7 Relations and Functions

In this section, we introduce the concept of relations and functions.

Relations
A **relation** R from a set A to a set B is a set of ordered pairs (a, b), where
• a is a member of A,
• b is a member of B,
• The set of all first elements (a) is the **domain** of the relation, and
• The set of all second elements (b) is the **range** of the relation.
Often we use the notation $a\ R\ b$ to indicate that a and b are related, rather than the order pair notation (a, b). We refer to a as the **input** and b as the **output.**

Example 7.1
Find the domain and range of the relation $R = \{(2, 3), (2, 4), (3, 7), (5, 2)\}$.

Solution.
The domain is the set $\{2, 3, 5\}$ and the range is the set $\{2, 3, 4, 7\}$.$\blacksquare$

Note that a relation R is just a subset of the Cartesian product $A \times B$.
We can also represent a relation as an **arrow diagram.** For example, the relation $\{(1, 2), (0, 1), (3, 4), (2, 1), (0, -2)\}$ can be represented by the diagram of Figure 7.1

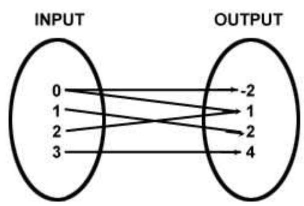


Figure 7.1

When a relation R is defined from a set A into the same set A then there are three useful properties to look at:

<u>**Reflexive Property:**</u>
A relation R on A is said to be **reflexive** if every element of A is related to itself. In notation, $a\ R\ a$ for all $a \in A$. Examples of reflexive relations include:
• "is equal to" (equality)
• "is a subset of" (set inclusion)
• "is less than or equal to" and "is greater than or equal to" (inequality)
• "divides" (divisibility).
An example of a non reflexive relation is the relation "is the father of" on a set of people since no person is the father of themself.
When looking at an arrow diagram, a relation is reflexive if every element of A has an arrow pointing to itself. For example, the relation in Figure 7.2 is a

reflexive relation.

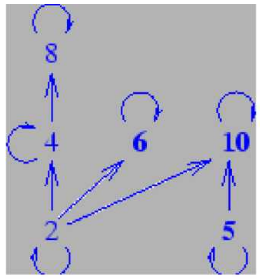


Figure 7.2

Symmetric Property

A relation R on A is **symmetric** if given $a\ R\ b$ then $b\ R\ a$. For example, "is married to" is a symmetric relation, while, "is less than" is not.The relation "is the sister of" is not symmetric on a set that contains a brother and sister but would be symmetric on a set of females.

The arrow diagram of a symmetric relation has the property that whenever there is a directed arrow from a to b, there is also a directed arrow from b to a. See Figure 7.3.

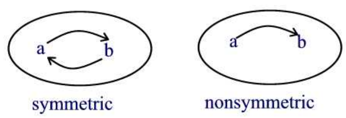


Figure 7.3

Transitive Property

A relation R on A is **transitive** if given $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$. Examples of reflexive relations include:

- "is equal to" (equality)
- "is a subset of" (set inclusion)
- "is less than or equal to" and "is greater than or equal to" (inequality)
- "divides" (divisibility).

On the other hand, "is the mother of" is not a transitive relation, because if Alice is the mother of Brenda, and Brenda is the mother of Claire, then Alice is not the mother of Claire.

The arrow diagram of a transitive relation has the property that whenever there are directed arrows from a to b and from b to c then there is also a directed arrow from a to c. See Figure 7.4.

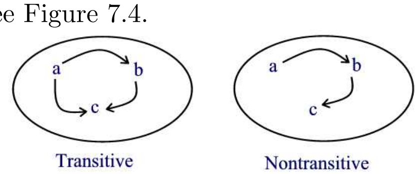
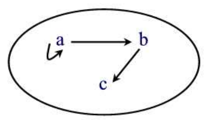


Figure 7.4

A relation that is reflexive, symmetric, and transitive is called an **equivalence relation** on A. Examples of equivalence relations include
• The equality ("=") relation between real numbers or sets.
• The relation "is similar to" on the set of all triangles.
• The relation "has the same birthday as" on the set of all human beings.
On the other hand, the relation " $\subseteq$ " is not an equivalence relation on the set of all subsets of a set A since this relation is not symmetric.

Practice Problems

Problem 7.1
Express the relation given in the arrow diagram below in its ordered-pair representation.



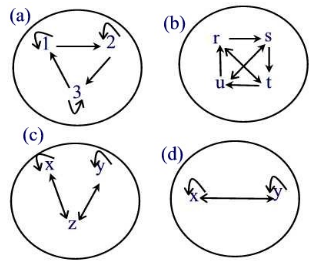
Problem 7.2
Consider the relation "is a factor of" from the set $A = \{2, 3, 4, 5, 6\}$ to the set $B = \{6, 8, 10, 12\}$. Make an arrow diagram of this relation.

Problem 7.3
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) $\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$
(b) $\{(1,2),(1,3),(2,3),(2,1),(3,2),(3,1)\}$
(c) $\{(1,1),(1,3),(2,2),(3,2),(1,2)\}$
(d) $\{1,1),(2,2),(3,3)\}$.

Problem 7.4
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) "less than" on the set $\mathbb{N}$
(b) "has the same shape as" on the set of all triangles
(c) "is a factor of" on the set $\mathbb{N}$
(d) "has the same number of factors as" on the set $\mathbb{N}$.

Problem 7.5
List all the ordered pairs of each of the following relations on the sets listed. Which, if any, is an equivalence relation?
(a) "has the same number of factors as" on the set $\{1, 2, 3, 4, 5, 6\}$
(b) "is a multiple of " on the set $\{2, 3, 6, 8, 10, 12\}$
(c) "has more factors than" on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Problem 7.6
Determine whether the relations represented by the following diagrams are reflexive, symmetric, or transitive. Which relations are equivalence relations?



Problem 7.7
Consider the relations R on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ defined by the equation $a + b = 11$. Determine all the ordered pairs (a, b) that satisfy the equation. Is this relation an equivalence relation?

Problem 7.8
True or false?
(a) "If a is related to b then b is related to a" is an example of a reflexive relation.
(b) The ordered pair $(6, 24)$ satisfies the relation "is a factor of".

Problem 7.9
Let R be a relation on the set $A = \{a, b, c\}$. As a list of ordered pairs the relation has five elements. One of the element is (a, b). What are the remaining elements if R is both reflexive and symmetric?

Problem 7.10
If the relation $\{(1, 2), (2, 1), (3, 4), (2, 4), (4, 2), (4, 3)\}$ on the set $\{1, 2, 3, 4\}$ is to be altered to have the properties listed, what other ordered pairs, if any, are needed?
(a) Reflexive (b) Symmetric (c) Transitive (d) Reflexive and transitive.

Functions
Note that the definition of a relation does not say that each element from A needs to be associated with one (or more) elements from B. It is sufficient if some associations between elements of A and B are defined. In contrast, there is the definition of a function:
A relation is a **function** if and only if every element of A occurs once and only once as a first element of the relation. That is, if every input of A has exactly one output in B. We call A the **domain** and B the **codomain.**

Example 7.2
Let $A = \{1, 2, 3, 4\}, B = \{14, 7, 234\}, C = \{a, b, c\}$, and $\mathbb{R} = real\ numbers$. Define the following relations:
(a) R_1 is the relation between A and B that associates the pairs

$$1\ R\ 234,\ \ 2\ R\ 7,\ \ 3\ R\ 14,\ \ 4\ R\ 234,\ \ 2\ R\ 234$$

(b) R_2 is the relation between A and C given by $\{(1, c), (2, b), (3, a), (4, b)\}$
(c) R_3 is the relation between A and C given by $\{(1, a), (2, a), (3, a)\}$.
Which of those relations are functions ?

Solution.
(a) R_1 is not a function since the element 2 is associated with two elements of B, namely 7 and 234.
(b) R_2 is a function since every member of A is associated to exactly one member of B. Note that members of A can be associated to same elements of B.
(c) R_3 is not a function since the element 4 from the domain A has no element associated with it.$\blacksquare$

Functions can be named using **function notation.** For example, the function represented symbolically by the equation:

$$y = x^2 + 1$$

might be named $f(x)$. In this case, the equation would be written as:

$$f(x) = x^2 + 1.$$

Note that the parentheses in the notation $f(x)$ do not indicate multiplication. $f(x)$ is "f of x", not "f times x".
With this notation, we define the **range** of f to be the set $\{(x, f(x)) | x \in A\}$.

Example 7.3
In stores that sell athletic shoes of various kinds, the cost of doing business includes fixed expenses C_0(like rent and pay for employees) and variable expenses m (like the number of pairs of shoes bought from manufacturers). Operating cost of any store would be a function of those two main factors. Express this function using function notation. Use n to denote the number of shoes bought from the manufacturer.

Solution.
If $C(n)$ denote the total cost of manufacturing n shoes than $C(n) = mn + C_0$.$\blacksquare$

Example 7.4
Find $f(2)$ if $f(x) = 3x - 4$.

Solution.
Replacing x by 2 to obtain $f(2) = 3(2) - 4 = 2$.$\blacksquare$

Describing and Visualizing Functions
Functions as Machines
You can make an analogy between a function and a machine (like a meat grinder). The purpose of this analogy is to link together the abstract symbols used in function notation with a mechanical device that you are already

very familiar with. If you ever get stuck on or confused by some function notation, try to think of what each symbol present would represent in "meat grinder terms."

• $x-$ this is the unprocessed meat that goes into the meat grinder.
• $f-$ this is the name of the machine that is being used (the meat grinder itself)
• $f(x)-$ this is the stuff (ground meat) that comes out of the machine.

Function notation is represented pictorially in Figure 7.5.

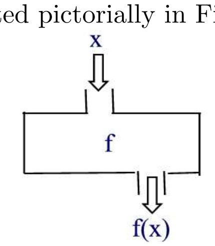
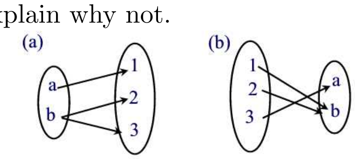


Figure 7.5

Functions as Arrow Diagrams
When the domain and codomain of a function are finite sets then one can represent the function by an arrow diagram. Remember that an arrow diagram represents a function if exactly one arrow must leave each element of the domain and point to one element in the codomain.

Example 7.5
Which of the following arrow diagrams represent functions? If one does not represent a function, explain why not.



Solution.
(a) The diagram does not represent a function since two arrows leave the same element b.
(b) The diagram represents a function since exactly one arrow leaves every element in the domain.■

Functions as Formulas(Symbolic Form)
Consider, for example, a circle of radius r, where the variable r is any positive integer. The formula $A(r) = \pi r^2$ defines the function area that expresses the area of the circle as a function of the radius r.

Functions in Tabular Form
Functions can be represented by tables. For example, the following table gives the grades of three students on a math quiz.

Student	Grade
Mark	8
Stve	7
Mary	10

Functions as Ordered Pairs
If the domain of the function is finite then one can represent the function by listing all the ordered pairs. If the domain is infinite then the function can be represented by ordered pairs using the set-builder notation. For example, the function that squares every real number can be represented by the set

$$\{(x, x^2) | x \in \mathbb{R}\}.$$

Example 7.6
Which of the following are functions from x to y? Assuming that the entire set of ordered pairs is given.
(a) $\{(1,2),(2,2),(3,4),(4,5)\}$
(b) $\{((1,3),(5,1),(5,2),(7,9)\}$.

Solution.
(a) The set satisfies the definition of a function.
(b) This is not a function since the element 5 is associated to two outputs 1 and 2.■

Functions as Graphs
A function whose domain and range are sets of numbers can be graphed on a set of $x-$ and $y-$axes: if $f(x) = y$, plot the points (x, y) for all x in the domain of the function. If the domain is finite then the graph consists of dots whereas if the domain is infinite then the graph is usually a curve. For example, the graph of the function $\{(1,2),(2,2),(3,4),(4,5)\}$ is given in Figure 7.6(a) whereas the graph of the function $y = 2x$ is given in Figure 7.6(b).

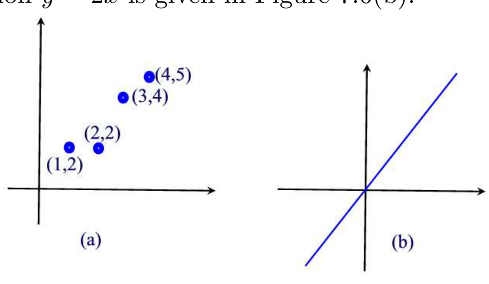


Figure 7.6

Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 7.7
Which of the graphs (a), (b), (c) in Figure 7.7 represent y as a function of x?

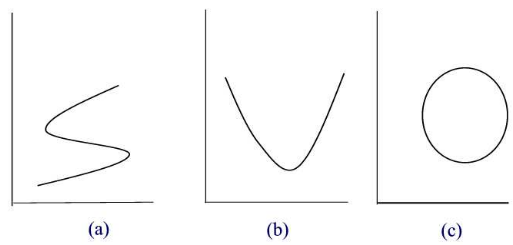


Figure 7.7

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

Practice Problems

Problem 7.11
List the ordered pairs for these functions using the domain specified. Find the range for each function.
(a) $C(t) = 2t^3 - 3t$, with domain $\{0, 2, 4\}$
(b) $a(x) = x + 2$, with domain $\{1, 2, 9\}$
(c) $P(n) = \left(\frac{n+1}{n}\right)$, with domain $\{1, 2, 3\}$.

Problem 7.12
Find the value of $\frac{f(x+h)-f(x)}{h}$ given that $f(x) = x^2$.

Problem 7.13
Given $f(x) = -x^2 + 2x + 6$, find $f(-4)$.

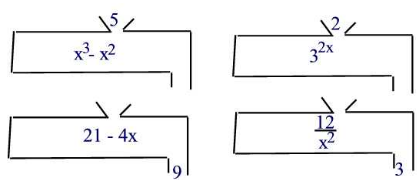
Problem 7.14
A function f on the set of real numbers $\mathbb{R}$ is defined as

$$f(x) = \frac{3x+2}{x-1}.$$

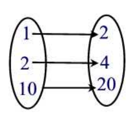
Find:
(a) the domain of f
(b) the range of f
(c) the image of -2 under f
(d) x when $f(x) = -3$.

Problem 7.15
Which of the following relations, listed as ordered pairs, could belong to a function? For those that cannot, explain why not.
(a) $\{(7,4),(6,3),(5,2),(4,1)\}$
(b) $\{((1,1),(1,2),(3,4),(4,4)\}$
(c) $\{(1,1),(2,1),(3,1),(4,1)\}$
(d) $\{(a,b),(b,b),(d,e),(b,c),(d,f)\}$.

Problem 7.16
Using the function machines, find all possible missing whole-number inputs and outputs.



Problem 7.17
The following functions are expressed in one of the following forms: a formula, an arrow diagram, a table, or a set of ordered pairs. Express each function in each of the other forms.
(a) $f(x) = x^3 - x$ for $x \in \{0,1,4\}$.
(b) $\{(1,1),(4,2),(9,3)\}$
(c)



(d)

x	$f(x)$
5	55
6	66
7	77

Problem 7.18
(a) The function $f(n) = \frac{9}{5}n + 32$ can be used to convert degrees Celsius to degrees Fahrenheit. Calculate, $f(0), f(100), f(5)$, and $f(-40)$.
(b) The function $g(n) = \frac{5}{9}(n - 32)$ can be used to convert degrees Fahrenheit to degrees Celsius. Calculate, $g(32), g(212), g(104)$, and $g(-40)$.
(c) Is there a temperature where the degrees Celsius equals the degrees Fahrenheit? If so, what is it?

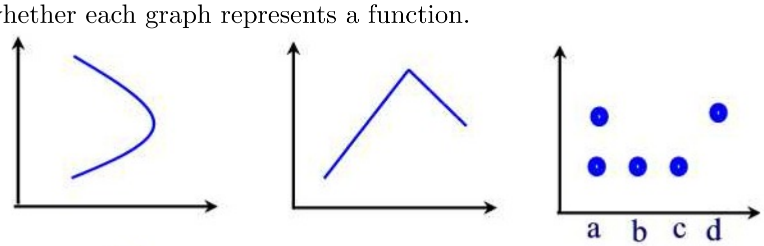
Problem 7.19
A fitness club charges an initiation fee of \$85 plus \$35 per month.
(a) Write a formula for a function, $C(x)$, that gives the total cost for using the fitness club facilities after x months.
(b) Calculate $C(18)$ and explain in words its meaning.
(c) When will the total amount spent by a club member first exceed \$1000?

Problem 7.20
If the interest rate of a \$1000 savings account is 5% and no additional money is deposited, the amount of money in the account at the end of t years is given by the function $a(t) = (1.05)^t \cdot 1000$.
(a) Calculate how much will be in the account after 2 years, 5 years, and 10 years.
(b) What is the minimum number of years that it will take to more than double the account?

Problem 7.21
A function has the formula $P(N) = 8n - 50$. The range for P is $\{46, 62, 78\}$. What is the domain?

Problem 7.22
Which of the following assignments creates a function?
(a) Each student in a school is assigned a teacher for each course.
(b) Each dinner in a restaurant is assigned a price.
(c) Each person is assigned a birth date.

Problem 7.23
Tell whether each graph represents a function.



(a) (b) (c)