruent
(b) SSS property guarantees that the two triangles are congruent
(c) Conditions do not necessarily imply that the two triangles are congruent ■

Problem 46.16
Given that $\Delta RST \cong \Delta JLK$, complete the following statements.
(a) $\Delta TRS \cong \Delta$____

(b) $\Delta SRT \cong \Delta$____
(c) $\Delta TSR \cong \Delta$____
(d) $\Delta JKL \cong \Delta$____

Solution.
(a) $\Delta TRS \cong \Delta KJL$
(b) $\Delta SRT \cong \Delta LJK$
(c) $\Delta TSR \cong \Delta KLJ$
(d) $\Delta JKL \cong \Delta RTS$ ■

Problem 46.17
You are given ΔRST and ΔXYZ with $m(\angle S) = m(\angle Y)$.
(a) To show $\Delta RST \cong \Delta XYZ$ by the SAS congruence property, what more would you need to know?
(b) To show that $\Delta RST \cong \Delta XYZ$ by the ASA congruence property, what more would you need to know?

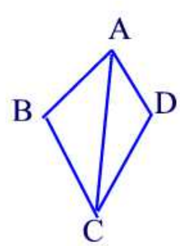
Solution.
(a) $RS = XY$ and $ST = YZ$
(b) $RS = XY$ and $m(\angle R) = m(\angle X$ ■

Problem 46.18
You are given ΔABC and ΔGHI with AB = GH. To show that $\Delta ABC \cong \Delta GHI$ by the SSS congruence property, what more would you need to know?

Solution.
$AC = GI$ and $BC = HI$ ■

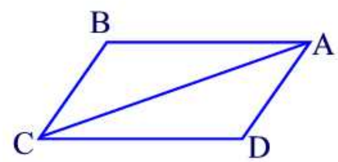
Problem 46.19
Suppose that ABCD is a kite with AB = AD and BC = DC. Show that the diagonal $\overline{AC}$ divides the kite into two congruent triangles.

Solution.
Since $AB = AD, BC = DC$, and $AC = AC$ then $\Delta ABC \cong \Delta ADC$ by SSS ■



Problem 46.20
(a) Show that the diagonal of a parallelogram divides it into two congruent triangles.
(b) Use part (a) to show that the opposite sides of a parallelogram are congruent.
(c) Use part (a) to show that the opposite angles of a parallelogram are congruent.

Solution.
(a) Since $m(\angle CAD) = m(\angle BCD$ and $m(\angle ACD) = m(\angle BAC$ then by ASA $\Delta ABC \cong \Delta CDA$. A similar argument shows that $\Delta BCD \cong \Delta BAD$
(b) By (a) $AB = CD$ and $BC = DA$
(c) By (a) $m(\angle B) = m(\angle D)$ and $m(\angle A) = m(\angle C$ ■



Problem 47.1
Which of the following triangles are always similar?
(a) right triangles
(b) isosceles triangles
(c) equilateral triangles

Solution.
(a) No. For example, consider the two right triangles ΔABC and ΔDEF with $m(\angle A) = m(\angle C) = 45°, m(\angle B) = 90°, m(\angle D) = 35°, m(\angle F = 55,$ and $m(\angle E) = 90°$
(b) Yes by the SAS property.
(c) Yes by the AA property since all angles are the same ■

Problem 47.2
Show that if $\Delta ABC \sim \Delta A'B'C'$ and $\Delta A'B'C' \sim \Delta A''B''C''$ then $\Delta ABC \sim \Delta A''B''C''$.

Solution.
Suppose that $\Delta ABC \sim \Delta A'B'C'$ then $m(\angle A) = m(\angle A'), m(\angle B) = m(\angle B'), m(\angle C) = m(\angle C')$ and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

Similarly, if $\Delta A'B'C' \sim \Delta A''B''C''$ then $m(\angle A') = m(\angle A"), m(\angle B') = m(\angle B"), m(\angle C') = m(\angle C")$ and

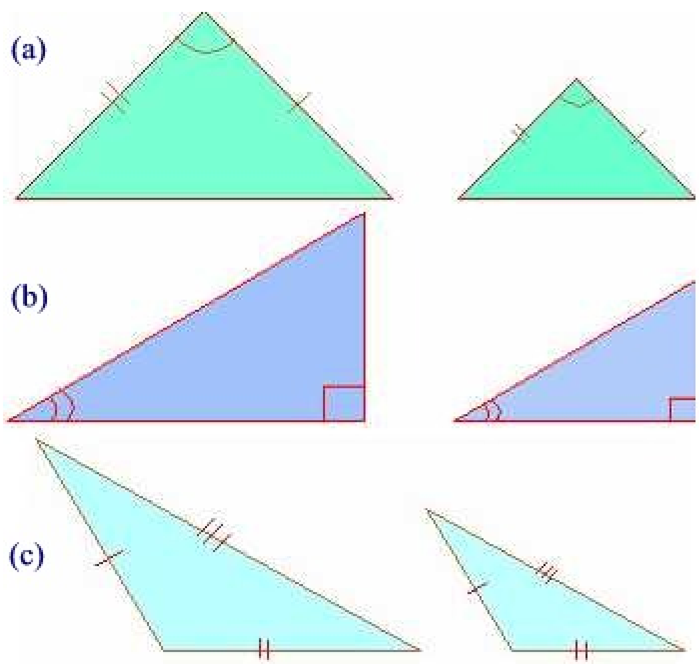
$$\frac{A'B'}{A"B"} = \frac{A'C'}{A"C"} = \frac{B'C'}{B"C"}$$

Thus, $m(\angle A) = m(\angle A"), m(\angle B) = m(\angle B"), m(\angle C) = m(\angle C")$ and

$$\frac{AB}{A"B"} = \frac{AC}{A"C"} = \frac{BC}{B"C"}$$

This shows that $\Delta ABC \sim \Delta A''B''C''$ ∎

Problem 47.3
Each pair of triangles is similar. By which test can they be proved to be similar ?



Solution.
(a) SAS
(b) AA
(c) SSS ∎

Problem 47.4
Suppose $\Delta ABC \sim \Delta DEF$ with scaled factor k.
(a) Compare the perimeters of the two triangles.
(b) Compare the areas of the two triangles.

Solution.
(a) The perimeter of ΔABC is $AB+BC+CA$. The perimeter of ΔDEF is $DE+EF+FE$. But $DE=kAB, DF=kAC$, and $EF=kBC$. Thus, the perimeter of ΔDEF is k times larger than the perimeter of ΔABC.
(b) Let h be a height of ΔABC with base BC and h' be a height if DEF with base EF. Then

$$\frac{\frac{h \cdot BC}{2}}{\frac{h' \cdot EF}{2}} = k^2 \blacksquare$$

Problem 47.5
Areas of two similar triangles are 144 sq.cm. and 81 sq.cm. If one side of the first triangle is 6 cm then find the corresponding side of the second triangle.

Solution.
By the previous exercise the scaled factor is $k=\sqrt{\frac{144}{81}}=\frac{4}{3}$. Now, if we assume $\Delta ABC \sim \Delta DEF$ and $AB = 6\ cm$ then $DE = kAB = \frac{4}{3}\cdot 6 = 12\ cm$ $\blacksquare$

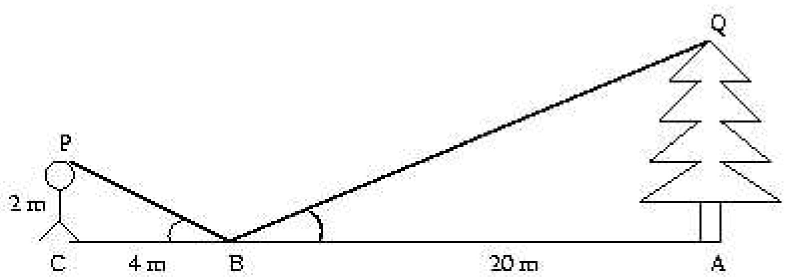
Problem 47.6
The side of an equilateral triangle ΔABC is 5 cm. Find the length of the side of another equilateral triangle ΔPQR whose area is four times area of ΔABC.

Solution.
Since the two triangles are equilateral then $\Delta ABC \sim \Delta PQR$. Since the area of ΔPQR is four times the area of ΔABC then $k=2$. Since $AB=2PQ$ then $PQ=\frac{5}{2}=2.5\ cm$ $\blacksquare$

Problem 47.7
The corresponding sides of two similar triangles are 4 cm and 6 cm. Find the ratio of the areas of the triangles.

Solution.
We are given that $k=\frac{6}{4}=1.5$. Thus, the ratio of the areas of the two triangles is $k^2=\frac{9}{4}$ $\blacksquare$

Problem 47.8
A clever outdoorsman whose eye-level is 2 meters above the ground, wishes to find the height of a tree. He places a mirror horizontally on the ground 20 meters from the tree, and finds that if he stands at a point C which is 4 meters from the mirror B, he can see the reflection of the top of the tree. How high is the tree?



Solution.
By the AA test we have $\Delta BPC \sim \Delta BQA$. Thus,

$$\frac{QA}{PC} = \frac{BA}{BC} = \frac{20}{4} = 5.$$

Hence, $QA = 5PC = 10\ m$ ■

Problem 47.9
A child 1.2 meters tall is standing 11 meters away from a tall building. A spotlight on the ground is located 20 meters away from the building and shines on the wall. How tall is the child's shadow on the building?

Solution.
By the AA property $\Delta ABC \sim \Delta APQ$. Thus,

$$\frac{AC}{AQ} = \frac{BC}{PQ}$$

Hence,

$$\frac{9}{20} = \frac{1.2}{PQ}$$

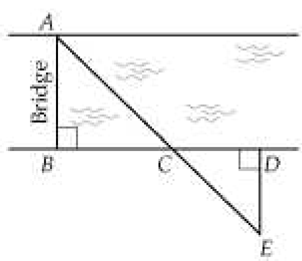
Solving for Q we find $PQ = \frac{24}{9} \approx 2.67\ m$ ■

Problem 47.10
On a sunny day, Michelle and Nancy noticed that their shadows were different lengths. Nancy measured Michelle's shadow and found that it was 96 inches long. Michelle then measured Nancy's shadow and found that it was 102 inches long.

(a) Who do you think is taller, Nancy or Michelle? Why?
(b) If Michelle is 5 feet 4 inches tall, how tall is Nancy?
(c) If Nancy is 5 feet 4 inches tall, how tall is Michelle?

Solution.
(a) Let M be the height of Michelle and N that of Nancy. Then using similar triangles, we have
$$\frac{M}{N} = \frac{96}{102} < 1.$$
This shows that Nancy is taller than Michelle.
(b) If $M = 64\ in$ then $N = \frac{64 \cdot 102}{96} = 68\ in$. That is, Nancy is 5 feet 8 inches tall.
(c) If $N = 64\ in$ then $M = \frac{96 \cdot 64}{102} \approx 60\ in$ or about 5 feet ■

Problem 47.11
An engineering firm wants to build a bridge across the river shown below. An engineer measures the following distances: BC = 1,200 feet, CD = 40 feet, and DE = 20 feet.



(a) Prove ΔABC is similar to ΔEDC.
(b) Railings cost \$4 per foot. How much will it cost to put railings on both sides of the bridge?

Solution.
(a) Since $m(\angle B) = m(\angle D) = 90°$ and $m(\angle BCA) = m(\angle DCE)$ (vertical

angles) then by the AA test $\Delta ABC \sim \Delta EDC$.
(b) Finding the length of AB we have

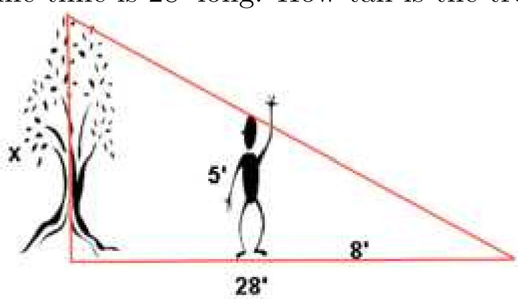
$$\frac{AB}{DE} = \frac{BC}{CD}$$

or

$$\frac{AB}{20} = \frac{1200}{40}$$

Solving for AB we find $AB = 600\ ft$. Thus, it costs $1200 \times 4\$4,800$ to put railings on both sides of the bridge ■

Problem 47.12
At a certain time of the day, the shadow of a $5'$ boy is $8'$ long. The shadow of a tree at this same time is $28'$ long. How tall is the tree?



Solution.
Using similar triangles we can write

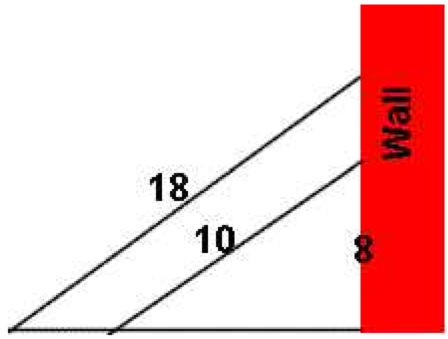
$$\frac{height\ of\ tree}{height\ of\ boy} = \frac{shadow\ of\ tree}{shadow\ of\ boy}$$

That is,

$$\frac{height\ of\ tree}{5} = \frac{28}{8}$$

Hence, the height of the tree is $5 \cdot \frac{28}{8} = 17.5'$ ■

Problem 47.13
Two ladders are leaned against a wall such that they make the same angle with the ground. The $10'$ ladder reaches $8'$ up the wall. How much further up the wall does the $18'$ ladder reach?

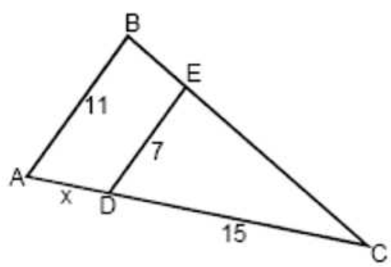


Solution.
By the AA test the two triangles are similar so that

$$\frac{top\ of\ 18'\ ladder\ from\ ground}{8} = \frac{18}{10}$$

Hence, the top of 18' ladder from the ground is $8 \cdot \frac{18}{10} = 14.4'$ ■

Problem 47.14
Given that lines DE and AB are parallel in the figure below, determine the value of x, i.e. the distance between points A and D.



Solution.
By the AA test we have $\Delta CED \sim \Delta CBA$ so that

$$\frac{CD}{CA} = \frac{ED}{AB}$$

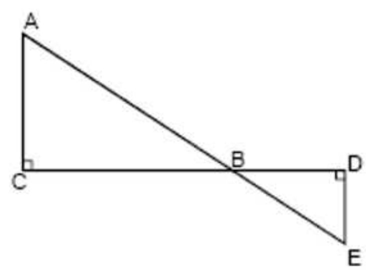
or

$$\frac{15}{x+15} = \frac{7}{11}$$

Solving this equation for x we find

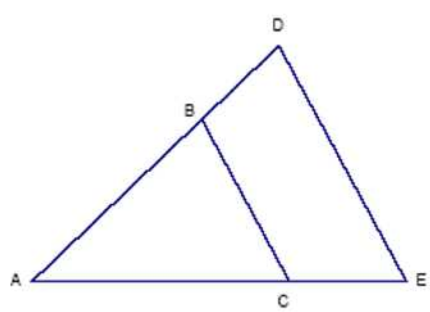
$$\begin{array}{rcl} \frac{15}{x+15} & = & \frac{7}{11} \\ 7(x+15) & = & 15 \cdot 11 \\ 7x + 105 & = & 165 \\ 7x + 105 - 105 & = & 165 - 105 \\ 7x & = & 60 \\ x & = & \frac{60}{7} \approx 8.57 \blacksquare \end{array}$$

Problem 47.15
In the figure below, lines AC and DE are vertical, and line CD is horizontal. Show that $\Delta ABC \sim \Delta EBD$.

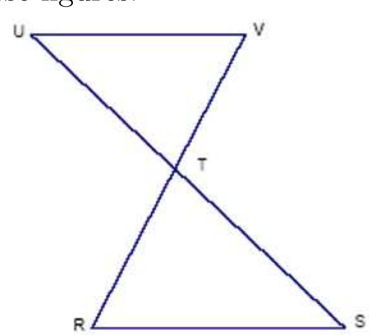


Solution.
We have: $m(\angle C) = m(\angle D) = 90°$ and $m(\angle CBA) = m(\angle DBE)$ (vertical angles). Hence, by the AA test we have $\Delta ABC \sim \Delta EBD$ $\blacksquare$

Problem 47.16
Find a pair of similar triangles in each of these figures:



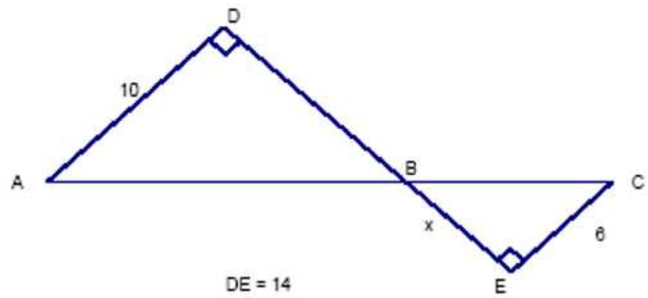
$\overline{BC} \parallel \overline{DE}$



$\overline{UV} \parallel \overline{RS}$

Solution.
$\Delta ABC \sim \Delta ADE$ by the AA test. Similarly, $\Delta TUV \sim \Delta TSR$ ■

Problem 47.17
Find x :



Solution.
Since $m(\angle E) = m(\angle D) = 90°$ and $m(\angle DBA) = m(\angle EBC)$ (vertical angles) then $\Delta BDA \sim \Delta BEC$. This implies that

$$\frac{DB}{x} = \frac{10}{6}$$

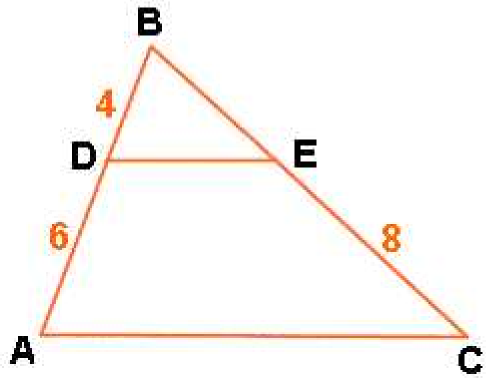
or

$$\frac{14 - x}{x} = \frac{10}{6}$$

Solving for x we find

$$\begin{array}{rcl} \frac{14-x}{x} & = & \frac{10}{6} \\ 10x & = & 6(14 - x) \\ 10x & = & 84 - 6x \\ 16x & = & 84 \\ x & = & \frac{84}{16} = 5.25 \blacksquare \end{array}$$

Problem 47.18
In the diagram, DE is parallel to AC. Also, BD = 4, DA = 6 and EC = 8. Find BC to the nearest tenth.



Solution.
By the AA test we have $\Delta BDA \sim BAC$ so that

$$\frac{BC}{BE} = \frac{BA}{BD}$$

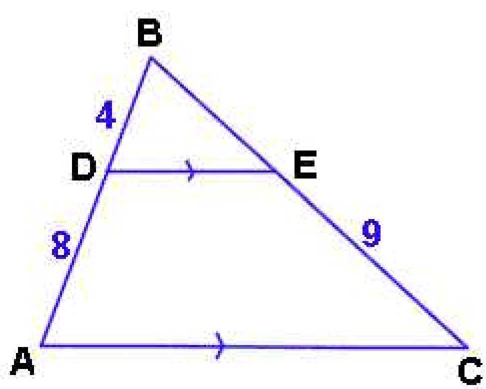
or

$$\frac{BC}{BC-8} = \frac{10}{4}$$

Solving this equation for BC we find

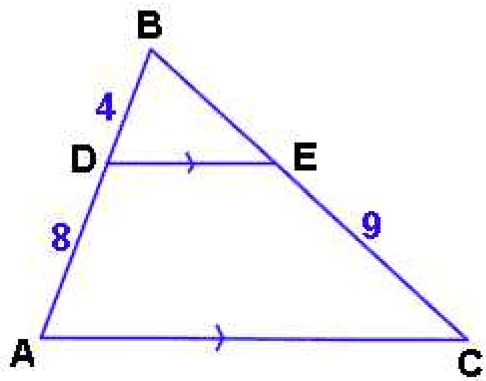
$$\begin{array}{rcl} \frac{BC}{BC-8} & = & \frac{10}{4} \\ 4BC & = & 10(BC-8) \\ 4BC & = & 10BC - 80 \\ 6BC & = & 80 \\ BC & = & \frac{80}{6} = 13.3 \blacksquare \end{array}$$

Problem 47.19
Find BC.



Solution.
See the previous exercise ■

Problem 47.20
Find BE



Solution.
Since $\Delta BDE \sim \Delta BAC$ then

$$\frac{BE}{BC} = \frac{4}{12}$$

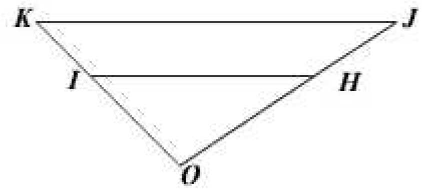
or

$$\frac{BE}{BE+9} = \frac{4}{12}$$

Solving for BE we find

$$\begin{array}{rcl} \frac{BE}{BE+9} & = & \frac{1}{3} \\ 3BE & = & BE+9 \\ 2BE & = & 9 \\ BE & = & 4.5 \blacksquare \end{array}$$

Problem 47.21
Copy and complete the given table. It is given that $\frac{OH}{HJ} = \frac{OI}{IK}$.



OH	HJ	OJ	OI	IK	OK
15	5	?	45	?	?

Solution.
We have: $OJ = OH + HJ = 15 + 5 = 20$. Also, $\frac{15}{5} = \frac{45}{IK}$. Solving for IK we find $IK = \frac{45}{3} = 15$. Finally, $OK = OI + IK = 45 + 15 = 60$ ■

OH	HJ	OJ	OI	IK	OK
15	5	20	45	15	15