5 Numeration Systems

Numeration and the Whole Numbers
If you attend a student raffle, you might hear the following announcement when the entry forms are drawn

> ”The student with identification number <u>50768-973</u> has just won <u>second</u> prize-<u>four</u> tickets to the big game this Saturday.”

This sentence contains three different types of numbers, each serving a different purpose.
• The number 50768-973 is an **identification** or **nominal number**. A nominal number is a sequence of digits used as a name or label. Telephone numbers, social security numbers, driver's license numbers are all examples of nominal numbers.
• The second type of numbers is called **ordinal numbers**. The words *first, second, third* and so on are used to describe the relative position of objects in an ordered sequence.
• The final use of number by the announcer is to tell how many tickets had been won. That is, the prize is a set of tickets and *four* tells us how many tickets are in the set. More generally, a **cardinal number** of a set is the number of objects in the set. If A is a finite set then we will denote the number of elements in A by $n(A)$. Some authors uses the notation $|A|$ for $n(A)$. So if $A = \{1, 2, \cdots, m\}$ then $n(A) = m$. We define $n(\emptyset) = 0$. The set of cardinal numbers of finite sets is called the set of **whole numbers** and is denoted by W. Thus, $W = \{0, 1, 2, 3, \cdots\}$.
We point out here that numbers can be represented verbally (in a language) or symbolically (in a numeration system). For example, the winner in the above student raffle story wins four tickets to the game. The word four is represented by the symbol 4 in the Hindu-arabic numeration system to be introduced below.

Example 5.1
True or false?

(a) Two equivalent sets are equal.
(b) Two equivalent sets have the same number of elements.

Solution.
(a) This is false. For example, $\{a, b, c\} \sim \{1, 2, 3\}$ but $\{a, b, c\} \neq \{1, 2, 3\}$.
(b) This is always true. That is, if $A \sim B$ then $n(A) = n(B)$.■

Example 5.2
For each set give the whole number that gives the number of elements in the set.

(a) $A = \{x|x\ is\ a\ month\ of\ the\ year\}$
(b) $B = \{n \in \mathbb{N}|n\ is\ square\ number\ between\ 70\ and\ 80\}$.
(c) $C = \{0\}$.

Solution.
(a) $n(A) = 12$.
(b) $n(B) = 0$ since $B = \emptyset$.
(c) $n(C) = 1$.■

Ordering the Whole Numbers
We often wish to relate the number of elements of two given sets. For example, if each child in the class is given one cupcake, and there are some cupcakes left over, we would know that there are more cupcakes than children. Notice that children have been matched to a proper subset of the set of cupcakes.
The order of the whole numbers can be defined in the following way: Let $n(A) = a$ and $n(B) = b$ be two whole numbers, where A and B are finite sets. If there is a one-to-one correspondence between A and a proper subset of B, we say that a **is less than** b and we write $a < b$. Equivalently, we can write $b > a$ which is read **"b is greater than a"**.
If $<$ or $>$ is combined with the equal sign we get the symbols $\leq$ and $\geq$.
There are three ways to compare whole numbers: (1) using sets, (2) counting chants, and (3) whole-number line as shown in the following example.

Example 5.3
Show that $4 < 7$ using the three methods introduced above.

Solution.
(1) Using sets: Figure 5.1(a) shows that a set with 4 elements is in a one-to-one correspondence with a proper subset of a set with 7 elements.

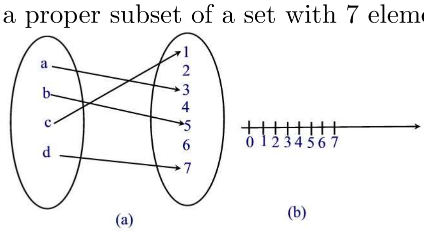


Figure 5.1

(2) Counting chant: one, two, three, four, five, six, seven. Since 4 precedes 7, 4 is is less than 7.
(3) Whole-Number Line: Figure 5.1(b) shows that 4 is to the left of 7 on the number line, so 4 is less than 7 or 7 is greater than 4.■

Practice Problems

Problem 5.1
Let A, B, and C be three sets such that $A \subset B \subseteq C$ and $n(B) = 5$.

(a) What are the possible values of $n(A)$?
(b) What are the possible values of $n(C)$?

Problem 5.2
Determine the cardinality of each of the following sets:

(a) $A = \{x \in \mathbb{N} | 20 \leq x < 35\}$
(b) $B = \{x \in \mathbb{N} | x + 1 = x\}$
(c) $C = \{x \in \mathbb{N} | (x-3)(x-8) = 0\}$.

Problem 5.3
Let A and B be finite sets.

(a) Explain why $n(A \cap B) \leq n(A)$.
(b) Explain why $n(A) \leq n(A \cup B)$.

Problem 5.4
Suppose B is a proper subset of C.

(a) If $n(C) = 8$, what is the maximum number of elements in B?
(b) What is the least possible elements of B?

Problem 5.5
Suppose C is a subset of D and D is a subset of C.

(a) If $n(C) = 5$, find $n(D)$.
(b) What other relationships exist between C and D?

Problem 5.6
Use the definition of less than to show each of the following:

(a) $2 < 4$ (b) $3 < 100$ (c) $0 < 3$.

Problem 5.7
If $n(A) = 4, n(B) = 5$, and $n(C) = 6$, what is the greatest and least number of elements in

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$?

Problem 5.8
True or false?If false give a counter example, i.e. an example that shows that the statement is false.

(a) If $n(A) = n(B)$ then $A = B$.
(b) If $n(A) < n(B)$ then $A \subset B$.

Problem 5.9
Suppose $n(A \cup B) = n(A \cap B)$. What can you say about A and B?

Problem 5.10
Let $U = \{1, 2, 3, \cdots, 1000\}$, F be a subset of U consisting of multiples of 5 and

S the subset of U consisting of multiples of 6.

(a) Find $n(S)$ and $n(F)$.
(b) Find $n(F \cap S)$.
(c) Label the number of elements in each region of a two-loop Venn diagram with universe U and subsets S and F.

Problem 5.11
Finish labeling the number of elements in the regions in the Venn diagram shown, where the subsets A, B, and C of the universe U satisfy the conditions listed. See Figure 5.2.

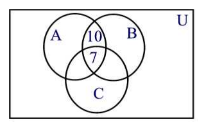


Figure 5.2

$$\begin{array}{rclrcl} n(U) & = & 100 & n(A) & = & 40 \\ n(B) & = & 50 & n(C) & = & 30 \\ n(A \cap B) & = & 17 & n(B \cap C) & = & 12 \\ n(A \cap C) & = & 15 & n(A \cap B \cap C) & = & 7 \end{array}$$

Problem 5.12
Let $S = \{s, e, t\}$ and $T = \{t, h, e, o, r, y\}$. Find $n(S), n(T), n(S \cup T), n(S \cap T), n(S \cap \overline{T})$, and $n(\overline{S} \cap T)$.

Problem 5.13
Suppose that $n(A) = m$ and $n(B) = n$. Find $n(A \times B)$.

Problem 5.14
Explain why $5 < 8$ using the definition of whole number inequality introduced in this section.

Problem 5.15
Let A and B be two sets in the universe $U = \{a, b, c, \cdots, z\}$. If $n(A) = 12, n(B) = 14$, and $n(A \cup B) = 21$, find $n(A \cap B)$ and $n(A \cap \overline{B})$.

Problem 5.16
Suppose that $n(A \times B) = 21$. What are all the possible values of $n(A)$?

Numeration Systems
A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically. We will examine various numeration systems that have been used througout history.

Tally Numeration System
This is the earliest numeration system. Suppose you want to count a group of things (sheep or trees, etc). You could use a vertical line to each object you want to count as shown is Figure 5.3.

Figure 5.3

One advantage of this system is its simplicity. However, a disadvantage of this system is its difficulty to read large numbers. For example, can you tell what number is represented by Figure 5.4?

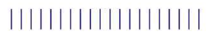


Figure 5.4

The tally system is improved by using **grouping.** The fifth tally mark was placed across every four to make a group of five. Thus, the number in Figure 5.4 is represented as shown in Figure 5.5.

Figure 5.5

Example 5.4
Write the numerals 1 - 10 using the tally numeration system.

Solution.
The numerals 1 - 10 are as follows:

$$|, \ ||, \ |||, \ ||||, \ |||||, \ ||||||, \ |||||||, \ ||||||||, \ |||||||||, \ ||||||||||. \blacksquare$$

Egyptian Numeration System (3400 BC)
This system uses grouping by ten and special symbols representing powers of 10 as shown in Figure 5.6.

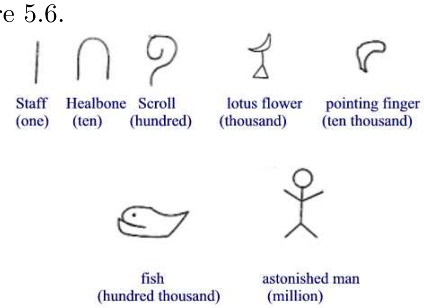


Figure 5.6

Thus, the number 10213 is represented by Figure 5.7.

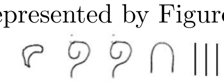


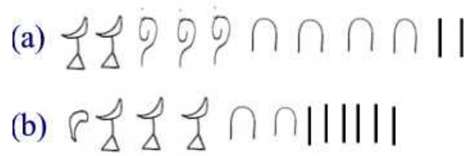
Figure 5.7

This system is called an **additive system** since the value of the symbols are added together to get the value of the number.
An advantage of this system is that fewer symbols are used than the tally system after you get to ten. The disadvantage is that this system is not easy when adding numbers.

Example 5.5
Write the following numbers, using Egyptian numerals:

(a) 2342 (b) 13,026.

Solution.



The Roman Numeration System (500 BC)
The Roman numeration system, an example of an additive system, was developed later than the Egyptian system. Roman numerals were used in most European countries until the eighteenth century. They are still commonly seen on buildings, on clocks, and in books. Roman numerals are selected letters of the Roman alphabet.
The basic Roman numerals are:

$$I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000.$$

Roman numbers are made up of combinations of these basic numerals. For example, $CCLXXXI = 281$.
The Roman system has two advantages over the Egyptian system. The first is that it uses the **subtraction principle** as well as the **addition principle.** Starting from the left, we add each numeral unless its value is smaller than the numeral to its right. In that case, we subtract it from that numeral. For example, the symbol DC represents 500 + 100, or 600, and CD represents 500 - 100, or 400. Similarly, MC represents 1000 + 100, or 1100, and CM represents 1000 - 100, or 900. This is called a **positional system** since the same symbols can mean different values depending on position.

Example 5.6
Write in our numeration system.

(a) CLXII (b) DCXLVI .

Solution.
(a) Since each numeral is larger than the next one on its right, no subtraction is necessary.

$$CLXII = 100 + 50 + 10 + 1 + 1 = 162.$$

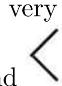
(b) Checking from left to right, we see that X has a smaller value than L. Therefore XL represents 50 - 10, or 40.

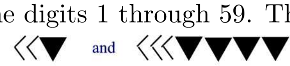
$$DCXLVI = 500 + 100 + (50 - 10) + 5 + 1 = 646. \blacksquare$$

In the roman numeral system, a symbol does not have to be repeated more than three consecutive times. For example, the number 646 would be written DCXLVI instead of DCXXXXVI.
The second advantage of the Roman numeration system over the Egyptian system is that it makes use of the **multiplication principle** for numbers over 1000. A bar over the symbol or group of symbols indicates that the symbol or symbols are to be multiplied by 1000. So, the number 5000 would be written as $\overline{V}$ and the number 40,000 would be written as $\overline{XL}$.

Example 5.7
(a) Convert 9,389 into a Roman numeral.
(b) Convert the Roman number MMCCCLXXXIX into our numeration system.

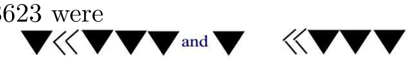
Solution.
(a) $9,389 = 9000+300+80+9 = \overline{IX}+CCC+LXXX+IX = \overline{IX}CCCLXXXIX.$
(b) $MMCCCLXXXIX = 2000 + 300 + 50 + 30 + (10 - 1) = 2,389. \blacksquare$

The Babylonian Numeration System (3000 BC)
The Babylonian numeration system was developed about the same time as the Egyptian system. This system introduced the notion of **place value**, in which the position of a symbol in a numeral determines its value. This made it possible to write numerals for even very large numbers using very few symbols. Indeed, the system utilized only two symbols,  for 1 and  for 10 and combined these additively to form the digits 1 through 59. Thus



respectively, represented 21 and 34. Beyond 59, the system was positional to base sixty, where the positions from right to left represented multiples of successive powers of 60 and the multipliers were composite symbols for 1 through 59. Thus,

represented $2 \cdot 60^1 + 33 = 153$ and $1 \cdot 60^2 + 31 \cdot 60^1 + 22 = 5482$.
A difficulty with the babylonian system was the lack of a symbol for zero. In writing numerals, a space was left if a certain value was not to be used and since spacing was not uniform this always lead to confusion. To be more specific, the notation for 83 and 3623 were



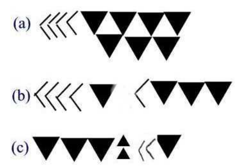
and these could easily be confused if the spacing were not clear. Indeed, there was no way even to indicate missing position values of the extreme right of a numeral, so  could represent 1 or $1 \cdot 60$ or $1 \cdot 60^2$ and so on. Eventually, the

Babylonians employed the symbol  as a **place holder** to indicate missing position values though they never developed the notion of zero as a number. Using this symbol,the numbers 83 and 3623 are now represented as

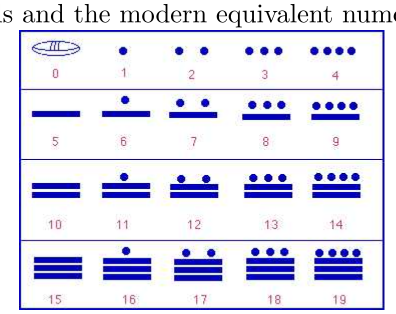
Example 5.8
Write the following numbers using Babylonian numeration:

(a) 47 (b) 2473 (c) 10,821.

Solution.



Mayan Numeration System (200 AD)
The Mayan number system was developed by the ancient Maya civilization in the region now known as the Yucatan Peninsula in Southeastern Mexico. The Maya seem to be the first people who used a place value system and a symbol for zero.
The Mayan numbers are based on three symbols: a dot, a bar, and a symbol for zero, or completion, usually a shell.
In the following table, you can see how the system of dots and bars works to create Mayan numerals and the modern equivalent numerals 0-19.



Like our numbering system, they used place values to expand this system to allow the expression of very large values. Their system has two significant differences from the system we use: 1) the place values are arranged vertically, and 2) they use a modified base 20 system. This means that, instead of the number in the second postion having a value 10 times that of the numeral (as in $21 = 2 \times 10 + 1$), in the Mayan system, the number in the second place has a value 20 times the value of the numeral. However, starting from the third place, the number in the third place has a value of $18 \cdot 20$ times the value of the numeral; and so on. The reason that $18 \cdot 20$ is used instead of $20 \cdot 20$ is that the main function of their number system was to keep track of time; their annual calendar consisted of 360 days. This above principle is illustrated in the example below.

Example 5.9
Write the following Mayan number in our numeration system.

Solution.
The number is: $6 \cdot 18 \cdot 20^2 + 0 \cdot 18 \cdot 20 + 14 \cdot 20 + 7 = 43,487.$■

Example 5.10
Write the number 27,408 in Mayan system.

Solution.
Since $27408 = 3 \cdot 18 \cdot 20^2 + 5808$; $5808 = 16 \cdot 18 \cdot 20 + 48$, and $48 = 2 \cdot 20 + 8$, the Mayan representation of 27408 is

Practice Problems

Problem 5.17
Write the following in Egyptian system.

(a) 11 (b) 597 (c) 1949.

Problem 5.18
Write the following Roman notation using the subtraction principle as appropriate.

(a) 9 (b) 486 (c) 1945.

Problem 5.19
Write the following numbers in Babylonian system.

(a) 251 (b) 3022 (c) 18,741.

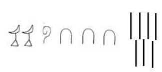
Problem 5.20
Write the following numbers in Mayan System.

(a) 12 (b) 584 (c) 12,473.

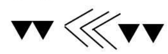
Problem 5.21
Write 2002, 2003, and 2004 in Roman numerals.

Problem 5.22
If the cornerstone represents when a building was built and it reads MCMXXII, when was this building built?

Problem 5.23
Write each of the following numbers in our numeration system, i.e. base 10.

(a) MDCCXXIX
(b) DCXCVII
(c) CMLXXXIV
(d) 

(e) 

(f) 
(g) 
(h) 
(i) 

Problem 5.24
Convert the Roman numeral DCCCCXXIV to Babylonian numeral.

Problem 5.25
Write the following numbers in the given system.

(a) Egyptian numeration:3275
(b) Roman numeration: 406
(c) Babylonian system: 8063
(d) Mayan numeration: 48

Problem 5.26
Represent the number 246 in the Mayan, Babylonian, and Egyptian numeration systems.

(a) In which system is the greatest number of symbols required?
(b) In which system is the smallest number of symbols required?

Problem 5.27
Some children go through reversal stage; that is they confuse 13 and 31, 27 and 72, 59 and 95. What numerals would give Roman children similar difficulties? How about Egyptian children?

Problem 5.28
After the credits of a film roll by, the Roman numeral MCMLXXXIX appears, representing the year in which the film was made. Express the year in our numeration system.

Problem 5.29
True of false?

(a) ||| is three in the tally system.
(b) $IV = VI$ in the Roman system.
(c) $\cap\,||| = |||\wedge$