6 The Hindu-Arabic System (800 BC)

Today the most universally used system of numeration is the **Hindu-Arabic system**, also known as the **decimal system** or **base ten system.** The system was named for the Indian scholars who invented it at least as early as 800 BC and for the Arabs who transmitted it to the western world. Since the base of the system is ten, it requires special symbols for the numbers zero through nine. The following list the features of this system:

(1) **Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9**. These symbols can be used in combination to represent all possible numbers.
(2) **Grouping by ten:** Grouping into sets of 10 is a basic principle of this system. Figure 6.1 shows how grouping is helpful when representing a collection of objects.

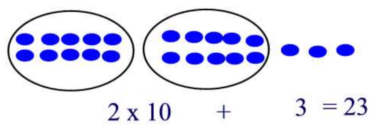


Figure 6.1

(3) **Place value:** The place value assigns a value of a digit depending on its placement in a numeral. To find the value of a digit in a whole number, we multiply the place value of the digit by its **face value,** where the face value is a digit. For example, in the numeral 5984, the 5 has place value "thousands", the 9 has place value "hundreds", the 8 has place value "tens", and 4 has place value "units".
(4) **Expanded form:** We can express the numeral 5984 as the sum of its digits times their respective place values, i.e. in **expanded form**

$$5984 = 5 \times 1000 + 9 \times 100 + 8 \times 10 + 4 = 5 \times 10^3 + 9 \times 10^2 + 8 \times 10^1 + 4.$$

Example 6.1
Express the number 45,362 in expanded form.

Solution.
We have: $45,362 = 4 \times 10^4 + 5 \times 10^3 + 3 \times 10^2 + 6 \times 10 + 2.$■

Word names for Hindu-Arabic numerals:
(1) unique names: 0 (zero), 1(one), 2(two), 3(three),4(four),5(five), 6(six), 7(seven), 8(eight), 9(nine), 10(ten), 11(eleven), 12(twelve).
(2) $13, 14, \cdots, 19$ teen (for ten). For example, $14 = (4 + 10)$ four-teen.
(3) $20, 21, \cdots, 99$ $57 = 5 \times 10 + 7$ fifty seven.
(4) $100, 101, \cdots, 999$ is the combination of hundreds and previous names. For example, 538 is read "five hundreds thirty eight."
(5) In numerals containing more than three digits, groups of three digits are sets off by commas. For example, the number 864,456,221,653,127,851 is read:

"eight hundred sixty four quadrillion four hundred fifty six trillion two hundred twenty one billion six hundred fifty three million one hundred twenty seven thousand eighthundred fifty one".

Nondecimal Numeration Systems
The decimal system discussed above is based on grouping by ten. Some other grouping are of interest such as grouping by two , three, four, etc. We apparently use base 10 because we (most of us) have ten fingers. Base 2 (also known as **binary**) is what computers use, internally. It has been joked that this is because they have two fingers (two electrical states, actually). In base 2, there are two different digits (0 and 1). And the first few numbers are $1, 10, 11, 100, 101, 110, 111, 1000, 1001, \cdots$.
It is important to label base two numbers (usually with a subscript 2) because they can be mistaken for base 10 numbers. For example $1010_{two} = 10_{ten}$.
Converting between binary and decimal numbers is fairly simple, as long as you remember that each digit in the binary number represents a power of two.

Example 6.2
Convert 101100101_{two} to the corresponding base-ten number.

Solution.
List the digits in order, and count them off from the RIGHT, starting with zero:

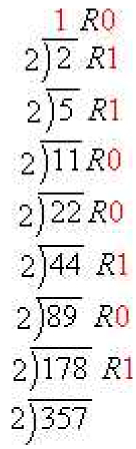
$$\begin{array}{rccccccccc} digits: & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ numbering: & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

Use this listing to convert each digit to the power of two that it represents: $1 \times 2^8 + 0 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
$= 256 + 64 + 32 + 4 + 1 = 357$. Thus, $101100101_{two} = 357_{ten}$.■

Converting decimal numbers to binary numbers is nearly as simple: just divide by 2 as illustrated by the example below.

Example 6.3
Convert 357_{ten} to the corresponding binary number.

Solution.
To do this conversion, you need to divide repeatedly by 2, keeping track of the remainders as you go.



These remainders tell us what the binary number is. Read the numbers from around the outside of the division, starting on top and wrapping your way around the right-hand side. As you can see:

$$357_{ten} = 101100101_{two}.\blacksquare$$

Conversions from any nondecimal system to base ten and vice versa, can be accomplished in a manner similar to that used for base two conversions.

Example 6.4
(a) Convert 11244_{five} to base ten.
(b) Convert 543_{ten} to base four.

Solution.
(a) Using the expanded notation we have

$$11244_{five} = 1 \times 5^4 + 1 \times 5^3 + 2 \times 5^2 + 4 \times 5 + 4 = 824.$$

(b) We use the process of repeated division by 4.

$$\begin{array}{rcl} 543 &=& 4 \times 135 + 3 \\ 135 &=& 4 \times 33 + 3 \\ 33 &=& 4 \times 8 + 1 \\ 8 &=& 4 \times 2 + 0 \\ 2 &=& 4 \times 0 + 2 \end{array}$$

Thus, $543_{ten} = 20133_{four}.\blacksquare$

Practice Problems

Problem 6.1
Write each of the following numbers in expanded form.
(a) 70 (b) 746 (c) 840,001.

Problem 6.2
Write each of the following expressions in standard place-value form. That is, $1 \times 10^3 + 2 \times 10^2 + 7 = 1207$.
(a) $5 \times 10^5 + 3 \times 10^2$.
(b) $8 \times 10^6 + 7 \times 10^4 + 6 \times 10^2 + 5$.
(c) $6 \times 10^7 + 9 \times 10^5$.

Problem 6.3
Write the following numerals in words.
(a) $2,000,000,000$
(b) $87,000,000,000,000$
(c) $52,672,405,123,139$.

Problem 6.4
Write each of the following base seven numerals in expanded notation.
(a) 15_{seven} (b) 123_{seven} (c) 5046_{seven}.

Problem 6.5
Convert each base ten numeral into a numeral in the base requested.
(a) 395 in base eight
(b) 748 in base four
(c) 54 in base two.

Problem 6.6
The base twelve numeration system has the following twelve symbols:0,1,2,3,4,5,6,7,8,9,T,E. Change each of the following numerals to base ten numerals.
(a) 142_{twelve} (b) 503_{twelve} (c) $T9_{twelve}$ (d) $ETET_{twelve}$.

Problem 6.7
Write each of the numerals in base six and base twelve.
(a) 128 (b) 74 (c) 2438.

Problem 6.8
Convert the following base five numerals into base nine numerals.
(a) 12_{five} (b) 204_{five} (c) 1322_{five}.

Problem 6.9
(a) How many different symbols would be necessary for a base twenty system?
(b) What is wrong with the numerals 85_{eight} and 24_{three}?

Problem 6.10
The set of even whole numbers is the set $\{0, 2, 4, 6, \cdots\}$. What can be said about the ones digit of every even number in the following bases?
(a) Ten (b) Four (c) Two (d) Five

Problem 6.11
Translate the following numbers from one base to the other:
(a) $38_{ten} = \underline{\qquad}_{two}$.
(b) $63_{ten} = \underline{\qquad}_{two}$.

Problem 6.12
Translate the following numbers from one base to the other:
(a) $1101_{two} = \_\_\_\_\_\__{ten}$.
(b) $11111_{two} = \_\_\_\_\_\__{ten}$.

Problem 6.13
The sum of the digits in a two-digit number is 12. If the digits are reversed, the new number is 18 greater than the original number. What is the number?

Problem 6.14
State the place value of the digit 2 in each numeral.
(a) 6234 (b) 5142 (c) 2178

Problem 6.15
(a) Write out the first 20 base four numerals.
(b) How many base four numerals precede 2000_{four}?

Problem 6.16
True or false?
(a) $7_{eight} = 7_{ten}$ (b) $30_{four} = 30_{ten}$ (c) $8_{nine} = 8_{eleven}$ (d) $30_{five} = 30_{six}$

Problem 6.17
If all the letters of the alphabet were used as our single-digit numerals, what would be the name of our base system?

Problem 6.18
Find the base ten numerals for each of the following.
(a) 342_{five} (b) $TE0_{twelve}$ (c) 101101_{two}

Problem 6.19
The **hexadecimal** system is a base sixteen system used in computer programming. The system uses the symbols:0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F. Change each of the following hexadecimal numerals to base ten numerals.
(a) $213_{sixteen}$ (b) $1C2B_{sixteen}$ (c) $420E_{sixteen}$

Problem 6.20
Write each of the following base ten numerals in base sixteen numerals.
(a) 375 (b) 2941 (c) 9520 (d) 24,274

Problem 6.21
Rod used base twelve to write the equation:

$$g36_{twelve} = 1050_{ten}.$$

What is the value of g?

Problem 6.22
For each of the following decimal numerals, give the place value of the underlined digit:
(a) $827,\underline{3}67$ (b) $8,421,0\underline{0}0$ (c) $9\underline{7},998$

Problem 6.23
A certain three-digit whole number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the digits is 10. What could the number be?

Problem 6.24
Find the number preceding and succeeding the number $EE0_{twelve}$.